The stationarity preserving part of a complete Boolean algebra

Takahashi, Joji

(Citation)
Kobe Journal of Mathematics,10(2):189-198

(Issue Date)
1993-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0003588

(URL)
https://hdl.handle.net/20.500.14094/E0003588

TAKAHASHI, J.
KOBE J. MATH.,
10 (1993) 189—198

THE STATIONARITY PRESERVING PART OF A COMPLETE BOOLEAN ALGEBRA

By Joji TAKAHASHI

(Received December 16, 1991)
(Revised September 4, 1992)

Abstract

An algebraic formula expressing the A-value of the statement that all stationary sets in $\mathbf{V}$ remain stationary in $\mathbf{V}^A$ is obtained, where A is an arbitrary complete Boolean algebra. It is also shown how this formula leads to some of the known characterizations of A being proper.

Forcing that preserves the stationarity of stationary sets in the ground model, named *proper forcing* by S. Shelah, has been a subject of considerable interest in set theory. Here, stationarity means that of subsets of $\mathcal{P}_{\leq\omega}(X) = \{M \subseteq X \mid \mathrm{Card}(M) \leq \omega\}$ with the base set X allowed to be arbitrary. In this paper, I will look at the Boolean-value

$$\| \text{ All stationary sets in } \widehat{\mathbf{V}} \text{ remain stationary } \|_A$$

for an arbitrary complete Boolean algebra A. Working with a complete Boolean algebra rather than with a poset, I will obtain an algebraic formula that expresses this value in terms of the infinite meet and join operations of A (Theorem 1.9). I will also show how this formula leads naturally to some of the known characterizations of A being proper (Theorems 2.1 and 2.3).

I am grateful to the referee for suggesting a number of improvements on my original manuscript.

§1. Preservation of stationarity

The main objective of this section is to derive the algebraic formula promised above (Theorem 1.9). Along the way, two preliminary results on the A-value

$$\|\widehat{S} \text{ is stationary in } \mathcal{P}_{\leq\omega}(\widehat{X})\|_A$$

for a given complete Boolean algebra A and given sets X and S ($S \subseteq \mathcal{P}_{\leq\omega}(X)$) will also be obtained (Propositions 1.6 and 1.8). Proposition 1.6 gives an upper bound of the A-value while Proposition 1.8 shows that it is possible to evaluate it precisely under a certain condition.

Let us begin by reviewing the material about club and stationary sets that we will need. For any set X, $\mathcal{P}_{<\omega}(X)$ denotes the set of all finite subsets of X, $\mathrm{Club}(X)$ the set of all closed unbounded subsets of $\mathcal{P}_{\leq\omega}(X)$, $\mathrm{Stat}(X)$ the set of all stationary subsets of $\mathcal{P}_{\leq\omega}(X)$, and $\mathrm{NStat}(X)$ the set of all nonstationary subsets of $\mathcal{P}_{\leq\omega}(X)$. If $\langle S_\alpha \mid \alpha \in X\rangle$ is a family of subsets of $\mathcal{P}_{\leq\omega}(X)$, then $\Delta\langle S_\alpha \mid \alpha \in X\rangle$ is its diagonal intersection. For any sets X and Y, $X \to Y$ denotes the set of all functions from X to Y.

The following two lemmas are standard. The reader should consult [2, III.1], [1, §1] or [3, III §1] for their proofs.

LEMMA 1.1. *Let X be an arbitrary set.*

(a) *If $S_\alpha \in \mathrm{Club}(X)$ for all $\alpha \in X$, then $\Delta\langle S_\alpha \mid \alpha \in X\rangle \in \mathrm{Club}(X)$.*

(b) *For any $S \subseteq \mathcal{P}_{\leq\omega}(X)$, the following two conditions are equivalent:*

(1.1.1) $\exists R \in \mathrm{Club}(X)\colon R \subseteq S$;

(1.1.2) $\exists f \in \mathcal{P}_{<\omega}(X) \to X\colon \{M \in \mathcal{P}_{\leq\omega}(X) \mid f''\mathcal{P}_{<\omega}(M) \subseteq M\} \subseteq S$.

LEMMA 1.2. *Let X and Y be sets such that $X \subseteq Y$.*

(a) $\forall S \in \mathrm{Stat}(X) : \{M \in \mathcal{P}_{\leq\omega}(Y) \mid M \cap X \in S\} \in \mathrm{Stat}(Y)$.

(b) $\forall T \in \mathrm{Stat}(Y) : \{M \cap X \mid M \in T\} \in \mathrm{Stat}(X)$.

In the remainder of this section, let A be an arbitrary complete Boolean algebra.

DEFINITION 1.3. By the *stationarity preserving part* of A, we mean the value

$$||\forall X \in \widehat{\mathbf{V}}\colon \mathrm{Stat}^{\widehat{\mathbf{V}}}(X) \subseteq \mathrm{Stat}(X)||_A.$$

We denoteit by $\mathrm{SPP}(A)$. (So A is proper in the sense of Shelah [3, III.1.9, p.76] if and only if $\mathrm{SPP}(A) = 1_A$.)

The notions of predense sets and $\langle M, A\rangle$-generic elements play an important role in standard accounts of proper forcing (e.g. [3, III], [1] and [2, III]). They are also highly relevant to my goal of obtaining an algebraic characterization of $\mathrm{SPP}(A)$ (Theorem 1.9). However, I prefer to use the term *A-covers* in referring to predense subsets of A, and taking advantage of the fact that A is not a plain poset but a complete Boolean algebra, I will work with $\mathrm{MJ}(A, M)$ defined below, which represents the greatest $\langle A, M\rangle$-generic element of A, rather than manipulate individual $\langle A, M\rangle$-generic elements.

DEFINITION 1.4. (a) We call a set $C \subseteq A$ an *A-cover* if $\bigsqcup^A C = 1_A$, and denote the set of all A-covers by $\mathrm{Cov}(A)$.

(b) For any set M, we can the value

$$\bigsqcap^A_{C \in \mathrm{Cov}(A) \cap M} \bigsqcup^A (C \cap M)$$

the *meet of joins* associated with $\langle A, M \rangle$, and denote it by $\mathrm{MJ}(A, M)$.

The value of $\mathrm{MJ}(A, M)$ depends only on $M \cap A$ and $M \cap \mathrm{Cov}(A)$. Therefore we have:

LEMMA 1.5. *For any sets M_1 and M_2 such that $M_1 \cap (A \cup \mathrm{Cov}(A)) = M_2 \cap (A \cup \mathrm{Cov}(A))$, it is the case that* $\mathrm{MJ}(A, M_1) = \mathrm{MJ}(A, M_2)$.

As a first step towards proving Theorem 1.9, let us obtain an upper bound of the elements of A that force S to be stationary in $\mathcal{P}_{\leq\omega}(\widehat{X})$ in the A-universe, where X and S are given sets such that $S \subseteq \mathcal{P}_{\leq\omega}(X)$.

PROPOSITION 1.6. *For any $X \supseteq A$ and any $S \subseteq \mathcal{P}_{\leq\omega}(X)$, we have*

$$\|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A \sqsubseteq \bigsqcup^A_{M \in S} \mathrm{MJ}(A, M). \tag{1.6.1}$$

PROOF. Let a^* denote the right-hand side of (1.6.1). Since the inequality may be rewritten as

$$-a^* \sqsubseteq \|\widehat{S} \in \mathrm{NStat}(\widehat{X})\|_A,$$

we will consider a $\mathbf{V}$-generic filter G in A such that

$$-a^* \in G, \tag{1.6.2}$$

and derive the conclusion that

$$S \in \mathrm{NStat}^{\mathbf{V}[G]}(X). \tag{1.6.3}$$

Let T denote the set of all $M \in \mathcal{P}^{\mathbf{V}[G]}_{\leq\omega}(X)$ such that

$$\forall C \in \mathrm{Cov}^{\mathbf{V}}(A) \cap M\colon\ C \cap M \cap G \neq \emptyset.$$

Clearly $T \in \mathbf{V}[G]$.

CLAIM 1. $S \cap T = \emptyset$.

PROOF. We have

$$-a^* = \prod_{M\in S}^{A} \coprod_{C\in \mathrm{Cov}^{\mathbf{V}}(A)\cap M}^{A} \prod_{a\in C\cap M}^{A} (-a).$$

So, by (1.6.2) and genericity of G,

$$\forall M \in S\colon \exists C \in \mathrm{Cov}^{\mathbf{V}}(A) \cap M\colon\ C \cap M \cap G = \emptyset,$$

which implies that $S \cap T = \emptyset$. //// (Claim 1)

CLAIM 2. $\exists R \in \mathrm{Club}^{\mathbf{V}[G]}(X)\colon\ R \subseteq T$.

PROOF. Pick a family $\langle T_C \mid C \in X\rangle \in \mathbf{V}[G]$ such that

$$\forall C \in X\colon\ T_C \in \mathrm{Club}^{\mathbf{V}[G]}(X)$$

and that for all $C \in \mathrm{Cov}^{\mathbf{V}}(A) \cap X$,

$$T_C = \{M \in \mathcal{P}^{\mathbf{V}[G]}_{\leq\omega}(X) \mid C \cap M \cap G \neq \emptyset\}.$$

(It is possible to pick such a family because $A \subseteq X$.) We then have $\Delta\langle T_C \mid C \in X\rangle \subseteq T$. //// (Claim 2)

Our goal (1.6.3) follows from Claims 1 and 2. //// (Proposition 1.6)

Let us next find a precise evaluation of the A-value $\|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A$. This requires

DEFINITION 1.7. Suppose that X is an arbitrary set and $S, S' \subseteq \mathcal{P}_{\leq\omega}(X)$. We will write $S \subseteq_{\mathrm{nst}} S'$ if $S - S' \in \mathrm{NStat}(X)$, or equivalently, if $S \cap R \subseteq S'$ for some $R \in \mathrm{Club}(X)$.

A simple argument using Lemma 1.1(b) shows that if $R \in \mathrm{Club}(X)$, then

$$\|\exists R' \in \mathrm{Club}(\widehat{X})\colon R' \cap \mathcal{P}^{\widehat{\mathbf{V}}}_{\leq\omega}(\widehat{X}) \subseteq \widehat{R}\|_A = 1_A.$$

It follows that if $S \subseteq_{\mathrm{nst}} S' (S, S' \subseteq \mathcal{P}_{\leq\omega}(X))$, then

$$\|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A \sqsubseteq \|\widehat{S'} \in \mathrm{Stat}(\widehat{X})\|_A.$$

Combining this fact with Proposition 1.6, we see that for any $S \subseteq \mathcal{P}_{\leq\omega}(X)$,

$$\|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A \sqsubseteq \prod_{S\subseteq_{\mathrm{nst}}S'\subseteq\mathcal{P}_{\leq\omega}(X)}^{A} \coprod_{M\in S'}^{A} \mathrm{MJ}(A, M).$$

We will show that equality holds here provided $A \cup \mathrm{Cov}(A) \subseteq X$.

PROPOSITION 1.8. *For any* $X \supseteq A \cup \mathrm{Cov}(A)$ *and any* $S \subseteq \mathcal{P}_{\leq\omega}(X)$, *we have*

$$\|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A = \prod^A_{S \subseteq_{\mathrm{nst}} S' \subseteq \mathcal{P}_{\leq\omega}(X)} \coprod^A_{M \in S'} \mathrm{MJ}(A, M). \tag{1.8.1}$$

PROOF. Let a^* denote the right-hand side of (1.8.1). In view of the observation made above, we need only prove that

$$a^* \sqsubseteq \|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A. \tag{1.8.2}$$

Choose a transitive set $Y \supseteq X$ that is closed under unordered pairs and unions. Let

$$T = \{M \in \mathcal{P}_{\leq\omega}(Y) \mid M \cap X \in S\}$$

and

$$b^* = \prod^A_{T \subseteq_{\mathrm{nst}} T' \subseteq \mathcal{P}_{\leq\omega}(Y)} \coprod^A_{M \in T'} \mathrm{MJ}(A, M).$$

By Lemma 1.2(b),

$$\|\widehat{T} \in \mathrm{Stat}(\widehat{Y})\|_A \sqsubseteq \|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A.$$

We obtain (1.8.2) from this and Claims 1 and 2 below.

CLAIM 1. $a^* = b^*$.

PROOF. Using Lemma 1.2, we easily see that the sets S' such that $S \subseteq_{\mathrm{nst}} S' \subseteq \mathcal{P}_{\leq\omega}(X)$ are precisely the sets of the form $\{M \cap X \mid M \in T'\}$, where $T \subseteq_{\mathrm{nst}} T' \subseteq \mathcal{P}_{\leq\omega}(Y)$. Hence

$$a^* = \prod^A_{T \subseteq_{\mathrm{nst}} T' \subseteq \mathcal{P}_{\leq\omega}(Y)} \coprod^A_{M \in T'} \mathrm{MJ}(A, M \cap X). \tag{1.8.3}$$

But $A \cup \mathrm{Cov}(A) \subseteq X$. So, by Lemma 1.5,

$$\forall M \subseteq Y\colon\ \mathrm{MJ}(A, M \cap X) = \mathrm{MJ}(A, M).$$

Therefore the right-hand side of (1.8.3) equals b^*. //// (Claim 1)

CLAIM 2. $b^* \sqsubseteq \|\widehat{T} \in \mathrm{Stat}(\widehat{Y})\|$.

PROOF. Recalling Lemma 1.1(b), let us consider an arbitrary A-name $\dot{f}$ such that

$$\|\dot{f} \in \mathcal{P}_{<\omega}(\widehat{Y}) \to \widehat{Y}\|_A = 1_A,$$

and show that

$$b^* \sqsubseteq \|\exists M \in \widehat{T}\colon \forall u \in \mathcal{P}_{<\omega}(M)\colon \dot{f}(u) \in M\|_A. \tag{1.8.4}$$

Let R be the set of all triples $\langle u, y, a\rangle$ such that

$$u \in \mathcal{P}_{<\omega}(Y) \wedge y \in Y \wedge a = \|\dot{f}(\widehat{u}) = \widehat{y}\|_A,$$

and put

$$T' = \{M \in T \mid M \prec \langle Y, \in, R\rangle\}.$$

(Note that the assumptions on Y assure that $R \subseteq Y$.)

CLAIM 2.1. $b^* \sqsubseteq \bigsqcup^A_{M \in T'} \mathrm{MJ}(A, M)$.

PROOF. Since T' is the intersection of T and a club subset of $\mathcal{P}_{\leq\omega}(Y)$, we have $T \subseteq_{\mathrm{nst}} T'$. //// (Claim 2.1)

CLAIM 2.2. $\forall M \in T'\colon \mathrm{MJ}(A, M) \sqsubseteq \|\forall u \in \mathcal{P}_{<\omega}(\widehat{M})\colon \dot{f}(u) \in \widehat{M}\|_A$.

PROOF. Consider an arbitrary $M \in T'$ and an arbitrary $u \in \mathcal{P}_{<\omega}(M)$. We must show that

$$\mathrm{MJ}(A, M) \sqsubseteq \|\dot{f}(\widehat{u}) \in \widehat{M}\|_A.$$

Let $C = \{\|\dot{f}(\widehat{u}) = \widehat{y}\|_A \mid y \in Y\}$. Clearly $C \in \mathrm{Cov}(A)$. Since $M \prec \langle Y, \in, R\rangle, u \in M$, and C is an element of Y that is definable in $\langle Y, \in, R\rangle$ from u, we have $C \in M$. Thus

$$C \in \mathrm{Cov}(A) \cap M. \tag{1.8.5}$$

Also

$$\forall y \in Y\colon [0_A \neq \|\dot{f}(\widehat{u}) = \widehat{y}\|_A \in M \Longrightarrow y \in M], \tag{1.8.6}$$

for if $y \in Y$ and $a = \|\dot{f}(\widehat{u}) = \widehat{y}\|_A \neq 0_A$, then y is definable in $\langle Y, \in, R\rangle$ from u and a as the unique element of Y such that $\langle u, y, a\rangle \in R$. Now

$$\begin{aligned} \mathrm{MJ}(A, M) &\sqsubseteq \textstyle\bigsqcup^A (C \cap M) && \text{(by (1.8.5))} \\ &\sqsubseteq \textstyle\bigsqcup^A_{y \in M} \|\dot{f}(\widehat{u}) = \widehat{y}\|_A && \text{(by (1.8.6))}, \end{aligned}$$

showing that $\mathrm{MJ}(A,M) \sqsubseteq \|\dot{f}(\widehat{u}) \in \widehat{M}\|_A$. //// (Claim 2.2)

Claims 2.1 and 2.2 imply (1.8.4). //// (Claim 2)

The proof of Proposition 1.8 is now complete. //// (Proposition 1.8)

We are now ready to characterize SPP(A).

THEOREM 1.9. *For any set $X \supseteq A \cup \mathrm{Cov}(A)$, we have*

$$\mathrm{SPP}(A) = \prod_{S \in \mathrm{Stat}(X)}^{A} \cdot \coprod_{M \in S}^{A} \mathrm{MJ}(A,M). \tag{1.9.1}$$

PROOF. By Lemma 1.2,

$$\begin{gathered} \forall X \in \mathbf{V}\colon \forall S \in \mathrm{Stat}(X)\colon \exists Y \supseteq A \cup \mathrm{Cov}(A)\colon \exists T \in \mathrm{Stat}(Y)\colon \\ \|\widehat{T} \in \mathrm{Stat}(\widehat{Y})\|_A \sqsubseteq \|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A. \end{gathered} \tag{1.9.2}$$

Also, if we denote the right-hand side of (1.9.1) by a^*_X, then we see by the same argument that we used in proving Claim 1 in the proof of Proposition 1.8 that

$$\forall X, Y \supseteq A \cup \mathrm{Cov}(A)\colon a^*_X = a^*_Y. \tag{1.9.3}$$

Now

$$\begin{aligned} \mathrm{SPP}(A) &= \prod_{X \in \mathbf{V}}^{A} \prod_{S \in \mathrm{Stat}(X)}^{A} \|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A \\ &= \prod_{X \supseteq A \cup \mathrm{Cov}(A)}^{A} \prod_{S \in \mathrm{Stat}(X)}^{A} \|\widehat{S} \in \mathrm{Stat}(\widehat{X})\|_A \text{ (by 1.9.2))} \\ &= \prod_{X \supseteq A \cup \mathrm{Cov}(A)}^{A} a^*_X \text{ (by Proposition 1.8).} \end{aligned}$$

Thus, by (1.9.3), $\mathrm{SPP}(A) = a^*_X$ for all $X \supseteq A \cup \mathrm{Cov}(A)$. //// (Theorem 1.9)

§2. Characterizations of properness

Throughout this section, let A be an arbitrary complete Boolean algebra and $\sqsubseteq$ the natural partial order on it.

Various necessary and sufficient conditions for A to be proper are known [3, III]. It is perhaps of some interest to see how Theorem 1.9 can be used to obtain some of them.

THEOREM 2.1. *For each set X, let S_X denote the set of all $M \in \mathcal{P}_{\leq\omega}(X)$ such that*

$$\forall a \in A \cap M(a \neq 0_A)\colon\ a \sqcap \mathrm{MJ}(A, M) \neq 0_A.$$

Then the following conditions are equivalent:

(2.1.1) A *is proper*;

(2.1.2) $\forall X \supseteq A \cup \mathrm{Cov}(A)\colon \exists R \in \mathrm{Club}(X)\colon R \subseteq S_X$;

(2.1.3) $\exists X \supseteq A \cup \mathrm{Cov}(A)\colon \exists R \in \mathrm{Club}(X)\colon R \subseteq S_X$;

(2.1.4) $\exists R \in \mathrm{Club}(A \cup \mathrm{Cov}(A))\colon R \subseteq S_{A\cup\mathrm{Cov}(A)}$.

PROOF. (2.1.1) $\Rightarrow$ (2.1.2): Assume (2.1.1) and consider an arbitrary set $X \supseteq A \cup \mathrm{Cov}(A)$. For each $a \in X$, put

$$S'_a = \{M \in \mathcal{P}_{\leq\omega}(X) \mid a \in A - \{0_A\} \Longrightarrow a \sqcap \mathrm{MJ}(A, M) \neq 0_A\}.$$

CLAIM. $\forall a \in X\colon \exists R \in \mathrm{Club}(X)\colon R \subseteq S'_a$.

PROOF. Consider an arbitrary $a \in X$. Our goal is to show that

$$\exists R \in \mathrm{Club}(X)\colon R \subseteq S'_a,$$

or equivalently, that

$$\forall S \in \mathrm{Stat}(X)\colon S \cap S'_a \neq \emptyset.$$

So let $S \in \mathrm{Stat}(X)$. The case $a \notin A - \{0_A\}$ being trivial, we further assume that $a \in A - \{0_A\}$. Since $\mathrm{SPP}(A) = 1_A$, it follows from Theorem 1.9 that $\bigsqcup^A_{M\in S} \mathrm{MJ}(A, M) = 1_A$. So there is an $M \in S$ such that $a \sqcap \mathrm{MJ}(A, M) \neq 0_A$. We have $M \in S \cap S'_a$, whence $S \cap S'_a \neq \emptyset$. //// (Claim)

Note that $S_X = \Delta\langle S'_a \mid a \in X\rangle$. Hence, by the claim above, S_X has a subset that is club in $\mathcal{P}_{\leq\omega}(X)$.

(2.1.2) $\Rightarrow$ (2.1.4): Trivial.

(2.1.4) $\Rightarrow$ (2.1.3): Trivial.

(2.1.3) $\Rightarrow$ (2.1.1): Assume (2.1.3) and choose an $X \supseteq A \cup \mathrm{Cov}(A)$ and an $R \in \mathrm{Club}(X)$ so that $R \subseteq S_X$. By Theorem 1.9, it suffices to prove that

$$\forall S \in \mathrm{Stat}(X)\colon \bigsqcup^A_{M\in S} \mathrm{MJ}(A, M) = 1_A.$$

So consider an arbitrary $S \in \mathrm{Stat}(X)$ and an arbitrary $a \in A - \{0_A\}$. We now have to show that

$$\exists M \in S\colon\ a \sqcap \mathrm{MJ}(A, M) \neq 0_A.$$

Since $\{M \in R \mid a \in M\} \in \mathrm{Club}(X)$, there is an $M \in R \cap S$ such that $a \in M$. Since $M \in R$, we have $M \in S_X$. So, since $a \in A \cap M$ and $a \neq 0_A$, we conclude that $a \sqcap \mathrm{MJ}(A, M) \neq 0_A$. //// (Theorem 2.1)

Thus, if A is proper ad $A \cup \mathrm{Cov}(A) \subseteq X$, then $R \subseteq S_X$ for some $R \in \mathrm{Club}(X)$. However, the proof of Theorem 2.1 only produces a rather faceless such set R that bears no easily recognizable relation to A and X. Let us now see that if X is endowed with sufficiently rich set-theoretic structure, we can use the club set

$$\{M \in \mathcal{P}_{\leq\omega}(X) \mid \langle A, \sqsubseteq\rangle \in M \prec \langle X, \in\rangle\}$$

for determining whether A is proper or not.

DEFINITION 2.2. By a *miniuniverse*, we mean a nonempty transitive set X that satisfies the following conditions:

(i) X is closed under unordered pairs, unions, and countable sequences;
(ii) For each $D \in X$, the sets $\{\{x, y\} \mid x, y \in D\}$ and $\{\bigcup x \mid x \in D\}$ are both elements of X;
(iii) For each $D \in X$, $\mathcal{P}(D) \subseteq X$.

Miniuniverses are quite easy to come by. Both the sets $\mathbf{V}(\alpha) = \{x \mid x$ is of rank $< \alpha\}$ (α is an ordinal of uncountable cofinality) and the sets $H(\kappa) = \{x \mid \mathrm{Card}(\mathrm{TC}(\kappa)) < \kappa\}$ (κ is a regular uncountable cardinal) are miniuniverses.

Elementary arguments show that the following facts hold for all miniuniverses X:

(a) ω is a definable element of $\langle X, \in\rangle$;
(b) For any $D, E \in X$, we have $D \times E \in X$, $D \to E \subseteq X$, and $\mathcal{P}_{<\omega}(D) \in X$;
(c) For any $D \in X$, membership in $\mathcal{P}_{<\omega}(D)$ and that in $\mathcal{P}_{\leq\omega}(D)$ are definable in $\langle X, \in\rangle$ from D.

THEOREM 2.3. *Let S_X be as in Theorem 2.1. Then the following two conditions are equivalent:*

(2.3.1) *A is proper;*

(2.3.2) *For any miniuniverse X with $\mathcal{P}(A) \in X$,*

$$\{M \in \mathcal{P}_{\leq\omega}(X) \mid \langle A, \sqsubseteq\rangle \in M \prec \langle X, \in\rangle\} \subseteq S_X.$$

PROOF. (2.3.2) $\Rightarrow$ (2.3.1): Assume (2.3.2) and choose a miniuniverse X so that $\mathcal{P}(A) \in X$. Then $A \cup \mathrm{Cov}(A) \subseteq X$ and

$$\{M \in \mathcal{P}_{\leq\omega}(X) \mid \langle A, \sqsubseteq\rangle \in M \prec \langle X, \in\rangle\} \in \mathrm{Club}(X).$$

Therefore condition (2.1.3) holds, and A is proper.

(2.3.1) $\Rightarrow$ (2.3.2): Assume (2.3.1), and consider an arbitrary miniuniverse X with $\mathcal{P}(A) \in X$ and an arbitrary set $M \in \mathcal{P}_{\leq\omega}(X)$ such that

$$\langle A, \sqsubseteq\rangle \in M \prec \langle X, \in\rangle.$$

Let $Y = A \cup \mathrm{Cov}(A)$. What we must show is that $M \in S_X$, but since we have

$$\forall M\colon [M \in S_X \Longleftrightarrow M \cap Y \in S_Y]$$

as we easily see using Lemma 1.5, it suffices to prove that

$$M \cap Y \in S_Y.$$

Condition (2.1.4) holds. So, by Lemma 1.1(b), there is a function $f \in \mathcal{P}_{<\omega}(Y) \to Y$ such that

$$\forall N \in \mathcal{P}_{\leq\omega}(Y)\colon [f''\mathcal{P}_{<\omega}(N) \subseteq N \Longrightarrow N \in S_Y]. \tag{2.3.3}$$

Note that these conditions on f are definable in $\langle X, \in\rangle$ from $\langle A, \sqsubseteq\rangle$. Hence we may further assume that $f \in M$. It then easily follows that

$$f''\mathcal{P}_{<\omega}(M \cap Y) \subseteq M \cap Y.$$

Therefore, by (2.3.3), $M \cap Y \in S_Y$. //// (Theorem 2.3)

References

[1] K,J. Devlin: The Yorkshireman's guide to proper forcing, "Surveys in set theory", Cambridge University Press, 1983, pp. 60–115.

[2] T. Jech: "Multiple forcing", Cambridge University Press, 1986.

[3] S. Shelah: "Proper forcing", Lecture Notes in Mathematics 940, Springer-Verlag, 1982.

Faculty of Education
Kobe University
Nada-ku, Kobe 657
Japan