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ON A GENERALIZATION OF V-RINGS AND SF-RINGS

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We call a ring R a left SGPF ring if every simple left R-module is either GP-injective or flat. Let R be a left SGPF ring. It is shown that (1) $RdR = R$ for each non-zero-divisor d of R; (2) If R is semiprime, then the centre of R is von Neumann regular; (3) If R is reduced, then R is fully left and right idempotent. Some known results of R. Yue Chi Ming are improved.

1. Introduction

Rings whose simple left R-modules are injective (flat), called left V-rings (SF-rings), are studied by many authors (cf. [1, 2, 4, 5]). In this paper we consider rings whose every simple left R-module is either GP-injective or flat (called left SGPF rings). Clearly both left V-rings and left SF-rings are left SGPF rings, but the converse is not true in either case (see Example 2 below). Hence the class of rings considered here is broader than the class of left V-rings or left SF-rings, but some properties shared by V-rings and SF-rings are still available. Let R denote a left SGPF ring. We prove that (1) $RdR = R$ for each non-zero-divisor d of R; (2) If R is semiprime, then the centre of R is a von Neumann regular ring; (3) If R is reduced, then R is fully left and right idempotent. New characteristic properties of strongly regular rings are also given. Some known results of R. Yue Chi Ming are generalized.

Throughout this paper, R represents an associative ring with identity and R-modules are unitary. For an element x of R, $\mathrm{l}(x)$ ($\mathrm{r}(x)$) denotes the left (right) annihilator of x in R. Recall that (1) R is fully left (right) idempotent [2] if every left (right) ideal is idempotent. (2) R is MELT [12] if every maximal essential left ideal of R is an ideal. (3) R is called weakly right duo (abbreviated WRD) [8] if for any $a \in R$ there exists a positive integer n such that $a^n R$ is an ideal of R. (4) R is called strongly regular if for each $a \in R$ there exists $b \in R$ such that $a = a^2 b$. Note that strong regularity is a left-right symmetric property. (5) R is reduced if it contains no nonzero nilpotent element. (6) A left (right) R-module M is called GP-injective (generalized p-injective) [13] if for any $a \in R$ there exists a positive integer m such that any left (right) R-homomorphism from Ra^m ($a^m R$)

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to M extends to an R-homomorphism from R to M.

2. Definition and main results

DEFINITION 1. R is called a left (right) SGPF ring if every simple left (right) R-module is either GP-injective or flat.

The next example shows that SGPF rings generalize effectively V-rings and SF-rings.

EXAMPLE 2. Let $R = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a,\ b,\ c \in \mathbb{Z}/2\mathbb{Z} \right\}$ and $M = \left\{ \begin{bmatrix} 0 & 0 \\ b & c \end{bmatrix} \mid b,\ c \in \mathbb{Z}/2\mathbb{Z} \right\}$. Then it is easily verified that M is a maximal left ideal of R. Now $m = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in M$, but there is no $t \in M$ such that $m = mt$. Thus R/M is not flat, whence R is not a left SF-ring. R is not a left V-ring either since the Jacobson radical $J(R) = \left\{ \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \mid b \in \mathbb{Z}/2\mathbb{Z} \right\} \neq 0$. It is also easy to see that every element of R is either nilpotent or idempotent or invertible. Thus every left R-module is GP-injective, in particular, R is a left SGPF ring.

We note that left SGPF rings need not be semiprime (cf. Example 2 above or [7, p.297]). However, the next theorem (motivated by [4, Lemma 2.3], [5, Proposition 3.3] and [9, Proposition 2]) holds.

THEOREM 3. *Let R be a left* SGPF *ring, then R has the following properties.*

(1) *$R = RcR$ for every non-zero-divisor c of R.*

(2) *If R is semiprime, then the centre of R is von Neumann regular.*

PROOF. (1) Let c be a non-zero-divisor of R. Suppose that $RcR \neq R$ and let M be a maximal left ideal of R containing RcR. Since R/M is simple, R/M is either GP-injective or flat. First assume that R/M is GP-injective. Then there exists a positive integer n such that any left R-homomorphism from Rc^n to R/M extends to an R-homomorphism from R to R/M. Now we define $f : Rc^n \to R/M$ via $f(xc^n) = x + M$ for $x \in R$. Since c is a non-zero-divisor, f is well-defined. Thus there exists $d \in R$ such that $1 + M = c^n(d + M)$, whence $1 \in M$ (because $c^n d \in RcR \subseteq M$), a contradiction. Next assume that R/M is flat. Since $c \in M$, there exists $y \in M$ such that $c = cy$, and so $1 = y \in M$ (for c is a non-zero-divisor), again a contradiction. Thus $R = RcR$.

(2) Let C denote the centre of R and $c \in C$. If $Rc + 1(c) \neq R$, then it must be contained in a maximal left ideal M of R. By hypothesis, R/M is either GP-injective or flat. First suppose that R/M is GP-injective. Then there exists

a positive integer m such that any R-homomorphism $Rc^m \to R/M$ extends to an R-homomorphism $R \to R/M$. Define $g : Rc^m \to R/M$ by $g(xc^m) = x + M$. Suppose that $xc^m = 0$. If $m = 1$, then $x \in \mathrm{l}(c) \subseteq M$, and hence g is well-defined. If $m > 1$, then $xc^{m-1}Rxc^{m-1} = 0$ (because $c \in C$). Thus $xc^{m-1} = 0$ since R is semiprime. Similarly, $xc^{m-2} = 0, \ldots, xc = 0$. Therefore $x \in \mathrm{l}(c) \subseteq M$, whence g is also well-defined. Consequently, there exists $e \in R$ such that $1 + M = c^m(e + M)$, and so $1 \in M$ since $c^m e = ec^m \in Rc \subseteq M$, a contradiction. Next suppose that R/M is flat. Since $c \in M$, there is $y \in M$ such that $c = cy$. Thus $(1 - y)c = c(1 - y) = 0$, i.e., $1 - y \in \mathrm{l}(c) \subseteq M$, whence $1 \in M$, again a contradiction. So we have $Rc + \mathrm{l}(c) = R$. Let $1 = tc + s$ with $t \in R$ and $s \in \mathrm{l}(c)$. Then $c = tc^2$, and hence

$$c = t(tc^2)c = t^2c^3 = (t^2c)c^2.$$

Put $v = t^2c$, we claim that $v \in C$. In fact, let $x \in R$, then

$$(xv - vx)c^2 = xt^2c^3 - t^2c^3x = xc - cx = 0$$

since $c \in C$, and hence $xv - vx \in l(c^2)$. But $c = tc^2 = c^2t$, and so $l(c^2) = l(c)$. Thus $xv - vx \in \mathrm{l}(c)$, whence $(xt^2 - t^2x)c^2 = 0$. Therefore $(xt^2 - t^2x)c = 0$, and hence $xv = vx$, i.e., $v \in C$. Consequently, $c = vc^2 = cvc$ for some $v \in C$. That is, C is von Neumann regular. This completes the proof.

REMARK 4. It is well known that for any ring R every simple R-module is either singular or projective, but not both [3, Proposition 1.24]. Hence Theorem 3(1) improves [10, Proposition 3(ii)]. We note that Theorem 3 also improves [14, Proposition 8(2)(3)].

As is known to all, a left V-ring is fully left idempotent [2], and under given conditions a left SF-ring is also fully left idempotent (cf. [1, Proposition 1]). Here we have the following.

THEOREM 5. *If R is a reduced left* SGPF *ring, then R is fully left and right idempotent.*

PROOF. Let $b \in R$. If $RbR + \mathrm{l}(b) \neq R$, let M be a maximal left ideal of R containing $RbR + \mathrm{l}(b)$. Then R/M is either GP-injective or flat. First assume that R/M is GP-injective, then there exists a positive integer n such that any left R-homomorphism from Rb^n to R/M extends to an R-homomorphism from R to R/M. We define $f : Rb^n \to R/M$ by $f(xb^n) = x + M$ for $x \in R$. Since R is reduced, $l(b^n) = l(b)$. Hence f is well-defined. Thus there is $c \in R$ such that $1 + M = b^n(c + M)$, whence $1 \in M$ (because $b^nc \in RbR \subseteq M$), a contradiction. Next assume that R/M is flat. Since $b \in M$, there exists $t \in M$ such that $b = bt$, i.e., $1 - t \in r(b)$. But, since R is reduced, $\mathrm{l}(b) = \mathrm{r}(b)$. Thus $1 - t \in \mathrm{l}(b) \subseteq M$, which

leads to $1 \in M$, again a contradiction. So we have $RbR + \mathrm{l}(b) = R$. Let $1 = x + y$ with $x \in RbR$ and $y \in \mathrm{l}(b)$. Then $b = xb + yb = xb \in (Rb)^2$, which implies that R is fully left idempotent. On the other hand, since $b = xb$, $1 - x \in \mathrm{l}(b) = \mathrm{r}(b)$, thus $b = bx \in (bR)^2$, which proves that R is fully right idempotent. This completes the proof of Theorem 5.

REMARK 6. In the Theorem 5 above, R need not be von Neumann regular since there exist reduced V-rings that are not von Neumann regular (cf., e.g., [2, Example 2]). In view of [15, Lemma 2], Theorem 5 improves [15, Corollary 2.1].

Next we shall give some new characterizations of strongly regular rings.

COROLLARY 7. *The following are equivalent for a ring R.*

1) *R is a strongly regular ring.*
2) *R is a reduced,* MELT *and left (or right)* SGPF *ring.*
3) *R is a reduced,* WRD *and left (or right)* SGPF *ring.*

PROOF. Obviously, 1) implies 2) and 3).

2) $\rightarrow$ 1). By Theorem 5 (and its proof), R is fully left and right idempotent. Thus 1) follows from [12, Theorem 10].

3) $\rightarrow$ 1). R is fully left and right idempotent by the foregoing proof. Since R is a WRD ring, for any $a \in R$, there exists a positive integer n such that $a^n R = Ra^n R$. Thus $Ra^n = (Ra^n)^2 = a^n Ra^n$, whence $a^n = a^n u a^n$ for some $u \in R$. Therefore $1 - a^n u \in l(a^n) = l(a)$ (for R is reduced), so that $a = a^n ua = aya$, where $y = a^{n-1} u \in R$. Hence R is von Neumann regular, and so R is strongly regular (since R is reduced). The proof is complete.

REMARK 8. We note that Corollary 7 improves [11, Theorem 13(2)(5)(6)].

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