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FIXED POINT THEOREMS FOR CONDENSING AND COMPACT MAPS

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Abstract

In this paper fixed point theorems for condensing and compact maps are established which extend corresponding results of Janos, Leader, Shih and Yeh.

1. Introduction

Let f be a self-map of a metric space (X,d). Set $C_f = \{g : X \to X \mid fg = gf\}$. Jungck [2] established some fixed point theorems for C_f. Janos, Ko and Tan obtained an interesting result on fixed point (see [3, Corollary 2]). Shih and Yeh [1] attempted to extend the result of Janos, Ko and Tan to a more general case. In this paper we define the families of maps H_f and CIS_f and show fixed point theorems for H_f and CIS_f. In Section 3, we establish the existence of fixed point for condensing maps and show that Theorem 3 in [1] is unfortunately false by providing a counterexample and give a correct version of the result. In Section 4, two general fixed point theorems for compact maps are proved. Our results extend results of Janos [4], Leader [5], Shih and Yeh [1].

2. Preliminaries

The following definition was introduced by Nussbaum in [7].

DEFINITION 2.1. Let X be a topological space, $f : X \to X$ a map, and M a nonempty subset of X. M is an attractor for compact sets under f if (I) M is compact and $f(M) \subset M$, and (II) given any compact set $C \subset X$ and any open neighborhood U of M, there exists an integer k such that $f^n(C) \subset U$ for $n \geq k$.

Throughout the rest of the paper, we assume that (X,d) is a metric space, $M(X)$ denotes the set of all metrics on X that are topologically equivalent to d. For $S \subset X$, $d(S)$ and $\overline{S}$ denote the diameter and closure of S respectively. A self-map f of a metric space (X,d) is compact if there exists a compact subset Y of X such that $f(X) \subset Y$. Denote by $\alpha(A)$ the Kuratowski measure of noncompactness of a subset A of a bounded metric space (X,d). We say that $f : X \to X$ is condensing if f is continuous and for any nonempty nontotally bounded subset A

of X, $\alpha(f(A)) < \alpha(A)$. Let N denote the set of positive integers and $N_0 = N \cup \{0\}$. For $f : X \to X$, define

$$H_f = \left\{ g : X \to X \mid g\left(\bigcap_{n=0}^{\infty} f^n(X)\right) \subset \bigcap_{n=0}^{\infty} f^n(X) \right\}$$

$$CIS_f = \{g : X \to X \mid \text{for every nonempty compact } f\text{-invariant subset } A \text{ of } X,\ g(A) \subset A\}.$$

Obviously, $H_f \supset C_f \supset \{f^n \mid n \in N_0\}$ and $CIS_f \supset \{f^n \mid n \in N_0\}$. If f is a continuous self-map of a compact metric space (X, d), then $H_f \supset CIS_f$.

The following result appears in [6].

LEMMA 2.2. *Let f be a continuous map of a metric space (X, d) into itself such that*

(a) *f has a unique fixed point v in X;*
(b) *the sequence of iterates $\{f^n x\}$ converges to v for all x in X;*
(c) *there exists an open neighborhood U of v with the property that given any open set V containing v there exists a positive integer k such that $f^n(U) \subset V$ for all $n \geq k$.*

Then for any $r \in (0, 1)$, there exists a metric $d^* \in M(X)$ such that $d^*(fx, fy) \leq r d^*(x, y)$ for all x, y in X.

3. A fixed point theorem for condensing maps

THEOREM 3.1. *Let f be a continuous self-map of a bounded complete metric space (X, d). Suppose that there exists $m \in N$ such that f^m is condensing and*

$$d(f^m x, f^m y) < d(\{gz \mid z \in \{x, y\} \text{ and } g \in CIS_f\}) \tag{1}$$

for all x, y in X with $x \neq y$. Then

(i) *f has a unique fixed point v in X such that $f^n x \to v$ for every x in X;*
(ii) *for every nonempty compact f-invariant subset Y of X, $\bigcap_{n=0}^{\infty} f^n(Y) = \{v\}$;*
(iii) *there exists a bounded complete metric d^* on X which is equivalent to d such that f is contractive with respect to d^*, i.e., $d^*(fx, fy) < d^*(x, y)$ for all x, y in X with $x \neq y$.*
(iv) *CIS_f has a unique fixed point v.*

PROOF. For $x \in X$, put $O(x) = \{f^n x \mid n \in N_0\}$. Clearly

$$\begin{aligned} \alpha(O(x)) &= \max\{\alpha(\{x, fx, \ldots, f^{m-1}x\}), \alpha(O(f^m x))\} \\ &= \alpha(O(f^m x)) = \alpha(f^m(O(x))) \end{aligned}$$

Since f^m is condensing, $O(x)$ is totally bounded. From the completeness of (X, d), $\overline{O(x)}$ is compact. By the continuity of f, we conclude that $f(\overline{O(x)}) \subset \overline{f(O(x))} \subset \overline{O(x)}$; i.e., $\overline{O(x)}$ is f-invariant. Hence $f^{n+1}(\overline{O(x)}) \subset f^n(\overline{O(x)})$ for $n \in N$. From the compactness of $\overline{O(x)}$ and the continuity of f, it follows that $\{f^n(O(x)) \mid n \in N\}$ has the finite intersection property; thus $\bigcap_{n=1}^{\infty} f^n(\overline{O(x)})$ is a nonempty compact subset of $\overline{O(x)}$. Let $D = \bigcap_{n=1}^{\infty} f^n(\overline{O(x)})$. We now prove that $f(D) = D$. For every v in D, there exists $x_n \in f^{n-1}(\overline{O(x)})$ such that $fx_n = v$, $n \in N$. From the compactness of $\overline{O(x)}$ we may (by selecting a subsequence, if necessary) assume that x_n converges to some point $u \in \overline{O(x)}$. Since $\{x_{n+1}, x_{n+2}, \cdots\} \subset f^n(\overline{O(x)})$ and $f^n(\overline{O(x)})$ is compact, $u \in f^n(\overline{O(x)})$ for $n \in N$. So that $u \in D$ and by $fu = v$, we have $D \subset f(D)$. Clearly $f(D) \subset D$. Hence $f(D) = D$. Thus $f^m(D) = D$. We assert that D is a singleton. Otherwise $d(D) > 0$, so by compactness there exist two distinct points $p, q \in D$ such that $d(p, q) = d(D)$. Since $f^m(D) = D$, there exist two distinct points x, y in D such that $p = f^m x$, $q = f^m y$; i.e., $d(f^m x, f^m y) = d(D)$. Obviously, $\{gz \mid z \in \{x, y\}$ and $g \in CIS_f\} \subset D$. From (1) we have

$$0 < d(D) = d(f^m x, f^m y) < d(\{gz \mid z \in \{x, y\} \text{ and } g \in CIS_f\}) \leq d(D)$$

a contradiction. Thus D is a singleton, say, $D = \{v\}$. Hence v is a fixed point of f.

To prove the uniqueness, if possible let $b(\neq v)$ be another fixed point of f. Put $A = \{v, b\}$. It is easy to see that $\{gz \mid z \in \{v, b\}$ and $g \in CIS_f\} = A$. By (1)

$$d(v, b) = d(f^m v, f^m b) < d(\{gz \mid z \in \{v, b\} \text{ and } g \in CIS_f\}) = d(v, b)$$

which is a contradiction. Hence v is the unique fixed point of f.

Since $f^n x \in f^n(\overline{O(x)})$ and $\bigcap_{n=1}^{\infty} f^n(\overline{O(x)}) = \{v\}$, $d(f^n x, v) \leq d(f^n(\overline{O(x)}) \to 0$ as $n \to \infty$; i.e., $f^n x \to v$ as $n \to \infty$. This proves (i).

Similarly we can show that (ii) holds.

We now prove that (iii) holds. Let C be any nonempty compact subset of X. Then

$$\alpha\left(\bigcup_{n=0}^{\infty} f^n(C)\right) = \max\left\{\alpha\left(\bigcup_{n=0}^{m-1} f^n(C)\right), \alpha\left(\bigcup_{n=m}^{\infty} f^n(C)\right)\right\}$$

$$= \alpha\left(\bigcup_{n=m}^{\infty} f^n(C)\right) = \alpha\left(f^m\left(\bigcup_{n=0}^{\infty} f(C)\right)\right)$$

Thus $\bigcup_{n=0}^{\infty} f^n(C)$ is totally bounded since f^m is condensing. Put $Y = \overline{\bigcup_{n=0}^{\infty} f^n(C)}$. By the completeness of X and the continuity of f, we conclude that Y is compact and $f(Y) \subset \overline{f(\bigcup_{n=0}^{\infty} f^n(C))} \subset Y$. It follows from (ii) that $\bigcap_{n=0}^{\infty} f^n(Y) = \{v\}$.

This implies $d(f^n(Y)) \to 0$ as $n \to \infty$. For every open neighborhood U of v, there exists an open ball $B(v,\varepsilon) = \{x \mid x \in X \text{ and } d(x,v) < \varepsilon\}$ such that $B(v,\varepsilon) \subset U$. Since $d(f^n(Y)) \to 0$ as $n \to \infty$, there exists some positive integer k such that $d(f^n(Y)) < \varepsilon$ for $n \geq k$. For any x in $f^n(Y)$ and $n \geq k$, we have $d(x,v) \leq d(f^n(Y)) < \varepsilon$; i.e., $f^n(Y) \subset B(v,\varepsilon)$. Hence $f^n(C) \subset f^n(Y) \subset B(v,\varepsilon) \subset U$ for $n \geq k$. This shows that $\{v\}$ is an attractor for compact sets under f. Thus (iii) follows from Theorem and Remark 1 of Janos, Ko and Tan [3].

We next prove that (iv) holds. For any $g \in CIS_f$, it follows from (i) and the definition of CIS_f that $g(\{v\}) \subset \{v\}$; i.e., $g(v) = v$. Hence v is a fixed point of CIS_f. Note that $f \in CIS_f$. By (i), v is the only fixed point of CIS_f. This completes the proof.

COROLLARY 3.2. *Let f be a continuous self-map of a bounded complete metric space (X,d). Suppose that there exists $m \in N$ such that f^m is condensing and*

$$d(f^m x, f^m y) < d(\{f^n z \mid z \in \{x,y\} \text{ and } n \in N_0\}) \tag{2}$$

for all x,y in X with $x \neq y$. Then (i), (ii) *and* (iii) *hold.*

PROOF. Since (2) implies (1), Corollary 3.2 follows from Theorem 3.1.

REMARK 3.1. Corollary 2 of Janos, Ko and Tan is a special case of Corollary 3.2 of this paper.

In [1, Theorem 3] Shih and Yeh proclaimed the following result:

THEOREM SY. *Let (X,d) be a bounded complete metric space, and let $f : X \to X$. Suppose that there exists $m \in N$ such that f^m is condensing and*

$$(*) \quad d(f^m x, f^m y) < \max\{d(x,y), d(x,f^m x), d(y,f^m y), d(x,f^m y), d(y,f^m x)\}$$

for all x,y in X with $x \neq y$. Then (iii) *holds.*

The following is a counterexample to Theorem SY:

EXAMPLE. Let $X = [0,1]$ with the usual metric d. Define $f : X \to X$ by $f0 = \frac{1}{2}$, $fx = 1$ for $x \in (0,1]$. Then f is not continuous and $f^2 x = 1$ for $x \in [0,1]$. Clearly f^2 satisfies $(*)$ and is condensing. Thus the conditions of Theorem SY hold. We assert that the conclusion of Theorem SY does not hold. Otherwise there is a bounded complete metric d^* on X which is equivalent to d such that $d^*(fx,fy) < d^*(x,y)$ for all x,y in X with $x \neq y$. It follows that f is continuous with respect to d^*. Since d^* is equivalent to d, so f is also continuous with respect to d. This is a contradiction.

By Corollary 3.2 we have

COROLLARY 3.3. *Let f be a continuous self-map of a bounded complete metric space (X,d). Suppose that there exists $m \in N$ such that f^m is condensing and* (*) *holds. Then* (iii) *holds.*

4. Fixed point theorems for compact maps

The results of this section are inspired by three papers of Jungck [2], Janos [4] and Leader [5].

THEOREM 4.1. *Let f be a continuous self-map of a metric space (X,d). Suppose that there exists $m \in N$ such that f^m is compact and*

$$d(f^m x, f^m y) < d(\{gz \mid z \in \{x,y\} \text{ and } g \in H_f\}) \tag{3}$$

for all x, y in X with $x \neq y$. Then (i) *and the following* (v) *and* (vi) *hold:*

(v) *for any $c \in (0,1)$ there exists a metric $d^* \in M(X)$ relative to which f is a contraction with Lipschitz constant c;*

(vi) *H_f has a unique fixed point v.*

PROOF. Since f^m is a compact map, so there exists a compact subset Y of X such that $Y \supset f^m(X)$. It follows that

$$\begin{aligned} X \supset Y \supset f^m(X) \supset f^m(Y) \supset \cdots \\ \supset f^{mn}(X) \supset f^{mn}(Y) \supset f^{m(n+1)}(X) \supset f^{m(n+1)}(Y) \supset \cdots \end{aligned} \tag{4}$$

Put $A = \bigcap_{n=0}^{\infty} f^{mn}(Y)$. By the continuity of f and compactness of Y, $f^{mn}(Y)$ is compact for $n \in N_0$. Clearly A is a nonempty compact subset of X. We now prove that $f(A) = A$. From (4) we have

$$\bigcap_{n=0}^{\infty} f^{mn}(X) = \bigcap_{n=1}^{\infty} f^{mn}(X) \subset \bigcap_{n=0}^{\infty} f^{mn}(Y) \subset \bigcap_{n=0}^{\infty} f^{mn}(X)$$

which implies $A = \bigcap_{n=0}^{\infty} f^{mn}(X)$. By $f(X) \subset X$, $f^{mn}(X) \subset f^n(X)$ for every $n \in N_0$. It follows that $\bigcap_{n=0}^{\infty} f^{mn}(X) \subset \bigcap_{n=0}^{\infty} f^n(X)$. Clearly $\bigcap_{n=0}^{\infty} f^{mn}(X) \supset \bigcap_{n=0}^{\infty} f^n(X)$. Hence $\bigcap_{n=0}^{\infty} f^{mn}(X) = \bigcap_{n=0}^{\infty} f^n(X)$. Since $f(A) = f(\bigcap_{n=0}^{\infty} f^n(X)) \subset \bigcap_{n=1}^{\infty} f^n(X) \subset A$, $f^m(A) \subset f(A) \subset A$. Take $y \in A$, $y \in f^{m(n+1)}(Y)$ for $n \in N_0$. Thus there exists $x_n \in f^{mn}(Y)$ such that $f^m x_n = y$. The sequence $\{x_n \mid n \in N_0\}$ has a cluster point x in the compact subset Y of X. Since f is continuous, $f^m x$ is a cluster point of $\{f^m x_n \mid n \in N_0\}$. It follows that $f^m x = y$. Now $x_n \in f^{mk}(Y)$ for $n \geq k$, and $f^{mn}(Y)$ is closed. So $x \in f^{mk}(Y)$ for every

$k \in N_0$; i.e., $x \in A$. This implies $f^m(A) \supset A$. Hence $A \subset f^m(A) \subset f(A) \subset A$ and it follows that $A = f(A) = f^m(A)$. We assert that A is a singleton. Otherwise $d(A) > 0$, so by compactness there exist distinct $u, w \in A$ such that $d(u, w) = d(A)$. Since $A = f^m(A)$, there exist distinct $x, y \in A$ such that $u = f^m x$, $w = f^m y$. Obviously $\{gz \mid z \in \{x, y\} \text{ and } g \in H_f\} \subset A$. Using (3) we obtain

$$0 < d(A) = d(f^m x, f^m y) < d(\{gz \mid z \in \{x, y\} \text{ and } g \in H_f\}) \leq d(A)$$

which is a contradiction. So A is a singleton, say, $A = \{v\}$. It follows that $fv = v$. Now if $b = fb$, then $b = f^n b$ for $n \in N$; i.e., $b \in A = \{v\}$. Hence v is the only fixed point of f.

By $\bigcap_{n=0}^{\infty} f^n(X) = \{v\}$ and $f^n x \in f^n(X)$ for x in X and n in N, we have $d(f^n x, v) \leq d(f^n(X)) \to 0$ as $n \to \infty$; i.e., $f^n x \to v$ as $n \to \infty$.

Take $U = X$. Since $d(f^n(X))$ diminishes to zero as $n \to \infty$, $f^n(U)$ sequeezes into any neighborhood of v. Thus (v) follows by Lemma 2.2.

We next prove that (vi) holds. Note that $\bigcap_{n=0}^{\infty} f^n(X) = \{v\}$. The remaining portion of the proof can be derived as in Theorem 3.1 (iv). This completes the proof.

As a consequence of Theorem 4.1 we have

COROLLARY 4.2. *Let f be a continuous self-map of a metric space (X, d). Suppose that there exists $m \in N$ such that f^m is compact and satisfies* (2) *for all x, y in X with $x \neq y$. Then* (i) *and* (v) *hold.*

REMARK 4.1. Theorem 1.1 of Janos [4] is a special case of Corollary 4.2 of this paper.

THEOREM 4.3. *Let f be a continuous self-map of a compact metric space (X, d). If for some $m \in N$, f^m satisfies* (3) *for all x, y in X with $x \neq y$, then* (i) *and* (v) *hold.*

PROOF. Put $A = \bigcap_{n=0}^{\infty} f^n(X)$. By Proposition 2 of Leader [5], A is compact, $d(f^n(X)) \downarrow d(A)$ as $n \to \infty$, and $f(A) = A \neq \emptyset$. Clearly $f^m(A) = A$. As in the proof of Theorem 4.1 we can show that A is a singleton. Theorem 4.3 follows from Theorem 1 of Leader [5].

As a particular case of Theorem 4.3 we have

COROLLARY 4.4. *Let f be a continuous self-map of a compact metric space (X, d). If for some $m \in N$, f^m satisfies* (2) *for all x, y in X with $x \neq y$, then* (i) *and* (v) *hold.*

REMARK 4.2. Our Corollary 4.4 generalizes Corollaries 2 and 3 of Leader

[5] and Theorem 2 of Shih and Yeh [1].

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