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(Citation)

Kobe Journal of Mathematics, 11(2):179-186

(Issue Date)

1994-12

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0003606>

(URL)

<https://hdl.handle.net/20.500.14094/E0003606>



MINIMAL IMMERSIONS OF RP^2 INTO RP^{2m}

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(Received September 22, 1993)

(Revised March 11, 1994)

1. Introduction

Let RP^{2m} be the $2m$ -dimensional real projective space of constant curvature 1, RP^2 topologically a real projective plane, and $\tilde{x} : RP^2 \rightarrow RP^{2m}$ a minimal immersion with respect to the induced metric. It is known that the area $A(\tilde{x})$ of $\tilde{x}$ must be a multiple of 2π , equal to or larger than $m(m+1)\pi^{[1,2]}$. A natural question is: How many minimal immersions of RP^2 into RP^{2m} are there that have a same area?

To this question, N. Ejiri got an answer for the least possible area value $m(m+1)\pi^{[3]}$. He proved that only one such immersion has area $m(m+1)\pi$.

In this paper, we first give a more general uniqueness theorem (Theorem 3.1) which, combined with a result of Barbosa [2], recovers the result above as a simple corollary. Then, we prove a nonexistence theorem (Theorem 3.4) for the second least value $[m(m+1) + 2]\pi$.

We also give some examples. These examples and the nonexistence theorem suggest a conjecture concerned with the area values (see Section 3).

2. Preliminary lemmas

Let S^2 be a 2-sphere, and S^{2m} the $2m$ -dimensional sphere of constant curvature 1. Then S^2 and S^{2m} are universal coverings of RP^2 and RP^{2m} respectively, and any full minimal immersion $\tilde{x} : RP^2 \rightarrow RP^{2m}$ is induced by a full minimal immersion $x : S^2 \rightarrow S^{2m}$. Fix a stereographic projection, we can identify S^2 with $\mathbb{C} \cup \{\infty\}$. It is trivial to check that the deck transformation of the covering $S^2 \rightarrow RP^2$ is given by setting z to $-\frac{1}{\bar{z}}$ (see also [3]), for each $z \in \mathbb{C}$. Thus a full minimal immersion x of S^2 into S^{2m} induces a minimal immersion of RP^2 into RP^{2m} if and only if for all $z \in \mathbb{C}$,

$$x\left(-\frac{1}{\bar{z}}\right) = \varepsilon x(z), \quad \varepsilon = \pm 1 \quad (2.1)$$

On the other hand, it is known^[2] that any full minimal immersion $x : S^2 \rightarrow S^{2m}$ is uniquely determined by its directrix Ξ , a totally isotropic, full holomorphic

curve in CP^{2m} of genus zero. If g is the associated map of x (For definition, see [3]), a local lift of Ξ to $\mathbb{C}^{2m+1}$, then (2.1) is equivalent to that

$$g(z) = (-1)^m z^{2m} \overline{g\left(-\frac{1}{\bar{z}}\right)} \quad (2.2)$$

LEMMA 2.1 [4]. *Let Ξ be the directrix of a minimal immersion $x : S^2 \rightarrow S^{2m}$. If $\text{Area}(x) = 2\pi[m(m+1) + 2]$, and $z = 0$ is a stationary point of Ξ , then Ξ has a local lift as*

$$\xi = a_0 + a_1 z + \cdots + a_{2m+1} z^{2m+1} + a_{2m+2} z^{2m+2}$$

in which, $a_0, a_2, \dots, a_{2m}, a_{2m+2}$ are linearly independent, and

$$a_1 = \lambda a_0, \quad a_{2m+1} = u_0 a_0 + u_2 a_2 + \cdots + u_{2m} a_{2m} \quad (2.3)$$

for some complex numbers $\lambda_0, u_0, u_2, \dots, u_{2m}$; Also,

$$(a_{m+1-s}, a_{m+1+s}) = (-1)^s \frac{m^2 - s^2}{m^2} \cdot \frac{(m+1)!(m+1)!}{(m+1-s)!(m+1+s)!} (a_{m+1}, a_{m+1}),$$

$$s = 1, 2, \dots, m+1 \quad (2.4)$$

where $(*, *)$ denotes the canonical symmetric product of $\mathbb{C}^{2m+1}$.

LEMMA 2.2 [2]. *Given any two $\mathbb{C}^{2m+1}$ -valued, totally isotropic and linearly full polynomials*

$$\begin{aligned} \xi &= a_0 + a_1 z + \cdots + a_{2m} z^{2m} \\ \eta &= b_0 + b_1 z + \cdots + b_{2m} z^{2m}, \end{aligned}$$

with $(a_{m+1}, a_{m+1}) = (b_{m+1}, b_{m+1}) = 1$, there is a transformation $\mathbf{A} \in SO(2m+1, \mathbb{C})$ such that $\eta = \pm \mathbf{A}\xi$.

LEMMA 2.3. *Let $\tilde{x} : RP^2 \rightarrow RP^{2m}$ be a full minimal immersion induced by a minimal immersion $x : S^2 \rightarrow S^{2m}$, and Ξ be the directrix of x with degree n . If $n = 2k$ or $2k+1$, then for any nonzero polynomial lift of Ξ :*

$$\xi = \sum_{i=0}^n a_i z^i, \quad \text{satisfying } (a_k, a_k) = 1,$$

we have

$$\xi(z) = cz^n \overline{\xi\left(-\frac{1}{\bar{z}}\right)}, \quad (2.5)$$

where c is a number independent of ξ ; when $n = 2k$, $c = (-1)^k$.

PROOF. Let g be as above the associated map of x . Barbosa proved in [3] that g has only isolated poles as its singularities with respect to any complex coordinate. In our present coordinate, g is well defined on the total of $\mathbf{C}$. By analyzing the change of g under the transformation from z to $\omega = \frac{1}{z}$, we see readily that g , as a $\mathbf{C}^{2m+1}$ -valued function of z , is rational, i.e., all its components are rational functions of z . Since g is nonzero on $\mathbf{C}$, we can assume, for some nonnegative integers n, r , complex numbers b_i and $\mathbf{C}^{2m+1}$ -valued vectors a'_j , that

$$g = \frac{1}{\sum_{i=0}^r b_i z^i} \sum_{j=0}^n a'_j z^j, \quad (a'_k, a'_k) = 1, \quad z \in \mathbf{C}. \quad (2.6)$$

Then

$$g\left(-\frac{1}{\bar{z}}\right) = \frac{\bar{z}^{r-n}}{\sum_{i=0}^r b_i (-1)^i \bar{z}^{r-i}} \sum_{j=0}^n a'_j (-1)^j \bar{z}^{n-j} \quad (2.7)$$

This with (2.2) and (2.6) gives that

$$\frac{(-1)^m z^{r+2m-n}}{\sum_{i=0}^r (-1)^i \bar{b}_i z^{r-i}} \sum_{j=0}^n \bar{a}'_j (-1)^j z^{n-j} = \frac{1}{\sum_{i=0}^r b_i z^i} \sum_{j=0}^n a'_j z^j \quad (2.8)$$

Note that both $\sum_{j=0}^n \bar{a}'_j (-1)^j z^{n-j}$ and $\sum_{j=0}^n a'_j z^j$ are nonzero polynomial lifts of Ξ , one sees that

$$c \sum_{j=0}^n \bar{a}'_j (-1)^j z^{n-j} = \sum_{j=0}^n a'_j z^j \quad (2.9)$$

with some number $c \neq 0$. This and (2.8) imply

$$(-1)^m z^{r+2m-n} \sum_{i=0}^r b_i z^i = c \sum_{i=0}^r \bar{b}_i (-1)^i z^{r-i}$$

from which we can see that $n = 2m + r + p$ for some $p \geq 0$, and

$$cz^p \sum_{i=0}^r \bar{b}_i (-1)^i z^{r-i} = (-1)^m \sum_{i=0}^r b_i z^i. \quad (2.10)$$

(2.9) is equivalent to that

$$c(-1)^j \bar{a}'_j = a'_{n-j}, \quad j = 0, 1, 2, \dots, n. \quad (2.11)$$

In particular,

$$c(-1)^k \bar{a}'_k = a'_k, \text{ if } n = 2k; c(-1)^k \bar{a}'_k = a'_{k+1} \text{ if } n = 2k + 1. \quad (2.12)$$

When $n = 2k$, by the second expression in (2.6), we can see that

$$c^2 = (-1)^{2k} c^2 (\bar{a}'_k, \bar{a}'_k) = (a'_k, a'_k) = 1,$$

i.e., $c = \pm 1$. (2.9) can be rewritten as

$$\sum_{j=0}^n a'_j z^j = cz^n \overline{\left[\sum_{j=0}^n a'_j \left(-\frac{1}{\bar{z}} \right)^j \right]} \quad (2.13)$$

Now, since there is a number $c_1 \neq 0$ such that

$$\xi = \sum_{j=0}^n a_j z^j = c_1 \sum_{j=0}^n a'_j z^j$$

we get $a_k = c_1 a'_k$ hence $c_1 = \pm 1$ by $(a_k, a_k) = (a'_k, a'_k) = 1$. Thus

$$\xi(z) = \pm \sum_{j=0}^n a'_j z^j = \pm cz^n \overline{\left[\sum_{j=0}^n a'_j \left(-\frac{1}{\bar{z}} \right)^j \right]} = cz^n \overline{\left[\sum_{j=0}^n a_j \left(-\frac{1}{\bar{z}} \right)^j \right]} = cz^n \overline{\left(-\frac{1}{\bar{z}} \right)}.$$

The independence of c is obvious. Suppose $n = 2k$, then from the first identity in (2.12) with $c = \pm 1$, we can conclude that a'_k is either real or imaginary. But the assumption that $(a'_k, a'_k) = 1$ excludes the latter possibility. So $\bar{a}'_k = a'_k$ and $c = (-1)^k$.

REMARK 2.4. We believe that, in Lemma 2.3, case $n = 2k + 1$ can not occur, that is, n must be even.

3. Main results

THEOREM 3.1. *Let $x_0 : S^2 \rightarrow S^{2m}$ be any full minimal immersion and $\mathcal{S} = \{\mathbf{A}x_0; \mathbf{A} \in SO(2m+1, \mathbb{C})\}$. Then modulo isometries, there is at most one $x \in \mathcal{S}$ which induces a minimal immersion of RP^2 into RP^{2m} .*

PROOF. Let x, y be any two that induce minimal immersions of RP^2 into RP^{2m} , and Ξ, H be their directrices with polynomial lifts ξ, η respectively. Let $\mathbf{A} \in SO(2m+1, \mathbb{C})$ be the one such that $y = \mathbf{A}x$, which is, by definition, in the sense that

$$c_1 \eta = \mathbf{A} \xi, \quad (3.1)$$

with c_1 a nonzero holomorphic function. We usually assume that both ξ and η are nonzero on the total of $\mathbf{C}$ and in this case, c_1 is a nonzero number, while $\deg \xi = \deg \eta = n$. Here, no loss of generality, we can also assume that ξ and η satisfy conditions in Lemma 2.3. It follows that $c_1 = \pm 1$.

Hence, we can apply Lemma 2.3 to obtain (Note: c_1 is real and c is invariant here)

$$cz^n \overline{\mathbf{A}\xi\left(-\frac{1}{z}\right)} = cz^n \overline{c_1 \eta\left(-\frac{1}{z}\right)} = c_1 cz^n \overline{\eta\left(-\frac{1}{z}\right)} = c_1 \eta(z) = \mathbf{A}\xi(z) = cz^n \overline{\mathbf{A}\xi\left(-\frac{1}{z}\right)}.$$

By the linear fullness of ξ ,

$$\overline{\mathbf{A}} = \mathbf{A}.$$

So $\mathbf{A}$ is real and hence belongs to $SO(2m+1, \mathbf{R})$. This indicates that Ξ, H are isometric and, by Barbosa [2], so are x, y . Q.E.D.

REMARK 3.2. A related result on the finiteness of such immersions from RP^2 to S^{2m} is proved in [5].

It is known that the standard immersion $x : S^2 \rightarrow S^{2m}$ given by spherical harmonic functions induces a minimal immersion $x : RP^2 \rightarrow RP^{2m}$ with constant Gauss curvature $K = \frac{2}{m(m+1)}$ and hence area $m(m+1)\pi$. Therefore from Barbosa's result, Lemma 2.2 and our uniqueness theorem above, we recover the following result of Ejiri [3]:

COROLLARY 3.3. Let $\tilde{x}$ be a full minimal immersion of RP^2 into RP^{2m} with area $m(m+1)\pi$. Then $\tilde{x}$ is induced by a standard minimal immersion.

Before stating the next theorem, we give first some examples.

EXAMPLE 3.4. Introduce the following numbers:

$$\lambda_0 = (-1)^m \frac{(m-1)!(m-1)!(k-m)!k}{2(k+m-1)!},$$

$$\lambda_{k-r} = (-1)^r \frac{k^2[(m-1)!]^2}{(k^2-r^2)(m+r-1)!(m-r-1)!}, \quad 1 \leq r \leq m-1$$

Choose an ordered orthonormal basis $e_1, \dots, e_m, e_0, e_{m+1}, \dots, e_{2m}$ for R^{2m+1} , and define

$$E_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}e_{m+i}), \quad i = 1, 2, \dots, m, \quad E_0 = e_0.$$

Then $E_1, \dots, E_m, E_0, \overline{E}_1, \dots, \overline{E}_m$ is a unitary basis for $\mathbf{C}^{2m+1}$ and the subspace spanned by $\{E_1, \dots, E_m\}$ is totally isotropic.

Now, for each integer $p \geq 0$, we set $k = m + 2p$, and define $a_0, a_{k-m+1}, \dots, a_{k-1}$ by

$$\alpha_0 = |\lambda_0|^{\frac{1}{2}}, \quad \alpha_i = |\lambda_i|^{\frac{1}{2}}, \quad k - m + 1 \leq i \leq k - 1.$$

Let

$$\begin{aligned} a_0 &= (-1)^k \alpha_0 E_1, \quad a_{k-m+1} = (-1)^{m-1} \alpha_{k-m+1} E_2, \dots, a_{k-1} = -\alpha_{k-1} E_m, \\ a_k &= E_0, \quad a_{k+1} = \alpha_{k-1} \bar{E}_m, \dots, a_{k+m-1} = \alpha_{k-m+1} \bar{E}_2, \quad a_{2k} = \alpha_0 \bar{E}_1 \\ a_1 &= \dots = a_{k-m} = a_{k+m} = \dots = a_{2k-1} = 0. \end{aligned}$$

Then by a proposition of Barbosa (Proposition 6.21 in [2]), we obtain for each p a totally isotropic holomorphic polynomial

$$\xi_p = \sum_{i=0}^{2k} a_i z^i,$$

which obviously satisfies (2.5). Hence we conclude, by direct verification, that the immersion x_p determined by ξ_p satisfies (2.1) and hence induces a minimal immersion $\tilde{x}$ of RP^2 into RP^{2m} . By Barbosa (see Proposition (6.15) in [2]),

$$\text{Area}(\tilde{x}) = \frac{1}{2} \text{Area}(x) = [m(m-1) + 2k]\pi = [m(m+1) + 4p]\pi.$$

Thus, for each possible area value $[m(m+1) + 4p]\pi$, we find a minimal immersion $\tilde{x}_p : RP^2 \rightarrow RP^{2m}$ which takes this value as its area.

These examples suggest that maybe $[m(m+1) + 4p]\pi$ are the only values that can be taken as the areas of minimal immersions of RP^2 into RP^{2m} . We are unable to give a general answer, but we can prove here the following result:

THEOREM 3.5. *There is no minimal immersion of RP^2 into RP^{2m} that is of area $[m(m+1) + 2]\pi$.*

PROOF. Let $x : S^2 \rightarrow S^{2m}$ be any full minimal immersion with directrix Ξ and area $2\pi[m(m+1) + 2]$, we shall prove that it does not induce a minimal immersion of RP^2 into RP^{2m} .

To this end, we suppose the contrary, and denote by $\tilde{x}$ the induced immersion by x of RP^2 into RP^{2m} . By Lemma 2.1., we know that, under a suitable coordinate z , Ξ has a local lift

$$\xi = \sum_{i=0}^{2m+2} a_i z^i$$

satisfying $(a_{m+1}, a_{m+1}) = 1$ and

$$a_1 = \lambda a_0, \quad a_{2m+1} = u_0 a_0 + \sum_{j=2}^{2m} u_j a_j \quad (3.2)$$

where $\lambda, u_0, u_j, 2 \leq j \leq 2m$, are complex numbers. By Lemma 2.3,

$$\xi(x) = (-1)^{m+1} \overline{\xi\left(-\frac{1}{\bar{z}}\right)} z^{2m+2},$$

that is

$$\sum_{i=0}^{2m+2} a_i z^i = (-1)^{m+1} \sum_{i=0}^{2m+2} \bar{a}_i (-1)^i z^{2m+2-i}.$$

This implies

$$a_i = (-1)^{m+i+1} \bar{a}_{2m+2-i}.$$

In particular,

$$a_1 = (-1)^m \bar{a}_{2m+1}, \quad a_0 = \bar{a}_{2m+2} (-1)^{m+1} \quad (3.3)$$

Thus from (3.2) it is seen that

$$(-1)^{m+1} \lambda \bar{a}_{2m+2} = a_1 = (-1)^m \bar{a}_{2m+1} = (-1)^m (\bar{u}_0 \bar{a}_0 + \sum_{j=2}^{2m} \bar{u}_j \bar{a}_j).$$

But we already know that the vectors $a_0, a_2, \dots, a_{2m}, a_{2m+2}$ are linearly independent, so

$$\lambda = u_0 = u_2 = \dots = u_{2m} = 0.$$

Set $s = m + 1$ in (2.4). We see that

$$(a_0, a_{2m+2}) = (-1)^m \frac{(2m+1)! [(m+1)!]^2}{m^2 (2m+2)!}.$$

But (3.3) indicates that the left-hand side is nothing but $(-1)^{m+1} |a_0|^2$. Thus we finally obtain

$$|a_0|^2 = -\frac{(2m+1)! [(m+1)!]^2}{m^2 (2m+2)!} < 0,$$

a contradiction which completes the proof of Theorem 3.5.

REMARK 3.6. Theorem 3.5 was already known to J. Bolton and L.M. Woodward [6].

The following proposition dues to the referee, which answers partially the question stated before Theorem 3.5.

PROPOSITION 3.7. *Let x be a full equivariant minimal immersion of RP^2 into RP^{2m} . Then the area of x is given by*

$$\text{Area}(x) = [m(m+1) + 4p]\pi, \quad p \geq 0.$$

PROOF. Let x be a full equivariant minimal immersion of type $(m_1, \dots, m_m)$ of RP^2 into RP^{2m} .

Case $m = 2k$ (even). By Proposition 4.2 of [3]

$$m_1 = 2k_1 - 1, \quad m_2 = 2k_2, \quad m_3 = 2k_3 - 1, \dots, m_m = 2k_m.$$

By Theorem 3.1 of [3],

$$\begin{aligned} \text{Area}(x) &= 2\pi[2(k_1 + \dots + k_m) - k] \\ &= \pi[m(m+1) + 4(k_1 + \dots + k_m) - m(m+1) - 2k] \\ &= \pi[m(m+1) + 4(k_1 + \dots + k_m) - 4k(k+1)] \\ &= \pi[m(m+1) + 4(k_1 + \dots + k_m) - k(k+1)] \end{aligned}$$

Case $m = 2k - 1$ (odd). The same argument applies.

Acknowledgment. The author is grateful to Professors An-min Li and Guosong Zhao for their kindness. He also gives much thanks to the referee for important comments and useful suggestions.

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