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THE M_B PROPERTY AND NORMALITY OF PRODUCT SPACES

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Abstract

In this paper we introduce the M_B property and show its role in normality of some product spaces in particular those products whose second factors are κ-metrizable.

1. Definitions and elementary concepts

Here every topological space is Hausdorff, and κ is an infinite regular cardinal.

DEFINITION 1.1. X is B_κ if and only if for every monotone family $\{F_\alpha : \alpha < \kappa\}$ ($F_\alpha \subset F_\beta$ if $\alpha > \beta$) of closed subsets of X, with $\bigcap_{\alpha<\kappa} F_\alpha = \emptyset$, there is a monotone family $\{U_\alpha : \alpha < \kappa\}$ of open subsets of X such that $F_\alpha \subset U_\alpha$ and $\bigcap_{\alpha<\kappa} \operatorname{cl} U_\alpha = \emptyset$.

Let $(\kappa+1)^-$ be $(\kappa+1)$ topologized using the singletons from κ and the intervals of the form $(\alpha, \kappa]$ for $\alpha < \kappa$, as basis. Yasui gave a generalization of Dowker's theorem that states, a normal space X is countably paracompact if and only if $X \times (\omega+1)^-$ is normal:

THEOREM 1.2 [6]. *If X is normal, the following are equivalent:*
(1) *X is B_κ.*
(2) *$X \times (\kappa+1)^-$ is normal.*

DEFINITION 1.3. X is $(\kappa, \lambda]$-*para-Lindelöf* if and only if every open cover of size λ has a localy $< \kappa$ refinement (every x in X has a neighborhood meeting $< \kappa$ many members from the refinement).

From Yasui [7] and Kammoun [1] we have the following:

THEOREM 1.4 [7], [1]. *X is B_κ if and only if X is $(\kappa, \kappa]$-para-Lindelöf if and only if $X \times (\kappa+1)^-$ is B_κ.*

2. On products with a κ-metrizable factor

Consider a set A, let $\alpha(y) = \{x \in A^\kappa : x_\beta = y_\beta,\ \beta \leq \alpha\}$, $\mathbf{B}_\alpha = \{\alpha(y) : y \in A^\kappa\}$ and $\mathbf{B} = \bigcup_{\alpha<\kappa} \mathbf{B}_\alpha$ be a basis for a topology $\mathbf{T}$ on A^κ. It is known that if $\kappa > \omega$, every κ-metrizable space Y is a subspace of some space $(A^\kappa, \mathbf{T})$ [4]. Observe that in the latter topology $\mathbf{B}_\alpha$ is a partition of Y, and if $\alpha(y) \cap \beta(z) \neq \emptyset$, then either $\alpha(y) \supset \beta(z)$ or $\beta(z) \supset \alpha(y)$, also $\mathbf{B}_\alpha$ refines $\mathbf{B}_\beta$ if $\beta < \alpha$.

In the following Y is κ-metrizable with a base $\mathbf{B}$ as above.

DEFINITION 2.1. A space X is M_B if for every monotone family $\{F_B : B \in \mathbf{B}\}$ of closed subsets of X with $\cap\{F_B : y \in B\} = \emptyset$ for each fixed y in Y, there is a monotone family $\{U_B : B \in \mathbf{B}\}$ of open subsets of X with $U_B \supset F_B$ and $\cap\{\mathrm{cl}\, U_B : y \in B\} = \emptyset$.

In [5] Przymusinski gave a characterization for the spaces whose products with a given metrizable space are normal, defining the M_B as above, but with $\mathbf{B}$ as a σ-locally finite basis.

THEOREM 2.2 [5]. *Let Y be metrizable. Then $X \times Y$ is normal if and only if X is normal M_B.*

One direction of the above theorem generalizes (we include the proof for completeness):

THEOREM 2.3 [2]. *Let Y be κ-metrizable. Then X is M_B if $X \times Y$ is normal.*

PROOF. It remains for the case when $\kappa > \omega$. Suppose $X \times Y$ is normal and $\{F_B : B \in \mathbf{B}\}$ is a monotone family of closed subsets of X such that $\cap\{F_B : y \in B\} = \emptyset$ if $y \in Y$. Let $\underline{\mathbf{B}} = (B \in \mathbf{B} : B \text{ is not discrete}\}$, then $\mathbf{B}\backslash\underline{\mathbf{B}}$ consists of singletons, so that $F_B = \emptyset$ if $B \in \mathbf{B}\backslash\underline{\mathbf{B}}$. Denote $\underline{\mathbf{B}} \cap \mathbf{B}_\alpha$ by $\underline{\mathbf{B}}_\alpha$. Claim: We can choose distinct points $p_{0,B}$ and $p_{1,B}$ from every $B \in \underline{\mathbf{B}}$ so that the set $\{p_{0,B} : B \in \underline{\mathbf{B}}\} \cup \{p_{1,B} : B \in \underline{\mathbf{B}}\}$ consists of distinct points.

Proof of the Claim. We have $\underline{\mathbf{B}} = \bigcup_{\alpha<\kappa} \underline{\mathbf{B}}_\alpha$. Well order $\underline{\mathbf{B}}$ in a way that the members of $\underline{\mathbf{B}}_\alpha$ come after all members of $\underline{\mathbf{B}}_\beta$ if $\beta < \alpha$. For each fixed $B \in \underline{\mathbf{B}}$ the set $\{G \in \underline{\mathbf{B}} : G \text{ precedes } B \text{ in the well order}\}$ is locally $< \kappa$ because if $y \in Y$ and $B \in \underline{\mathbf{B}}_\alpha$ for some $\alpha < \kappa$, then $\alpha(y) \cap G \neq \emptyset$ if and only if $G = \beta(y)$ for some $\beta \leq \alpha$. So assume $p_{i,G}$ where $i \leq 1$, have been chosen for all G preceding B. Let y be a nonisolated point of $\mathbf{B}$ and let V be a neighborhood of y, so that $V \subset B$ and containing only τ many $p_{i,G}$ for some $\tau < \kappa$. Now, one can pick $p_{i,B}$ for $i \leq 1$ distinct from the earlier choices.

To complete the proof of the theorem, let $K_i = \cup\{F_B \times \{p_{i,B}\} : B \in \underline{\mathbf{B}}\}$

for $i \leq 1$. Then the sets K_i are disjoint and closed. Indeed, letting $K_{i,\alpha} = \cup\{F_B \times \{p_{i,B}\} : B \in \underline{\mathbf{B}}_\alpha\}$ we then have $K_i = \bigcup_{\alpha<\kappa} K_{i,\alpha}$. The sets $K_{i,\alpha}$ are closed because $\underline{\mathbf{B}}_\alpha$ are discrete, which implies that $\{X \times \{p_{i,B}\} : B \in \underline{\mathbf{B}}_\alpha\}$ is discrete. Similarly, $\{X \times \{p_{i,B}\} : B \in \bigcup_{\beta\leq\alpha} \underline{\mathbf{B}}_\beta\}$ is locally $< \kappa$ (1). To show that K_i is closed, let $\langle x, y\rangle \notin K_i$, find $B \in \mathbf{B}$ such that $y \in B$ and $x \notin F_B$. There exists β with $B \supset \beta(y)$ and if $\alpha \geq \beta$ then $((X\backslash F_B) \times \beta(y)) \cap K_{i,\alpha} = \emptyset$, since otherwise $((X\backslash F_B) \times \beta(y)) \cap (F_G \times \{p_{i,G}\}) \neq \emptyset$ for some $G \in \underline{\mathbf{B}}_\alpha$, giving $\beta(y) \cap G \neq \emptyset$ which implies that $B \supset G$, $F_B \supset F_G$ and $((X\backslash F_B) \times \beta(y)) \cap (F_G \times \{p_{i,G}\}) = \emptyset$, a contradiction. Hence, $\langle x, y\rangle$ has a neighborhood disjoint from $\bigcup_{\alpha\geq\beta} K_{i,\alpha}$ and on the other hand $\bigcup_{\alpha\leq\beta} K_{i,\alpha}$ is closed, using (1).

Now let U_0 and U_1 be open with $U_i \supset K_i$ and $\operatorname{cl} U_0 \cap \operatorname{cl} U_1 = \emptyset$. If $B \in \underline{\mathbf{B}}$ define $U_B = \{x \in X : \langle x, p_{i,D}\rangle \in U_i$ for some $D \in \underline{\mathbf{B}}$ with $B \supset D\}$ and if $B \notin \underline{\mathbf{B}}$ let $U_B = \emptyset$. Then $\{U_B : B \in \mathbf{B}\}$ is monotone, $U_B \supset F_B$ and U_B is open. To see that $\bigcap_{y\in B} \operatorname{cl} U_B = \emptyset$ if $y \in Y$: Fix x in X if y is nonisolated in Y, there exists $i \leq 1$ such that $\langle x, y\rangle \notin \operatorname{cl} U_i$. Then there is a neighborhood V of x and $B \in \underline{\mathbf{B}}$ a neighborhood of y such that $(V \times B) \cap \operatorname{cl} U_i = \emptyset$, also $(V \times D) \cap \operatorname{cl} U_i = \emptyset$ for all $D \in \underline{\mathbf{B}}$ with $B \supset D$. Hence, $V \cap U_B = \emptyset$ implying that $x \notin \operatorname{cl} U_B$. If y is isolated, there exists $B \in \mathbf{B}$ with $B = \{y\}$, in this case $U_B = \emptyset$. So $\cap\{\operatorname{cl} U_B : y \in B\} = \emptyset$ for all $y \in Y$.

THEOREM 2.4. *Let Y be κ-metrizable, then X has the M_B property if $X \times Y$ is B_κ.*

PROOF. Let $\mathbf{B}$ be a basis for Y, as above, and let $\{F_B : B \in \mathbf{B}\}$ be a monotone family of closed subsets of X with $\cap\{F_B : y \in B\} = \emptyset$ for all $y \in Y$. Define $F_\alpha = \cup\{F_B \times B : B \in \mathbf{B}_\alpha\}$. Observe that the sets F_α are closed, since $\mathbf{B}_\alpha$ are discrete families of clopen sets, and $F_\beta \supset F_\alpha$ if $\beta < \alpha$. Moreover, $\bigcap_{\alpha<\kappa} F_\alpha = \emptyset$, for if $\langle x, y\rangle \in X \times Y$, then there is α large enough and $B \in \mathbf{B}_\alpha$ with $y \in B$ and $x \notin F_B$, so $\langle x, y\rangle \notin F_\alpha$ since B is the only member of $\mathbf{B}_\alpha$ containing y.

Let $\{U_\alpha : \alpha < \kappa\}$ be a monotone family of open subsets of $X \times Y$ with $U_\alpha \supset F_\alpha$ and $\bigcap_{\alpha<\kappa} \operatorname{cl} U_\alpha = \emptyset$ as provided by the hypothesis on $X \times Y$. Now for $B \in \underline{\mathbf{B}}_\alpha$ define $U_B = \pi(U_\alpha \cap (X \times B))$, where π is the projection map $\pi : X \times Y \to X$. We have $F_B \subset U_B$ and $U_B \subset U_D$ if $B \subset D$. Moreover, if $\langle x, y\rangle \in X \times Y$ then there is $\alpha < \kappa$ and $V \times W$ an open neighborhood of $\langle x, y\rangle$ with $(V \times W) \cap U_\alpha = \emptyset$. Let $B \in \underline{\mathbf{B}}_\nu$ such that $y \in B \subset W$ and $\nu \geq \alpha$. Then $(V \times B) \cap U_\alpha = \emptyset$ implying that $V \cap U_B = \emptyset$, hence $x \notin \operatorname{cl} U_B$. So $\cap\{\operatorname{cl} U_B : y \in B\} = \emptyset$.

THEOREM 2.5. *Let X be B_κ and locally κ-Lindelöf. Then X is M_B if Y is κ-metrizable and κ-Lindelöf (every open cover of size $\geq \kappa$ has a subcover of size $< \kappa$).*

PROOF. Since Y is κ-Lindelöf we may assume that $\mathbf{B} = \bigcup_{\alpha<\kappa} \mathbf{B}_\alpha$, $|\mathbf{B}_\alpha| < \kappa$ for each α, as basis for Y. Suppose $\{F_B : B \in \mathbf{B}\}$ is a monotone family of closed subsets of X with $\cap\{F_B : y \in B\} = \emptyset$ for all $y \in Y$. Fix $x \in X$, $y \in Y$ and a κ-Lindelöf neighborhood N of x. We then have, $\{X\backslash F_B : y \in B\}$ as an open cover of N, in particular there exists B such that $N \subset X\backslash F_B$. Let ζ^y be the least ($< \kappa$) ordinal for which $F_{\zeta(y)}$ misses N. Now, the family $\{\zeta^y(y) : y \in Y\}$ is a partition of Y of size $< \kappa$. Let $\alpha = \sup\{\zeta^y : y \in Y\}$, then $N \cap F_{\alpha(y)} = \emptyset$ for all $y \in Y$. Therefore $x \notin \mathrm{cl}_X(\cup\{F_B : B \in \mathbf{B}_\alpha\})$ and so if we let $F_\alpha = \mathrm{cl}_X(\cup F_B : B \in \mathbf{B}_\alpha\})$ for each $\alpha < \kappa$ then, $F_\alpha \subset F_\beta$ if $\beta < \alpha$ and $\bigcap_{\alpha<\kappa} F_\alpha = \emptyset$ by what we have just seen. Now, since X is B_κ there is a monotone family $\{U_\alpha : \alpha < \kappa\}$ of open subsets of X with $U_\alpha \supset F_\alpha$ and $\bigcap_{\alpha<\kappa} \mathrm{cl}\, U_\alpha = \emptyset$. If $B \in \mathbf{B}_\alpha$, let $U_B = U_\alpha$, then $\{U_B : B \in \mathbf{B}\}$ is monotone with $U_B \supset F_B$ and $\cap\{\mathrm{cl}\, U_B : y \in B\} = \emptyset$ for all $y \in Y$.

THEOREM 2.6. *X is M_B if in the hypothesis of* 2.5, *"locally κ-Lindelöf" is replaced by κ-additive (the intersection of fewer than κ-many open sets if open).*

PROOF. Let $\mathbf{B}$ be a basis of Y as described earlier. Since Y is κ-Lindelöf we can assume $|\mathbf{B}_\alpha| < \kappa$. Now, $F_\alpha = \cup\{F_B : B \in \mathbf{B}_\alpha\}$ is closed since it is the union of less than κ many closed subsets of the κ-additive space X. Fix $x \in X$ and for each $y \in Y$ pick the least $\zeta^y < \kappa$ so that $x \notin F_{\zeta(y)}$. Then the set of all such $\zeta^y(y)$ is a partition of Y of cardinality $< \kappa$, because Y is κ-Lindelöf. Let $\alpha = \sup\{\zeta^y : y \in Y\}$ then $\alpha < \kappa$, since κ is regular, and $x \notin F_B$ if $B \in \mathbf{B}_\alpha$, for if $x \in F_{\alpha(y)}$ then $x \in F_\zeta$ where $\zeta = \zeta^y \leq \alpha$, a contradiction. To complete the proof proceed as in the last paragraph of the proof of 2.5.

DEFINITION 2.7. A space X is M_κ if for every cardinal $\lambda < \kappa$ and every monotone family $\{F_\sigma : \sigma \in \lambda^{<\kappa}\}$ of closed subsets of X there is a monotone family $\{U_\sigma : \sigma \in \lambda^{<\kappa}\}$ of open subsets of X with $F_\sigma \subset U_\sigma$ and if $\bigcap_{\alpha<\kappa} F_{f|\alpha} = \emptyset$ then $\bigcap_{\alpha<\kappa} U_{f|\alpha} = \emptyset$ for each $f \in \lambda^\kappa$. For notational purposes let $[\sigma] = \{f \in \lambda^\kappa : f$ extends $\sigma\}$.

We have the following characterization (given by Morita in the early 1960's) of the spaces X whose products with every metrizable space are normal.

THEOREM 2.8 [5]. *$X \times Y$ is normal for every metrizable Y if and only if X is normal M_ω.*

The following generalizes one direction of 2.8:

THEOREM 2.9. *If $X \times Y$ is normal for every κ-metrizable Y, then X is M_κ.*

PROOF. Let $\{F_\sigma : \sigma \in \lambda^{<\kappa}\}$ be a monotone family of closed subsets of X. Consider $Y = \{f \in \lambda^\kappa : \bigcap_{\alpha<\kappa} F_{f|\alpha} = \emptyset\}$ as a subspace of λ^κ with the κ-metrizable topology, let $\mathbf{B} = \{[\sigma] \cap Y : \sigma \in \lambda^{<\kappa}\}$ be the basis for Y. Since $X \times Y$ is normal, X has the M_B property (use 2.3) and $\cap\{F_\sigma : f \in [\sigma]\} = \bigcap_{\alpha<\kappa} F_{f|\alpha} = \emptyset$ for each $f \in Y$. This implies the existence of a monotone family $\{U_\sigma : \sigma \in \lambda^{<\kappa}\}$ of open subsets of X with $F_\sigma \subset U_\sigma$ and $\cap\{\operatorname{cl} U_\sigma : f \in [\sigma]\} = \emptyset$ for each $f \in Y$.

3. More results

The following generalizes 2.2 in the class of κ-additive spaces:

THEOREM 3.1. *Let Y be κ-metrizable and X be κ-additive. Then $X \times Y$ is normal if and only if X is normal M_B.*

PROOF. 2.3 gives ($\rightarrow$). For the other direction, suppose X is κ-additive with the M_B property and K_i, $i \leq 1$, are two disjoint closed subsets of $X \times Y$. Let $\mathbf{B}$ be a basis of Y as above and define $F_{i,B} = \operatorname{cl} \pi(X \times B) \cap K_i)$, where $\pi : X \times Y \rightarrow X$ is the projection map, and let $F_B = F_{0,B} \cap F_{1,B}$. Then $\{F_B : B \in \mathbf{B}\}$ is a monotone family of closed subsets of X with $\cap\{F_B : y \in B\} = \emptyset$ if $y \in Y$. For if $\langle x, y\rangle \in X \times Y$, there is $i \leq 1$ and a neighborhood $N \times B$ of $\langle x, y\rangle$ missing K_i, implying that N misses $F_{i,B}$ and then $x \notin F_B$.

Now let $\{U_B : B \in \mathbf{B}\}$ be the expansion of F_B's provided by M_B. Let $W_{i,i,B}$ and $W_{1-i,i,B}$ be disjoint open sets containing $F_{i,B}$ and $F_{1-i,B} \backslash U_B$ respectively for $i \leq 1$. Define $V_{i,B} = W_{i,i,B} \cap W_{i,1-i,B}$ for $i \leq 1$, then $(F_{0,B} \cup F_{1,B}) \backslash U_B \subset V_{0,B} \cup V_{1,B}$ and $F_{i,B} \cap \operatorname{cl} V_{1-i,B} = \emptyset$ for each i. Let $\mathbf{V} = \bigcup_{\alpha<\kappa} \bigcup_{i\leq 1} \{V_{i,B} \times B : B \in \mathbf{B}_\alpha\}$, then $\mathbf{V}$ is a closure preserving open cover of $K_0 \cup K_1$ such that $K_i \cap (\operatorname{cl} V_{1-i,B} \times B) = \emptyset$: Observe that $\{V_{i,B} \times B : B \in \mathbf{B}_\alpha\}$ is discrete, and if $\langle x, y\rangle \in K_0 \cup K_1$ then there is B such that $y \in B$ and $x \notin \operatorname{cl} U_B$ implying that $x \in (F_{0,B} \cup F_{1,B}) \backslash U_B \subset V_{0,B} \cup V_{1,B}$. Let $V_{i,\alpha} = \cup\{V_{i,B} \times B : B \in \mathbf{B}_\alpha\}$ and $U_{i,\alpha} = V_{i,\alpha} \backslash (\bigcup_{\beta\leq\alpha} \operatorname{cl} V_{1-i,B})$, then $U_{i,\alpha}$ is open (using κ-additivity). Now the sets $U_i = \bigcup_{\alpha<\kappa} U_{i,\alpha}$, are disjoint open sets which contain K_0 and K_1, respectively.

THEOREM 3.2. *Let X be κ-additive. Then $X \times Y$ is normal for every κ-metrizable space Y if and only if X is normal M_κ.*

THEOREM 3.3. *Let X be κ-additive normal B_κ space and Y be κ-metrizable. Then $X \times Y$ is normal if it is B_κ.*

PROOF. If $X \times Y$ is B_κ then from 2.4 it follows that X is M_B. Now use ($\leftarrow$) of 3.1.

We have another analogue to Dowkers's theorem:

THEOREM 3.4. *If X is a κ-additive normal space, then the following are equivalent*:

(1) *X is B_κ.*

(2) *$X \times Y$ is normal for every κ-Lindelöf, κ-metrizable space Y.*

PROOF. If X is κ-additive and B_κ and Y is κ-metrizable κ-Lindelöf then 2.6 implies that X is M_B, where $\mathbf{B}$ is a basis of Y as above, which gives the normality of $X \times Y$. For the other direction, observe that $(\kappa+1)^-$ is a κ-metrizable κ-Lindelöf space. Now use 1.2 which states that the the normality of $X \times (\kappa+1)^-$ implies that X has the B_κ property.

THEOREM 3.5. *If X is κ-additive and Z is κ-Lindelöf, then $X \times Z$ is M_B whenever X is M_B.*

PROOF. We start by showing the claim: If $N \subset X \times Z$ is a neighborhood of $\{x\} \times Z$, then there exists $W \subset X$ a neighborhood of x such that $N \supset W \times Z$. We also have, the projection map $\pi : X \times Z \to X$ is closed. To see this, let us cover $\{x\} \times Z$ by basic elements $U \times V \subset N$. Since $\{x\} \times Z$ is $(\kappa, \lambda]$-Lindelöf for each λ, it can be covered by $< \kappa$ many such basic elements, say, $\{U_\alpha \times V_\alpha : \alpha < \kappa_1 < \kappa\}$. We assume that every $U_\alpha \times V_\alpha$ meets $\{x\} \times Z$. Let $W = \cap\{U_\alpha : \alpha < \kappa_1\}$, then W is open (since X is κ-additive) and $W \times Z \subset \cup\{U_\alpha \times V\alpha : \alpha < \kappa_1\} \subset N$.

Now let $F \subset X \times Z$ be closed and $x \notin \pi(F)$. As $\{x\} \times Z \subset (X \times Z) \backslash F$, it follows that x has a neighborhood W such that $(W \times Z) \cap F = \emptyset$. We then have $\pi(F) \cap W = \emptyset$ showing that $\pi(F)$ is closed.

To complete the proof of the theorem, let $\mathbf{B}$ be a basis for Y as above and let $\{F_B : B \in \mathbf{B}\}$ be a monotone family of closed subsets of $X \times Z$ with $\bigcap_{y \in B} F_B = \emptyset$ for each $y \in Y$. Define $\underline{F}_B = \pi(F_B)$, where $\pi : X \times Z \to X$ is the projection map, then $\underline{F}_B \subset \underline{F}_D$ if $B \subset D$. By the claim, $\underline{F}_B$ is closed for each $B \in \mathbf{B}$ and $\bigcap_{y \in B} \underline{F}_B = \emptyset$ for each $y \in Y$. Indeed, let $x \in X$ and fix $y \in Y$, then the family $\{(X \times Z) \backslash F_B : y \in B\}$ is an (increasing) open cover of the κ-Lindelöf space $\{x\} \times Z$. So there exists $B \in \mathbf{B}$ containing y such that $(\{x\} \times Z) \cap F_B = \emptyset$. Hence $x \notin \pi(F_B)$.

Using the M_B property of X, find a monotone open family $\{\underline{U}_B : B \in \mathbf{B}\}$ such that $\underline{F}_B \subset \underline{U}_B$ and $\bigcap_{y \in B} \operatorname{cl} \underline{U}_B = \emptyset$ for each $y \in Y$. Let $U_B = \pi^{\leftarrow} \underline{U}_B$, then $\{U_B : B \in \mathbf{B}\}$ is a monotone family of open subsets of $X \times Z$ with $U_B \supset F_B$ and $\bigcap_{y \in B} \operatorname{cl} U_B = \emptyset$ for each $y \in Y$: Observe that $U_B = \pi^{\leftarrow}(\underline{U}_B) \supset \pi^{\leftarrow}(\underline{F}_B) \supset F_B$. If $\langle x, y \rangle \in X \times Z$, find $B \in \mathbf{B}$ with $x \notin \operatorname{cl} \underline{U}_B$, then $\langle x, y \rangle \notin \operatorname{cl} \pi^{\leftarrow}(\underline{U}_B)$.

We obtain a generalization of Rudin-Starbird [5] theorem which states that

if Z is compact and Y is a metric with $X \times Y$ and $X \times Z$ being normal then $X \times Y \times Z$ is normal.

THEOREM 3.6. *Let Z be a κ-additive κ-Lindelöf space, X be κ-additive and Y be κ-metrizable. Then $X \times Y \times Z$ is normal if $X \times Y$ and $X \times Z$ are normal.*

PROOF. X is M_B since $X \times Y$ is normal. By 3.5 $X \times Z$ is M_B. Use ($\leftarrow$) of 3.1 to get the normality of $X \times Y \times Z$.

THEOREM 3.7. *Let X be κ-additive and Y be nondiscrete κ-metrizable. Then $X \times Y$ is B_κ if it is normal.*

PROOF. By 1.2, it suffices to show that if $X \times Y$ is normal then $X \times Y \times (\kappa+1)^-$ is normal. Since Y is not discrete then $X \times (\kappa+1)^-$ is homeomorphic to a closed subset of $X \times Y$ (Y has a closed subset homeomorphic to $(\kappa+1)^-$), hence it is normal. By 3.5, since $(\kappa+1)^-$ is κ-Lindelöf, $X \times (\kappa+1)^-$ is M_B. Now by 3.1 $X \times (\kappa+1)^- \times Y$ is normal.

We obtain a generalization of Rudin [5] which states that the product $X \times Y$ of a normal space X and a nondiscrete space Y is normal if and only if it is countably paracompact.

THEOREM 3.8. *Suppose X is a κ-additive normal space and Y is a nondiscrete κ-metrizable space. Then the following are equivalent*:

(1) *X is M_B.*
(2) *$X \times Y$ is B_κ.*
(3) *$X \times Y$ is normal.*

PROOF. Using 3.7 we get (3) $\rightarrow$ (2), 2.4 gives (2) $\rightarrow$ (1) and (1) $\rightarrow$ (3) follows from 3.1.

COROLLARY 3.9. *If Y is κ-lindelöf κ-metrizable and X is normal κ-additive B_κ, then $X \times Y^n$ is B_κ and normal for all $n < \omega$.*

PROOF. From 2.6 it follows that X is M_B. Use ($\leftarrow$) of 3.1 to get the normality of $X \times Y$ and then 3.8 to get the B_κ property on $X \times Y$. Now, we proceed by induction. Suppose that $X \times Y^j$ is B_κ for $j < \omega$. Use 2.6, 3.1 and 3.8 again to get the normality and the B_κ property of $X \times Y^{j+1}$.

In [5] it is shown that if Y is a nondiscrete metrizable space then $X \times Y$ is normal if and only if it is countably paracompact. On one hand in 2.4 we have, X is M_B if $X \times Y$ is B_κ, and in 1.4, X is B_κ if and only if $X \times (\kappa+1)^-$ is normal

B_κ. On the other hand 3.8 "extends" this to: For a normal κ-additive B_κ space X and a κ-metrizable Y, $X \times Y$ is normal if and only if it is B_κ. Can we extend 3.8 by dropping the κ-additivity on X for $\kappa > \omega_1$?

PROBLEM 1. *Let X be a normal B_κ space and Y be κ-metrizable. Suppose that $X \times Y$ has the B_κ property. Is $X \times Y$ normal?*

PROBLEM 2. *Let Y be κ-metrizable and X be a normal M_B space. Is $X \times Y$ normal?*

In 3.9 it is shown that $X \times Y^n$ is a B_κ normal space if X is a κ-additive normal B_κ space and Y is κ-metrizable κ-Lindelöf.

PROBLEM 3. *Suppose Y is κ-metrizable. κ-Lindelöf and X is a normal κ-additive B_κ space. Is $X \times Y^\omega$ normal? Is it B_κ?*

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