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A NOTE ON A COMMUTATIVITY OF AUTOMORPHISMS OF TOPOLOGICAL ALGEBRAS

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Abstract

In this note we prove that if A is a complete locally m-convex (lmc) algebra and α, β are continuous automorphisms of A satisfying the equation $\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x)$ for all $x \in A$, then α, β commute.

1. Introduction

During the last decade, a lot of work has been done on the operator equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1} \qquad (*)$$

where α and β are $*$-automorphisms of a von Neumann algebra. This operator equation in the commuting case (that is, when α, β commute) plays an important role in the Tomita–Takesaki theory [6, 7]. Recently, Brešar [2, 3] has obtained algebraic generalizations of some results of Batty [1] and Thaheem [12] concerning this operator equation in a general setting of semiprime rings. Among several other results, Thaheem [12] (see also Brešar [3]) has proved the following decomposition theorem:

THEOREM A. *Let M be a von Neumann algebra and α, β be $*$-automorphisms of M satisfying the equation $\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x)$ for all $x \in M$. If α, β commute then there is a central projection p in M such that $\alpha = \beta$ on Mp and $\alpha = \beta^{-1}$ on $M(1-p)$.*

Some situations have been identified where the operator equation $(*)$ itself implies the commutativity of α, β and consequently the additional assumption of commutativity of α, β may be dropped from Theorem A. For instance, it has been shown in [13] that if α, β are $*$-automorphisms of a C^*-algebra M satisfying the operator equation $(*)$ and if α (or β) is inner, then α, β commute (see also Brešar [2] for a similar result for semiprime rings).

Thaheem [12] proved the following commutativity theorem.

THEOREM B. *Let M be a commutative Banach algebra and α, β be automorphisms of M satisfying the equation $\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x)$ for all $x \in M$, then α, β commute.*

In view of considerable applications of (non-normed) topological algebras in other fields such as quantum mechanics (cf. Lassner [8]) it would be of interest to study the operator equation $(*)$ in a wider context of topological algebras. Our aim in this note is to extend Theorem B to a more general situation of topological algebras. We prove here the following theorem:

THEOREM 1. *Let A be a (unital) commutative semisimple complete lmc algebra and α, β be continuous automorphisms of A satisfying equation $\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x)$ for all $x \in A$, then α, β commute.*

Theorem 1 is proved in section 3 while section 2 contains some preliminaries required for the development of the proof of this result. For more information about the operator equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, we may refer to [11, 12] where further references can be found.

2. Preliminaries

By a topological algebra A we mean a linear associative algebra A over complex scalars possessing a multiplicative identity 1 which is also a Hausdorff topological vector space with jointly continuous multiplication. An m-calibration on a topological algebra A is a family $P = (p_\alpha)$ of seminorms on A determining its topology and satisfying $p_\alpha(xy) \le p_\alpha(x)p_\alpha(y)$, $p_\alpha(1) = 1$ for all α and all $x, y \in A$. A is called a locally multiplicatively convex (*lmc*) algebra if its topology is determined by an m-calibration P (see for instance [9, 10]). Let A be a topological algebra and A^* be the (topological) dual of A, the set of all continuous linear functionals on A. For a continuous automorphism α of A, we denote by α^* the (topological) adjoint of α on A^* defined by $\langle x, \alpha^*(x^*)\rangle = \langle \alpha(x), x^*\rangle = x^*(\alpha(x))$ for any $x^* \in A^*$. For some topological algebras where automorphisms are automatically continuous, we refer to Fragoulopoulou [4]. Let Δ be the set of all non-zero continuous multiplicative linear functionals on A. Then Δ endowed with relative topology from A^* is called the Gelfand space of A (see Mallios [9, p.139]). If A is commutative *lmc* algebra then Δ is nonempty and coincides with the set of all closed maximal ideals of A (see e.g. Fragoulopoulou [5, p.7]). We assume that Δ separates points on A. For each $x \in A$, we define $\widehat{x}$ (the Gelfand transform of x) by $\widehat{x}(f) = f(x)$, $f \in \Delta$. We denote by $\widehat{A}$, the set of all Gelfand transforms $\widehat{x}$, $x \in A$. By radical of A, we mean the set $\text{rad}(A) = \bigcap_{f\in\Delta} \ker(f)$, where $\ker(f)$ denotes the kernel of $f \in \Delta$. A is said to be semisimple if $\text{rad}(A) = \{0\}$. For other undefined notations and terminology used here, we refer to Mallios [9] and

Fragoulopoulou [4, 5].

3. A Commutativity Restult

Let A be a commutative semisimple *lmc* algebra and α, β be continuous automorphisms of A with α^* and β^* as their adjoints, respectively. It is easy to see that $\alpha^*(\Delta) = \Delta$, $\beta^*(\Delta) = \Delta$. Also $\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x)$ if and only if $\alpha^*(x^*) + \alpha^{*^{-1}}(x^*) = \beta^*(x^*) + \beta^{*^{-1}}(x^*)$ for all $x \in A$ and $x^* \in A^*$.

We now prove the following.

THEOREM 1. *Let A be a commutative semisimple complete lmc algebra and α, β be continuous automorphisms of A satisfying the equation $\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x)$ for all $x \in A$. Then α, β commute.*

PROOF. Put $G = \{x^* \in A^* : \alpha^*(x^*) = \beta^*(x^*)\}$ and $H = \{x^* \in A^* : \alpha^*(x^*) = \beta^{*^{-1}}(x^*)\}$. It is easy to see that G and H are both invariant under α^* and β^*. Put $L = G \cap \Delta$ and $K = H \cap \Delta$. Then L and K are also invariant under α^* and β^*. Also, $L \cup K \subseteq \Delta$. We show that $L \cup K = \Delta$. To prove this, let $f \in \Delta$. Then $\alpha^*(f) + \alpha^{*^{-1}}(f) = \beta^*(f) + \beta^{*^{-1}}(f)$ and for any $x \in A$, we have

$$\widehat{x}(\alpha^*(f)) + \widehat{x}(\alpha^{*^{-1}}(f)) = \widehat{x}(\beta^*(f)) + \widehat{x}(\beta^{*^{-1}}(f)) \tag{1}$$

If $f \notin L \cup K$, then

$$\left.\begin{array}{c} \alpha^*(f) \neq \beta^*(f) \\ \text{and} \quad \alpha^*(f) \neq \beta^{*^{-1}}(f) \quad \text{or} \quad \alpha^{*^{-1}}(f) \neq \beta^*(f) \end{array}\right\} \tag{2}$$

There are two possibilities for $\alpha^*(f)$ and $\alpha^{*^{-1}}(f)$, namely

$$\begin{array}{ll} \text{either} & \text{(a) } \alpha^*(f) = \alpha^{*^{-1}}(f) \\ \text{or} & \text{(b) } \alpha^*(f) \neq \alpha^{*^{-1}}(f) \end{array}$$

Since Δ separates points on A, therefore we can find some $\widehat{y} \in \widehat{A}$ $(y \in A)$ such that

$$\widehat{y}(\alpha^*(f)) = \widehat{y}(\alpha^{*^{-1}}(f)) \neq 0$$

and

$$\widehat{y}(\beta^*(f)) = \widehat{y}(\beta^{*^{-1}}(f)) = 0$$

This together with (2) contradicts (1). Similarly we can choose $\widehat{z} \in \widehat{A}$ $(z \in A)$ such that in case (b),

$$\widehat{z}(\alpha^*(f)) \neq 0 \quad \text{and} \quad \widehat{z}(\alpha^{*^{-1}}(f)) = \widehat{z}(\beta^*(f)) = \widehat{z}(\beta^{*^{-1}}(f)) = 0$$

Again this together with (2) contradicts (1). Therefore, in any case, we have $\alpha^*(f) = \beta^*(f)$ or $\alpha^*(f) = {\beta^*}^{-1}(f)$, a contradiction. Thus $f \in L \cup K$ and consequently we get that $L \cup K = \Delta$.

It is easy to see that α^* and β^* commute on Δ. Thus for any $f \in \Delta$, we have $f(\alpha\beta(x) - \beta\alpha(x)) = 0$ for all $x \in A$ and consequently $(\alpha\beta(x) - \beta\alpha(x)) \in \bigcap_{f\in\Delta} \ker(f) = \mathrm{rad}(A)$. That A is semisimple implies $\alpha\beta(x) = \beta\alpha(x)$ for all $x \in A$. This completes the proof.

We leave as a problem whether or not Theorem 1 holds for some sort of noncommutative topological algebras (e.g. p-commutative topological algebras).

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