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THE PURELY IMAGINARY POINT SPECTRUM OF THE LINEARIZED BOLTZMANN OPERATOR WITH A SINGULAR EXTERNAL POTENTIAL

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1. Introduction

The nonlinear Boltzmann equation with an external-force potential $\phi = \phi(x)$ describes the time evolution of rarefied gas acted upon by the external force $F = -\nabla\phi$. It has the form,

$$\partial f/\partial t + \Lambda f = Q(f,f), \tag{1.1}$$

where $f = f(t,x,\xi)$ denotes the unknown function. This unknown function represents the density of the gas particles at time $t \geqq 0$, at a point $x \in \Omega$, and with a velocity $\xi \in \mathbb{R}^3$. We denote by $\Omega \subseteq \mathbb{R}^3$ a domain in which the gas particles are confined. Further, Λ is defined as follows:

$$\Lambda \equiv \xi \cdot \nabla_x - \nabla_x\phi \cdot \nabla_\xi,$$

and $Q(\cdot,\cdot)$ is a bilinear form,

$$Q(g,h) \equiv (1/2)\int_{\xi'\in\mathbb{R}^3, s\in S^2} q(\theta, |\xi-\xi'|)$$
$$\times\{g(\eta)h(\eta') + g(\eta')h(\eta) - g(\xi)h(\xi') - g(\xi')h(\xi)\}d\xi' ds,$$

where g and h are functions of (t,x,ξ) and $\eta = \xi - ((\xi-\xi')\cdot s)s$, $\eta' = \xi' + ((\xi-\xi')\cdot s)s$ for each $s \in S^2$. We denote by S^2 the unit sphere whose center is the origin. The θ denotes the angle variable given by $\cos\theta = (\xi-\xi')\cdot s/|\xi-\xi'|$. The kernel $q(\theta, V)$ is a given function of $(\theta, V) \in [0,\pi] \times [0,+\infty)$ and assumed to be nonnegative. We impose the assumption of cut-off hard potentials in the sense of Grad (see [1–2]). Under this assumption, we can linearize (1.1) around the absolute Maxwellian state $M \equiv \exp(-E(x,\xi))$, where $E(x,\xi) \equiv \phi(x) + |\xi|^2/2$. Substituting $f = M + M^{1/2}u$ in (1.1), and dropping the nonlinear term, we obtain the linearized Boltzmann equation,

$$\partial u/\partial t = Bu, \tag{1.2}$$

where $B \equiv A + e^{-\phi(x)}K$ and $A \equiv -\Lambda + e^{-\phi(x)}(-\nu)$. The operator B is called *the linearized Boltzmann operator with the potential term.* $\nu = \nu(\xi)$ is a multiplication operator, and K is an integration operator with a symmetric kernel. These operators act only on ξ, and satisfy the following (see [1–2]):

LEMMA 1.1. (i) *There exists a positive constant* $c_{1.1}$ *such that* $0 < \nu(\xi) \leqq c_{1.1}(1 + |\xi|)$ *for each* $\xi \in \mathbb{R}^3$.

(ii) K *is a self-adjoint compact operator on* $L^2(\mathbb{R}_\xi{}^3)$.

(iii) $(-\nu + K)$ *is a self-adjoint nonpositive operator on* $L^2(\mathbb{R}_\xi{}^3)$.

(iv) *The point spectrum of* $(-\nu + K)$ *contains* 0, *and the null space is spanned by* $\xi_j \exp(-|\xi|^2/4)$, $j = 1, 2, 3$, $\exp(-|\xi|^2/4)$, *and* $|\xi|^2 \exp(-|\xi|^2/4)$, *where* ξ_j *is the j-th component of* ξ, *i.e.,* $\xi = (\xi_1, \xi_2, \xi_3)$.

It is important to investigate how solutions of (1.2) will decay (see [3, p. 768], [4, p. 241], and [5, p. 1827]). For this purpose we need to first inspect the purely imaginary point spectrum of B and the corresponding eigenspaces, because we can obtain estimates for the decaying of solutions of (1.2) only in function spaces perpendicular to the eigenspaces corresponding to the purely imaginary point spectrum of B (see [1–2]).

In [6] we have already studied the purely imaginary point spectrum of B and the corresponding eigenspaces when $\Omega = \mathbb{R}^3$. By making use of the result in [6], we have obtained decay estimates for solutions of (1.2) (cf. [3–4]). In the present paper, we will study the purely imaginary point spectrum of B when Ω is a domain whose boundary $\partial\Omega$ is sufficiently smooth and bounded. The boundary condition is the perfectly reflective boundary condition. We assume that the traces upon $\partial\Omega$ of functions contained in the domain of B are square-integrable with respect to the measure $|n(x) \cdot \xi| d\sigma_x d\xi$, where $n = n(x)$ denotes the outer unit normal of $\partial\Omega$ at x, and $d\sigma_x$ designates an infinitesimal surface element of $\partial\Omega$. We accept that $\phi = \phi(x)$ has a singularity at finite points of Ω. The main result is Theorems 4.1–2.

In [6], the eigenvalues of B and the corresponding eigenfunctions have only to satisfy the following:

$$\mu v = Bv. \tag{1.3}$$

In this paper, in the same way as [6], we obtain μ and v which satisfy (1.3), and moreover we need to examine whether v satisfies the perfectly reflective boundary condition or not. The forms of eigenfunctions of B are heavily restricted by this fact, and hence we have to perform more complicated calculations than those in [6].

However, because of this restriction, some eigenvalues of B in [6] are not eigenvalues of B in the present paper. As a result, the structure of the point spectrum is simplified; the purely imaginary point spectrum of B is only equal to $\{0\}$.

2. Preliminaries

If Ω is bounded, then we impose the following only:

ASSUMPTION 2.1. (i) *The boundary $\partial\Omega$ is sufficiently smooth.*

(ii) *There exists a finite sequence $\{x_j\}_{j=1,\dots,N} \subset \Omega$ such that $\phi = \phi(x)$ is sufficiently smooth and real-valued in $\Omega\backslash\{x_j\}_{j=1,\dots,N}$, and is continuous in $(\partial\Omega \cup \Omega)\backslash\{x_j\}_{j=1,\dots,N}$.*

(iii) *There exists a constant $c_{2.1}$ such that, for each $x, \phi(x) \geqq c_{2.1}$.*

If Ω is unbounded, then we impose the following in addition to the above assumption:

ASSUMPTION 2.2. *$\partial\Omega$ is bounded.*

REMARK 2.3. *From Assumptions* 2.1–2, *we see that* $\exp(-\phi(x)/2)$, $\phi(x)\exp(-\phi(x)/2)$, $|x|\exp(-\phi(x)/2) \in L^2(\partial\Omega_x)$.

We define $S_j \equiv \{(x,\xi) \in \partial\Omega \times \mathbb{R}^3; (-1)^j n(x)\cdot\xi < 0\}$, $j = 1,2$. We consider the operators in the complex Hilbert space $L^2(\Omega_x \times \mathbb{R}_\xi{}^3)$. By $L^2(S_j;\rho)$, we denote the space of square-integrable functions of $(x,\xi) \in S_j$ with respect to $\rho(x,\xi)d\sigma_x d\xi$, $j = 1,2$, where $\rho = \rho(x,\xi) \equiv |n(x)\cdot\xi|$.

By $\boldsymbol{D}(L)$ we denote the domain of an operator L. We define $\boldsymbol{D}(\Lambda) \equiv \{v = v(x,\xi) \in L^2(\Omega_x \times \mathbb{R}_\xi{}^3)$; $\Lambda v \in L^2(\Omega_x \times \mathbb{R}_\xi{}^3)$, and $v = v(x,\xi)$ satisfies the following boundary conditions:

$$\text{(SI)} \qquad (\gamma_j v(\cdot,\cdot))(x,\xi) \in L^2(S_j;\rho), \quad j = 1,2,$$

$$\text{(PRBC)} \qquad (\gamma_1 v(\cdot,\cdot))(x,\xi) = (\gamma_2 v(\cdot,\cdot))(x,\xi - 2(n(x)\cdot\xi)n(x)),$$

for each $(x,\xi) \in S_1\}$. γ_j, $j = 1,2$, denote the trace operators along the characteristic curves of Λ, which are defined by

$$dx/dt = \xi, \quad d\xi/dt = -\nabla\phi(x). \tag{2.1}$$

We similarly define $\boldsymbol{D}(A) \equiv \{v = v(x,\xi) \in L^2(\Omega_x \times \mathbb{R}_\xi{}^3)$; $Av \in L^2(\Omega_x \times \mathbb{R}_\xi{}^3)$, and $v = v(x,\xi)$ satisfies (SI) and (PRBC)$\}$. It follows from Assumption 2.1,(iii), and Lemma 1.1,(ii) that we can define $\boldsymbol{D}(B) \equiv \boldsymbol{D}(A)$.

By $a(\phi)$ ($a(\Omega)$, respectively) we denote the set of all axes of symmetry of $\phi = \phi(x)$ (Ω, respectively).

REMARK 2.4. *Performing calculations similar to those in obtaining the fact that if* $v, \xi \cdot \nabla_x v \in L^2(\Omega_x \times \mathbb{R}_\xi{}^3)$, *then* $v = v(x,\xi)$ *is absolutely continuous along the characteristic lines of* $\xi \cdot \nabla_x$, *we can deduce that if*

$$v, \Lambda v \in L^2(\Omega_x \times \mathbb{R}_\xi{}^3), \tag{2.2}$$

then $v = v(x,\xi)$ *is absolutely continuous along the characteristic curves of* Λ. *We can construct the trace operators* γ_j, $j = 1,2$. *In addition, combining* (SI) *and* (PRBC), *we deduce that if* $v \in \boldsymbol{D}(\Lambda)$, *then*

$$(v, \Lambda v) + (\Lambda v, v) = 0. \tag{2.3}$$

3. Necessary Conditions

LEMMA 3.1. *Suppose that* $v = v(x,\xi) \in \boldsymbol{D}(B)$ *is not identically equal to* 0, *and that* $\operatorname{Re}\mu \geqq 0$. *If* μ *and* v *satisfy* (1.3), *then*

$$\mu = 0, \tag{3.1}$$

$$\Lambda v = 0, \tag{3.2}$$

and v *has the form,*

$$v = \left(\sum_{j=1}^{3} a_j \xi_j + a_4 |\xi|^2 + a_5\right) \exp(-E(x,\xi)/2), \tag{3.3}$$

where $E(x,\xi) \equiv \phi(x) + |\xi|^2/2$. *The coefficients* $a_j = a_j(x)$, $j = 1, \ldots, 5$, *are complex-valued functions of* $x \in \Omega$ *which satisfy the following* (3.4–6):

$$a_j = \alpha_j + \sum_{k=1}^{3} \alpha_{jk} x_k, \quad j = 1,2,3, \tag{3.4}$$

(3.5) a_4 *is a complex constant,*

$$a_5 = 2a_4\phi(x) + \beta_0, \tag{3.6}$$

where β_0 *is a complex constant. The coefficients* α_j, α_{jk}, $j,k = 1,2,3$, *are complex constants which satisfy the following* (3.7–8):

$$\alpha_{jk} + \alpha_{kj} = 0, \quad j,k = 1,2,3. \tag{3.7}$$

(3.8) : *Define* $(\alpha, \beta) \equiv ((\operatorname{Re}\alpha_1, \operatorname{Re}\alpha_2, \operatorname{Re}\alpha_3), (\operatorname{Re}\alpha_{23}, \operatorname{Re}\alpha_{31}, \operatorname{Re}\alpha_{12})), ((\operatorname{Im}\alpha_1, \operatorname{Im}\alpha_2, \operatorname{Im}\alpha_3), (\operatorname{Im}\alpha_{23}, \operatorname{Im}\alpha_{31}, \operatorname{Im}\alpha_{12}))$. *If* $a(\phi) \cap a(\Omega)$ *is empty, then* $(\alpha, \beta) =$

$(0,0)$. If $\phi=\phi(x)$ and Ω have only one common axis of symmetry, i.e., if $a(\phi)\cap a(\Omega)=\{\ell\}$, then (α,β) satisfies $\beta//\ell$ and $\alpha=-\gamma\times\beta$ for each $\gamma\in\ell$. If both $\phi=\phi(x)$ and Ω are spherically symmetric with respect to $\gamma\in\mathbb{R}^3$, then (α,β) satisfies $\alpha=-\gamma\times\beta$.

REMARK 3.2. *From Assumptions 2.1–2, we easily see that if $a(\phi)\cap a(\Omega)$ is not empty, then only the following two cases exist: (1) $\phi=\phi(x)$ and Ω have only one common axis of symmetry. (2) $\phi=\phi(x)$ and Ω are spherically symmetric with respect to only one point.*

PROOF of LEMMA 3.1. Performing calculations similar to those in [6] with the aid of (2.3) and Lemma 1.1, we obtain (3.3), (3.5), and the following:

$$\mathrm{Re}\,\mu=0, \tag{3.9}$$

$$\mu v=-\Lambda v. \tag{3.10}$$

$$\mu a_5-\sum_{j=1}^{3}a_j\partial\phi/\partial x_j=0, \tag{3.11}$$

$$\mu a_j+\partial a_5/\partial x_j-2a_4\partial\phi/\partial x_j=0,\quad j=1,2,3, \tag{3.12}$$

$$\partial a_j/\partial x_k+\partial a_k/\partial x_j=0,\quad j\neq k,\quad j,k=1,2,3, \tag{3.13}$$

$$\mu a_4+\partial a_j/\partial x_j=0,\quad j=1,2,3, \tag{3.14}$$

where the derivatives are those in the sense of distribution. (3.5) and (3.9–14) are necessary conditions for (3.3) to satisfy (1.3).

By substituting (3.3) in (PRBC), we obtain the following necessary condition for (3.3) to satisfy (PRBC):

$$\nabla\psi\cdot a=0\ \text{ in }\ \partial\Omega, \tag{3.15}$$

where $a\equiv(a_j)_{j=1,2,3}$. $\psi=\psi(x)$ is a real-valued function of $x\in\mathbb{R}^3$ representing $\partial\Omega$ in such a way that $\partial\Omega=\{x\in\mathbb{R}^3;\psi(x)=0\}$.

Let us prove (3.1) by contradiction. Assume that $\mu\neq0$. (3.5) and (3.12) give $\partial a_j/\partial x_k-\partial a_k/\partial x_j=0$, $j,k=1,2,3$. It follows from these equalities and (3.13) that $\partial a_j/\partial x_k=0$, $j\neq k$, $j,k=1,2,3$. These equalities and (3.14) give

$$a_j=-\mu a_4x_j+\beta_j,\quad j=1,2,3, \tag{3.16}$$

where β_j, $j=1,2,3$, are complex constants. Substituting (3.16) in (3.15), we see that $\partial\Omega$ is an unbounded surface. This is contradictory to Assumptions 2.1–2.

Hence we obtain (3.1). (3.2) follows from (3.1) and (3.10). (3.4) and (3.6–7) can be obtained in the same way as [6].

Let us prove (3.8). Substituting (3.4) with (3.7) in (3.11) with $\mu = 0$ and in (3.15), and noting that $\phi = \phi(x)$ and $\psi = \psi(x)$ are real-valued, we conclude that $\phi = \phi(x)$ and $\psi = \psi(x)$ satisfy equations of the same form, $\nabla\phi \cdot (\alpha + x \times \beta) = 0$, $\nabla\psi \cdot (\alpha + x \times \beta) = 0$, where (α, β) is that in (3.8). Solving these equalities in the same way as [6], we can obtain (3.8).

4. The Main Theorem

We denote the point spectrum of B by σ_p.

THEOREM 4.1. *If Ω is bounded, then the following* (i–iv) *hold:*

(i) $\sigma_p \cap \{\mu \in \mathbb{C}; \operatorname{Re}\mu \geqq 0\} = \{0\}$.

(ii) *If $a(\phi) \cap a(\Omega)$ is empty, then the null space of B is spanned by*

$$e^{-E(x,\xi)/2}, E(x,\xi)e^{-E(x,\xi)/2}, \tag{4.1}$$

where $E(x,\xi) \equiv \phi(x) + |\xi|^2/2$.

(iii) *If $\phi = \phi(x)$ and Ω have only one common axis of symmetry, i.e., if $a(\phi) \cap a(\Omega) = \{\ell\}$, then the null space of B is spanned by*

$$e^{-E(x,\xi)/2}, E(x,\xi)e^{-E(x,\xi)/2}, ((x-\gamma) \times \xi)^{\ell} e^{-E(x,\xi)/2}, \quad \gamma \in \ell, \tag{4.2}$$

where we denote the projection of $(x - \gamma) \times \xi$ upon the line ℓ by $((x-\gamma) \times \xi)^{\ell}$.

(iv) *If $\partial\Omega$ is a sphere, and if $\phi = \phi(x)$ is spherically symmetric with respect to the center of $\partial\Omega$, then the null space of B is spanned by*

$$e^{-E(x,\xi)/2}, E(x,\xi)e^{-E(x,\xi)/2}, ((x-\gamma) \times \xi)_j e^{-E(x,\xi)/2}, \quad j = 1,2,3, \tag{4.3}$$

where $\gamma \in \mathbb{R}^3$ is the center of $\partial\Omega$. We denote the j-th component of $(x-\gamma) \times \xi$ by $((x-\gamma) \times \xi)_j$, $j = 1,2,3$.

PROOF. Write V as the set of all functions of the form (3.3) whose $a_j = a_j(x)$, $j = 1, \dots, 5$, satisfy (3.4–8). Making use of Lemma 3.1, we see that $\sigma_p \cap \{\mu \in \mathbb{C}; \operatorname{Re}\mu \geqq 0\} \subseteq \{0\}$ and that if $0 \in \sigma_p$, then the null space is contained in V. We easily see that $V \subseteq L^2(\Omega \times \mathbb{R}^3)$. Recalling Remark 2.3, we see that all elements of V satisfy (SI). Moreover, we easily deduce that if $v \in V$, then v satisfies (PRBC). By Lemma 1.1,(iv), we see that all elements of V satisfy (1.3) with $\mu = 0$. Hence, we deduce that $0 \in \sigma_p$ and that V is contained in the null space of B. It follows from (3.4–8) that if ϕ and Ω satisfy the conditions of (ii–iv) of the present theorem respectively, then V is spanned by (4.1–3) respectively. Hence we obtain the theorem.

THEOREM 4.2. *If Ω is unbounded, and if*

$$e^{-\phi(x)/2}, \phi(x)e^{-\phi(x)/2}, |x|e^{-\phi(x)/2} \in L^2(\Omega_x), \tag{4.4}$$

then the following (i–iv) *hold:*

(i) $\sigma_p \cap \{\mu \in \mathbb{C}; \operatorname{Re}\mu \geqq 0\} = \{0\}$.

(ii) *If $a(\phi) \cap a(\Omega)$ is empty, then the null space of B is spanned by* (4.1).

(iii) *If $\phi = \phi(x)$ and Ω have only one common axis of symmetry, i.e., if $a(\phi) \cap a(\Omega) = \{\ell\}$, then the null space of B is spanned by* (4.2).

(iv) *If $\partial\Omega$ is a sphere, and if $\phi = \phi(x)$ is spherically symmetric with respect to the center of $\partial\Omega$, then the null space of B is spanned by* (4.3).

PROOF. Making use of Lemma 3.1, we see that $\sigma_p \cap \{\mu \in \mathbb{C}; \operatorname{Re}\mu \geqq 0\} \subseteq \{0\}$ and that if $0 \in \sigma_p$, then the null space is contained in V. It follows from (4.4) that $V \subseteq L^2(\Omega \times \mathbb{R}^3)$. Recalling Remark 2.3, we see that all elements of V satisfy (SI). Moreover, we easily deduce that if $v \in V$, then v satisfies (PRBC) and (1.3) with $\mu = 0$. Hence, we deduce that $0 \in \sigma_p$ and that V is contained in the null space of B. In the same way as Theorem 4.1, we obtain the theorem.

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