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BOOK PRESENTATION OF THE COMPLETE n -PARTITE GRAPHS

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1. Introduction

Kobayashi defined the book presentation of spatial graphs, in [K]. Then in [K2], he introduced the concept of complete H -path book presentability as an intrinsic property of graphs with book presentation.

On the other hand, Böhme [B] has defined some spatial graphs that are strongly discatenated graphs. He, also, defined the concept of intrinsic property of graphs, that is, strong discatenability.

Kobayashi compared the two intrinsic properties, strong discatenability with complete H -path book presentability, in [K2]. He obtained the following Propositions 1.1 and 1.2.

PROPOSITION 1.1 ([K2]). *For a pseudo Hamilton graph G if G is strongly discatenable, then G is completely H -path book presentable.* $\square$

PROPOSITION 1.2 ([K2]). *Let G be an H -path extendable pseudo Hamilton graph. If G is completely H -path book presentable, then G is strongly discatenable.* $\square$

These propositions do not imply the equivalence of strong discatenability and complete H -path book presentability. But if we assume the graphs to be pseudo Hamilton and H -path extendable, then the two concepts are equivalent.

In section 3, we will show that the condition H -path extendability is fairly essential for the concept, complete H -path book presentability. If we remove the condition H -path extendability, there exists a counter example to the converse of Proposition 1.1.

Clearly, the complete graph on n vertices K_n is H -path extendable. But the complete n -partite graph $K_{m_1, m_2, \dots, m_n}$ is not always H -path extendable. In section 4, we will decide the complete n -partite graph which is H -path extendable as follows.

THEOREM 1.3. *Let G be a complete n -partite graph, that is $G = K_{m_1, m_2, \dots, m_n}$ where $m_1 \geq m_2 \geq \dots \geq m_n$, and let G be a pseudo Hamilton graph. Then G is H -path extendable if and only if m_i ($1 \leq i \leq n$) satisfies one of the following three conditions (1), (2) and (3).*

- (1) $n = 2$, $0 \leq m_1 - m_2 \leq 1$ and $m_1 \geq 3$
- (2) $n = 3$, $m_1 = m_2 \geq 3$ and $m_3 = 1$
- (3) $m_i = 2$ or 1

We checked the two concepts, strong discatenability and complete H -path book presentability for the complete n -partite graph $K_{m_1, m_2, \dots, m_n}$. Then we obtained the following theorem.

THEOREM 1.4. *Let G be a complete n -partite graph, that is $G = K_{m_1, m_2, \dots, m_n}$ where $m_1 \geq m_2 \geq \dots \geq m_n$, and a pseudo Hamilton graph. Then G is strongly discatenable if and only if G is completely H -path book presentable.*

So we conclude that, H -path extendability is not best possible to distinguish the two concepts.

In the last section, we introduce five transformations of graphs. We examine whether the graphs which are transformed by them, keep the properties of the pseudo Hamilton graphs discussed above.

2. Preliminaries

Throughout this paper, all graphs are assumed to be finite, simple and connected. Let G be a graph. We denote by $V(G)$ the set of vertices of G , and by $E(G)$ the set of edges of G .

We denote K_n the complete graph on n vertices, and $K_{m_1, m_2, \dots, m_n}$ the complete n -partite graph where $m_1 \geq m_2 \geq \dots \geq m_n$.

If H is a subgraph of G with $V(H) = V(G)$, and H has no isolated vertices, then H is called a spanning subgraph of G .

If a subgraph of the graph G are homeomorphic to a circle or a simple arc, we call it a cycle or a path of the graph G respectively. A k -cycle is a cycle with exactly k vertices.

A spanning cycle, and a spanning path of a graph G are called a Hamilton cycle, and a Hamilton path denoted by Δ_G , respectively. We call a graph containing a Hamilton path a pseudo Hamilton graph.

Let f be a tame embedding of a graph G into the 3-space R^3 . We call the image $f(G) = G_f$, usually written as $\tilde{G}$, a spatial graph or a spatial representation of G . We say that two spatial graphs G_f and G_g are ambient isotopic, denoted by $G_f \approx G_g$, if and only if there exists an orientation preserving homeomorphism of R^3 that maps G_f to G_g .

Now we introduce some definitions for the graphs on their spatial representations.

DEFINITION 2.1 ([K]). Let G be a pseudo Hamilton graph, and $B_p = \Xi \bigcup_{i=1}^p P_i$ a book with the binder Ξ and p sheets P_i ($i = 1, 2, \dots, p$). If an embedding $\psi : G \rightarrow B_p$ satisfies the following conditions (1), (2) and (3), then we call ψ (or $\psi(G)$) a book presentation with respect to a Hamilton path Δ_G of G (briefly B.P.H. Δ_G).

- (1) $\psi(\Delta_G) \subset \Xi$
- (2) for any edge e in $E(G) - E(\Delta_G)$, there is a sheet P_i with $\psi(\text{Int}(e)) \subset P_i$
- (3) for any sheet P_i , there is at least one edge $e \in E(G) - E(\Delta_G)$ with $\psi(\text{Int}(e)) \subset P_i$

DEFINITION 2.2 ([K2]). Let G be a pseudo Hamilton graph. G is completely H -path book presentable if and only if there is a spatial graph $\tilde{G}$ of G such that for any Δ_G , there is a B.P.H. $\Delta_G \psi : G \rightarrow B_p$ with $\psi(G) \approx \tilde{G}$ (ambient isotopic).

DEFINITION 2.3 ([B]). Let G be a graph. G is strongly discatenable if and only if there is a spatial graph $\tilde{G}$ such that for any simple closed curve $\tilde{C}$ on $\tilde{G}$, there is a disk $\tilde{D}^2$ in R^3 such that $\tilde{D}^2 \cap \tilde{G} = \partial \tilde{D}^2 \cap \tilde{G} = \partial \tilde{D}^2 = \tilde{C}$.

The spatial graph $\tilde{G}$ as above is called a strongly discatenated graph of G .

DEFINITION 2.4. G is said to be self-linked if and only if for any spatial graph $\tilde{G}$, there exists a non-trivial link in $\tilde{G}$.

We need the next two definitions for abstract graphs.

DEFINITION 2.5 ([K2]). Let G be a pseudo Hamilton graph. If for any path d in G , there is a Hamilton path Δ_G of G containing d , then we call G an H -path extendable graph.

DEFINITION 2.6. A graph H is a minor of a graph G , denoted by $H \preceq G$, if and only if there exists a subgraph G' of G and its edges $e_1, e_2, \dots, e_n$ such that H is isomorphic to the graph which is obtained from G' by a sequence of edge contractions along $e_1, e_2, \dots, e_n$.

3. Some properties of spatial graphs

We defined some concepts on spatial embeddings of graphs in the previous section.

Here we mention some facts about these concepts.

PROPOSITION 3.1. *Let G be a graph, and H a minor of G . If H is self-linked, then G is also self-linked.* $\square$

The following theorem is proved by Sachs [S], Conway and Gordon [C-G].

THEOREM 3.2 ([C-G, S]). *The following eight graphs K_6 , $K_{3,3,1}$, G_7 , G_8 , G_9 , $K_{4,4}$, $K_{4,4} - e$ and PG illustrated in Fig. 1 [K-S] are self-linked, where $K_{4,4} - e$ is a graph obtained from $K_{4,4}$ by removing an edge e , PG is Petersen graph.* $\square$

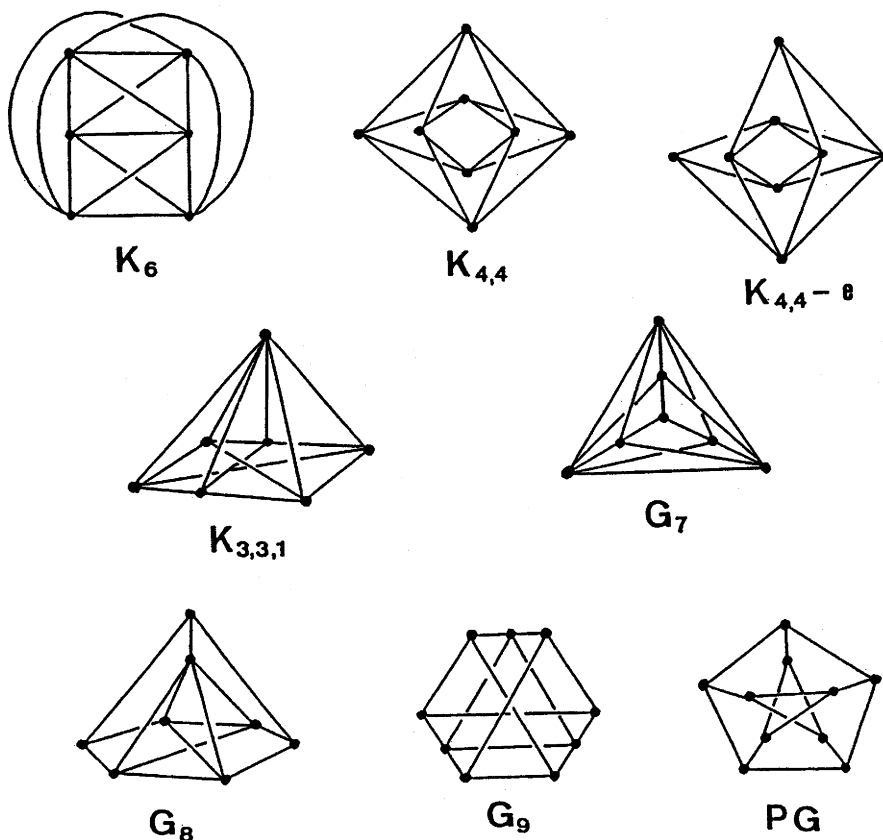


Fig. 1.

It is easy to see the following Proposition.

PROPOSITION 3.3. *Let G be a graph. If G is a self-linked graph, then G is not strongly discatenable.* $\square$

EXAMPLE. The complete graph on n vertices K_n is a Hamilton graph and H -path extendable.

- If $n \geq 6$, K_n is neither strongly discatenable nor completely H -path book presentable.
- If $n \leq 5$, K_n is strongly discatenable and completely H -path book presentable.

PROPOSITION 3.4. *The converse to Proposition 1.1 does not hold.*

PROOF. If G is H -path extendable, then the converse to Proposition 1.1 holds. Clearly, the following example G_1 (Fig. 2) is not H -path extendable.

Since G_1 has K_6 as a minor graph, G_1 is not strongly discatenable. Next we will show that G_1 is completely H -path book presentable.

We list the Hamilton paths Δ_{G_1} . Considering their symmetry, we obtain the Hamilton paths of three types $\Delta_{G_1}^1$, $\Delta_{G_1}^2$ and $\Delta_{G_1}^3$ in Fig. 3.

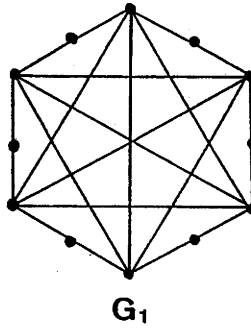


Fig. 2.

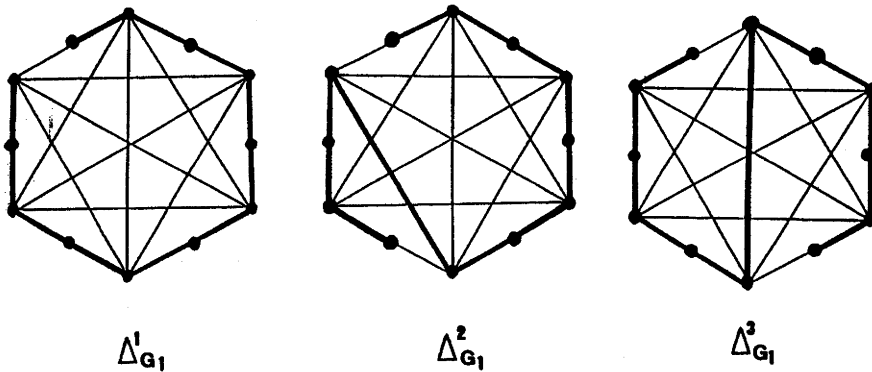


Fig. 3.

Here we take an embedding of the graph $G_1, \tilde{G}_1$ as follows (Fig. 4).

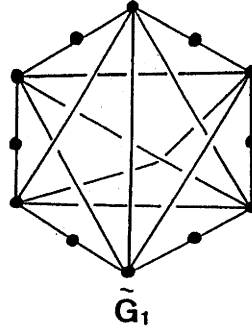


Fig. 4.

By the symmetry of the spatial graph $\tilde{G}_1$, it is easy to check if there exists the book presentations with respect to Hamilton path $\Delta_{G_1}^i, \psi_i : G_1 \rightarrow B_{p_i}$ such that $\psi_i(G_1)$ is ambient isotopic to $\tilde{G}_1$ ($\psi_i(G_1) \approx \tilde{G}_1, 1 \leq i \leq 3$). Then we see that $\tilde{G}_1$ has book presentations with respect to each Hamilton path of G_1 (Fig. 5). $\square$

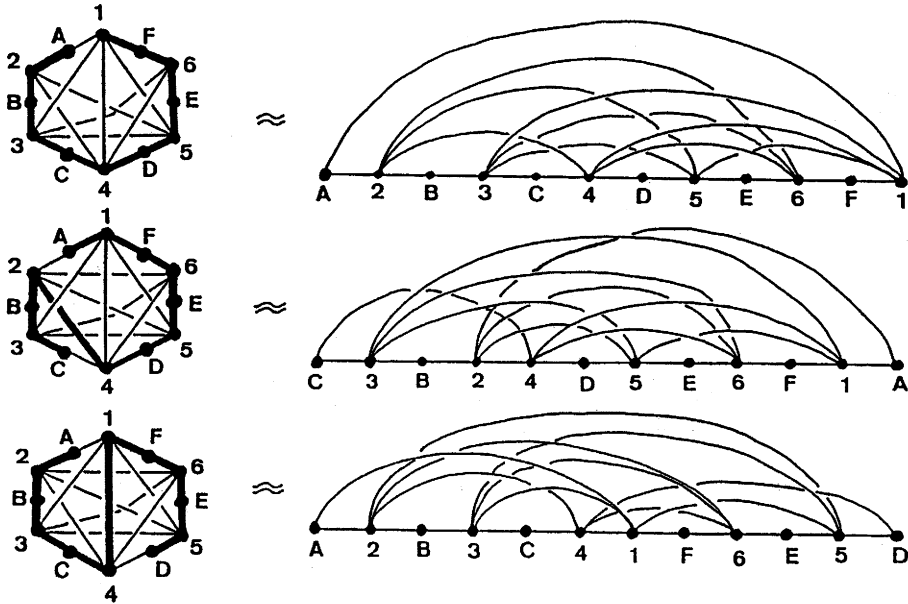


Fig. 5.

By the way, in the graph G_1 , there are six vertices which have valencies two. The vertex with valency two, is not essential topologically, but we can make another graph G_2 whose vertices have valencies at least three. The graph is drawn in Fig. 6 and Fig. 7 where for each tangle a, b, c, d, e and f in Fig. 6, we attach the tangle in Fig. 7.

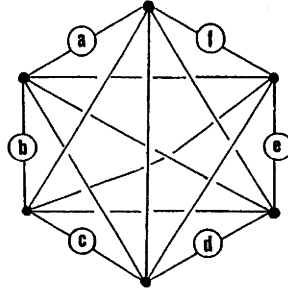


Fig. 6.

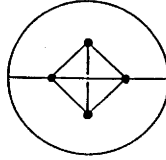


Fig. 7.

Similarly to the case of G_1 , we can show that this graph G_2 is also completely H -path book presentable.

4. Pseudo Hamilton graphs of $K_{m_1, m_2, \dots, m_n}$

In this section, we discuss the Hamilton path of complete n -partite graph $K_{m_1, m_2, \dots, m_n}$.

Let $G = K_{m_1, m_2, \dots, m_n}$ be a complete n -partite graph where $m_1 \geq m_2 \geq \dots \geq m_n$, and let V_i ($1 \leq i \leq n$) be a subset of vertices of G and the number of vertices of V_i denoted by $|V_i| = m_i$.

Now we remember a theorem on complete n -partite graph by Toba.

THEOREM 4.1 ([T2]). *Let $G = K_{m_1, m_2, \dots, m_n}$ be a complete n -partite graph where $m_1 \geq m_2 \geq \dots \geq m_n$. G is a pseudo Hamilton graph if and only if m_i*

$(1 \leq i \leq n)$ satisfies

$$m_1 \leq \sum_{i=2}^n m_i + 1.$$

□

To prove Theorem 1.4, we need the following lemma.

LEMMA 4.2. *Let $G = K_{m_1, m_2, \dots, m_n}$ be an n -partite graph and $m_1 \geq 3$. Let v_1, v_2 and v_3 be three vertices of V_1 , and put $V'_1 = V_1 - \{v_1, v_2, v_3\}$. Let $G' = K_{m_1-3, m_2, \dots, m_n}$. If G' is a pseudo Hamilton graph, then G is not H -path extendable.*

PROOF. Let v_4 and v_5 denote the endpoints of a Hamilton path $\Delta_{G'}$ of G' .

Case I. $v_4, v_5 \in V'_1$

There is no edge which connects one of $\{v_4, v_5\}$ with one of $\{v_1, v_2, v_3\}$.

Hence there is no Hamilton path Δ_G that contains Hamilton path $\Delta_{G'}$.

Case II. $v_4, v_5 \notin V'_1$

There is no edge which connects $\{v_i\}$ ($1 \leq i \leq 3$) with each other.

Hence there is no Hamilton path Δ_G that contains Hamilton path $\Delta_{G'}$.

Case III. $v_4 \in V'_1, v_5 \notin V'_1$.

There is no edge which connects the vertices $\{v_j\}$ ($1 \leq j \leq 4$).

Hence there is no Hamilton path Δ_G there contains Hamilton path $\Delta_{G'}$.

□

PROOF OF THEOREM 1.3.

From Theorem 4.1, m_i ($1 \leq i \leq n$) satisfies the condition $m_1 \leq \sum_{i=2}^n m_i + 1$ (*). To use Lemma 4.2, we consider two cases separated by the conditions of m_1 .

Case I. $m_1 \geq 4$

(1) $m_1 - 3 \geq m_2$

By Lemma 4.2, it is enough to consider only the case that $G' = K_{m_1-3, m_2, \dots, m_n}$ is not a pseudo Hamilton graph. Then m_i ($1 \leq i \leq n$) satisfies the following condition, $m_1 \geq \sum_{i=2}^n m_i + 5$ (**). But there is no m_i ($1 \leq i \leq n$) which satisfies both (*) and (**).

(2) $m_1 - 2 \leq m_2$

Similarly we consider only the case that m_i ($1 \leq i \leq n$) satisfies $m_2 \geq m_1 + \sum_{i=3}^n m_i - 1$.

By the relation $m_1 \geq m_2 \geq m_1 + \sum_{i=3}^n m_i - 1$, we obtain $1 \geq \sum_{i=3}^n m_i$.

Now we list the possible cases α , β and γ as follows.

$$\alpha: m_1 = m_2 \geq 4, m_3 = 1, m_i = 0 \quad (i \geq 4)$$

$$\beta: m_1 = m_2 \geq 4, m_j = 0 \quad (j \geq 3)$$

$$\gamma: m_1 = m_2 + 1 \geq 4, m_k = 0 \quad (k \geq 3)$$

In fact, it is easy to see that the graph G satisfying α , β or γ is H -path extendable.

Case II. $m_1 \leq 3$

(1) $m_1 = 3$

By a similar argument to Case I, we consider only the case that m_i satisfies $m_2 \geq \sum_{i=3}^n m_i + 2$. There are possible cases α and β .

$$\alpha: m_2 = 3, m_3 = 1 \text{ or } 0, \text{ and } m_4 = 0$$

$$\beta: m_2 = 2, m_3 = 0$$

For both cases, G is H -path extendable.

(2) $m_1 = 2$

In this case, the types of the graph G are α or β as follows.

$$\alpha. \quad G = \underbrace{K_{2,2,\dots,2}}_m \quad (1 \leq m \leq n)$$

That is, $G = K_m + K'_m - \{\bigcup_{i=1}^m v_i v'_i\}$ where K_m and K'_m are both the complete graphs on m vertices, $v_i \in V(K_m)$, $v'_i \in V(K'_m)$, $v_i, v'_i \in E(G)$, and $+$ denotes the join of graphs. So G is H -path extendable.

$$\beta. \quad G = \underbrace{K_{2,2,\dots,2}}_k \underbrace{1,1,\dots,1}_l \quad (1 \leq k+l \leq n)$$

That is, $G = K_k + K'_k + K_l - \{\bigcup_{i=1}^k v_i v'_i\}$ where K_k , K'_k and K_l are the complete graphs on k or l vertices, $v_i \in V(K_k)$, $v'_i \in V(K'_k)$, $v_i v'_i \in E(G)$ respectively. So G is H -path extendable.

(3) $m_1 = 1$

In this case, $G = K_p$ ($1 \leq p \leq n$). It is known that G is H -path extendable. $\square$

5. Proof of Theorem 1.4

Let L denote one of the complete graph on six vertices K_6 , the complete 3-partite graph $K_{3,3,1}$, and $K_{4,4} - e$, a subgraph of the complete bipartite graph.

Let $C_L = (C_i, C_j)_L$ be a pair of an i -cycle and a j -cycle of L . Let $V(C_i)$ be the set of vertices of an i -cycle C_i .

If C_k and the graph L are the subgraphs of the graph G , for an embedding $f : G \rightarrow R^3$, we denote the images of G , C_k and L , by $f(G) = \tilde{G}$, $f(C_k) = \tilde{C}_k$ and $f(L) = \tilde{L}$.

In this section, we prove Theorem 1.4 by using Lemma 5.1 and 5.2.

LEMMA 5.1. *For each graph L , and for every spatial graph $\tilde{L}$, there exists a Hamilton path Δ_L^* such that there is no book presentation with respect to the Hamilton path $\Delta_L^* \psi : L \rightarrow B_p$ which is ambient isotopic with $\tilde{L}$ ($\psi(L) \approx \tilde{L}$).*

PROOF. First we discuss the type of disjoint cycles in the graph L . There exist the pairs of disjoint cycles in L whose length types are as follows.

- There exists only the type of (3-cycle, 3-cycle) in K_6 , we denote them by $(C_{3_1}, C_{3_2})_{K_6}$.
- There exists only the type of (3-cycle, 4-cycle) in $K_{3,3,1}$, we denote them by $(C_{3_1}, C_{4_1})_{K_{3,3,1}}$.
- There exists only the type of (4-cycle, 4-cycle) in $K_{4,4} - e$, denoted by $(C_{4_1}, C_{4_2})_{K_{4,4}-e}$.

By the proof of Theorem 3.2, we know that, there is a pair of disjoint cycles C^*_L of L which satisfies the condition (A).

$$lk(\tilde{C}_L^*) \neq 0 \quad (\text{A})$$

In the case of $L = K_6$, for any pair of disjoint cycles $C_{K_6} = (C_{3_1}, C_{3_2})_{K_6}$, there exists a Hamilton path Δ_{K_6} satisfying the condition (B).

$$\Delta_{K_6} = u_1 u_2 u_3 v_1 v_2 v_3 \text{ where } u_k \in V(C_{3_1}), v_l \in V(C_{3_2}), 1 \leq k, l \leq 3 \quad (\text{B})$$

(B) means the condition of the Hamilton path Δ_{K_6} such that the three vertices in each cycle C_{3_1} and C_{3_2} mass together in the Hamilton path Δ_{K_6} .

It is easy to see that, there exists a Hamilton path $\Delta_{K_6}^*$ that satisfies the condition (B) for the pair of disjoint cycles $C^*_{K_6}$.

Now, we suppose that there exists a book presentation with respect to the Hamilton path $\Delta_{K_6}^*$. Then there exists $\tilde{D}_i^2$ ($i = 1, 2$) such that $\tilde{C}_{3_1} \cap \tilde{D}_1 = \partial \tilde{D}_1 = \tilde{C}_{3_1}$, $\tilde{C}_{3_2} \cap \tilde{D}_2 = \partial \tilde{D}_2 = \tilde{C}_{3_2}$ and $\tilde{C}_{3_1} \cap \tilde{D}_2 = \tilde{C}_{3_2} \cap \tilde{D}_1 = \emptyset$.

This contradicts to the condition (A). So we conclude Lemma 5.1, for the case that L is K_6 .

Similarly we can prove the cases, $L = K_{3,3,1}$ or $L = K_{4,4} - e$. $\square$

REMARK. On a pair of disjoint cycles C_{3_1}, C_{4_2} of $K_{3,3,1}$, the set of vertices $V(C_{3_1})$ is $\{a_{i1}, b_{j1}, c\}$ and $V(C_{4_2})$ is $\{a_{i2}, a_{i3}, b_{j2}, b_{j3}\}$ where $a_{ik} \in V_1$, $b_{jl} \in V_2$, $c \in V_3$, $1 \leq k, l \leq 3$.

On a pair of disjoint cycles C_{4_1}, C_{4_2} of $K_{4,4} - e$, the set of vertices $V(C_{4_1})$ is $\{a_{i1}, a_{i2}, b_{j1}, b_{j2}\}$, and $V(C_{4_2})$ is $\{a_{i3}, a_{i4}, b_{j3}, b_{j4}\}$ where $a_{im} \in V_1, b_{jn} \in V_2, 1 \leq m, n \leq 4$.

LEMMA 5.2. *Let L be one of three graphs $K_6, K_{3,3,1}$ or $K_{4,4} - e$. Let $G = K_{m_1, m_2, \dots, m_n}$ and $G' = K_{m'_1, m'_2, \dots, m'_n}$ be complete n -partite graphs with $m_1 \geq m_2 \geq \dots \geq m_n, m'_1 \geq m'_2 \geq \dots \geq m'_n$. Suppose that G contains L as a minor graph and G' is obtained from G by deleting the vertices of L . If there exists a Hamilton path of G' , the graph G is not completely H -path book presentable.*

From the definition of the complete n -partite graph, we can write $G = V_1 + V_2 + \dots + V_n, G' = V'_1 + V'_2 + \dots + V'_n$, where each V_i and V'_i are subsets of $V(G)$ and $V(G')$ respectively. Let $V_i = \{v_{ij}\}$ and $V'_i = \{v'_{ik}\}$ where $1 \leq i \leq n, 1 \leq j \leq m_i$ and $1 \leq k \leq m'_i$.

PROOF OF LEMMA 5.2.

We can easily see that each endpoint of the Hamilton path of the graph L lie in the distinct sets of vertices $\{V_i\}$ in G .

If there is a Hamilton path $\Delta_{G'}$ of G' , there is at least one edge which connects $\Delta'_{G'}$ and the Hamilton path Δ_L of L , for every Δ_L , because the each endpoint of L lie in the distinct sets of vertices $\{V_i\}$ in G . Hence there exists a Hamilton path Δ_G of G such that Δ_L is a minor graph of Δ_G ($\Delta_G \succeq \Delta_L$) for every Δ_L .

Here, we assume that G is completely H -path book presentable, then there exists a spatial graph $\tilde{L}$ of L such that for any Hamilton path Δ_L of L , there is a $B.P.H.\Delta_L\psi : L \rightarrow B_p$ which is ambient isotopic with $\tilde{L}$ ($\psi(L) \approx \tilde{L}$).

But this contradicts to Lemma 5.1. Hence G is not completely H -path book presentable. $\square$

PROOF OF THEOREM 1.4.

We discuss the pseudo Hamilton graph G . By Theorem 4.1, it is necessary that all m_i ($1 \leq i \leq n$) satisfy the inequality $m_1 \leq \sum_{i=2}^n m_i + 1$.

From Propositions 1.1 and 1.2, we know that Theorem 1.4 is true for the graph G which is H -path extendable. We examine the graph G of which m_i ($1 \leq i \leq n$) do not satisfy any of the conditions of Theorem 1.3.

In fact, the spatial graphs of six graphs $K_{4,1,1,1}, K_{4,2,1}, K_{3,1,1,1}, K_{3,2,2}, K_{3,2,1}$ and $K_{3,1,1}$ in Fig. 8 are strongly discatenated graphs. So we conclude that these graphs are strongly discatenable.

Clearly, if the graph G contains one of the three graphs $K_6, K_{3,3,1}$ or $K_{4,4} - e$, denoted by L as a minor graph, then G is not strongly discatenable. We shall check whether or not the graph G contains at least one of the three graphs denoted by L .

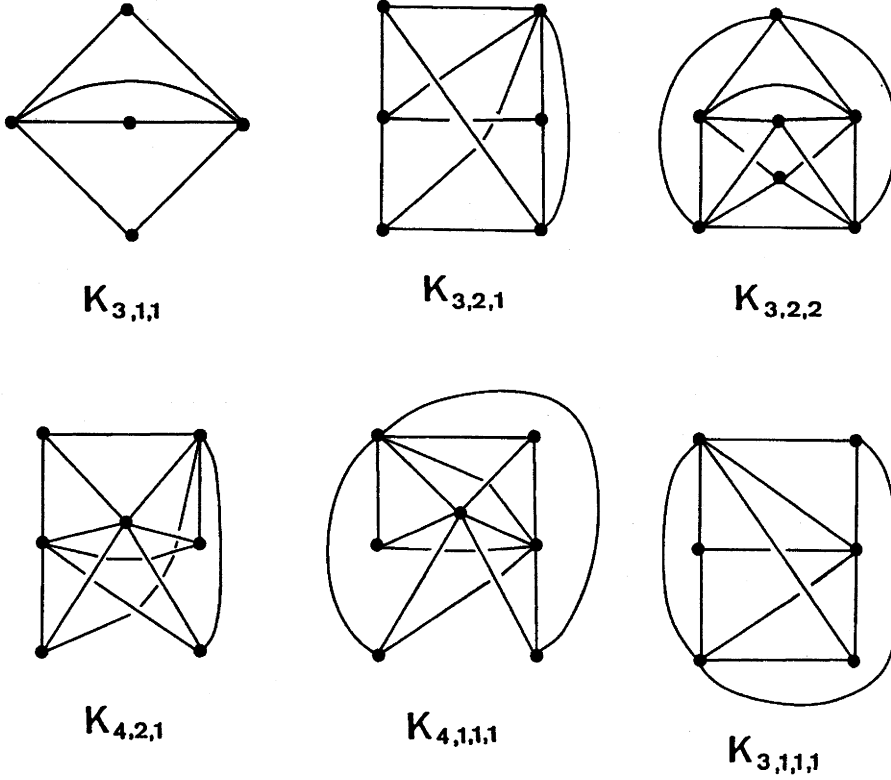


Fig. 8.

$K_{3,3,3}$ contains $K_{3,3,1}$ as a minor graph. For $K_{3,3,1}$, there is only one type of pairs of disjoint cycles $\{(3\text{-cycle} = C_3, 4\text{-cycle} = C_4)\}$. For all $C_{K_{3,3,1}} = (C_3, C_4)$ of $K_{3,3,1}$, there exists a Hamilton path $\Delta_{K_{3,3,1}}$ such that

$$\Delta_{K_{3,3,1}} = v_1 v_2 v_3 u_1 u_2 u_3 u_4, \text{ where } v_i \in V(C_3), u_j \in V(C_4), \\ 1 \leq i \leq 3, 1 \leq j \leq 4 (*).$$

For all spatial graphs of $K_{3,3,1}$, there exists $C^*_{K_{3,3,1}}$ such that $lk(\tilde{C}^*_{K_{3,3,1}}) \neq 0$, and a Hamilton path $\Delta^*_{K_{3,3,1}}$ which satisfies the condition (*).

We can assume $V(K_{3,3,1}) = \{v_{ij}, v_{31}\}$ ($i = 1$ or $2, 1 \leq j \leq 3$), $\{v_{31}\} = V_3$, and $v_1, u_4 \notin V_3$, without loss of generality. Then there exists a Hamilton path of $K_{3,3,3}$ such that there is no book presentation with respect to the Hamilton path $\Delta^*_{K_{3,3,3}} \psi : K_{3,3,3} \rightarrow B_p$ which is ambient isotopic with $\tilde{K}_{3,3,3}$, for any spatial graphs $\tilde{K}_{3,3,3}$.

To prove Theorem 1.4, we show that there exists a Hamilton path of G' which satisfy the condition of Lemma 5.2 for each G , excluding the seven graphs,

$K_{4,1,1,1}$, $K_{4,2,1}$, $K_{3,1,1,1}$, $K_{3,2,2}$, $K_{3,2,1}$, $K_{3,1,1}$ and $K_{3,3,3}$.

Case I. $n \geq 6$

- (1) If G satisfies $n = 6$ and $m_1 \leq \sum_{i=2}^6 m_i - 3$, or $n \geq 7$ and $m_1 \leq \sum_{i=2}^n m_i - 5$, we can regard L as K_6 where $V(L)$ is $\{v_{i1}\}$, $(n - 5 \leq i \leq n)$. Then we have $m'_i = m_i - 1$, $m'_j = m_j$ ($j \neq i$), $m_1 \geq m'_k$ for all integer k where $1 \leq k \leq n$, and $m'_1 \leq \sum_{i=2}^n m'_i + 1$. Hence there is a Hamilton path $\Delta_{G'}$ of G' .
- (2) If G satisfies the following conditions (α) or (β) , we can regard L as $K_{4,4} - e$.

$$(\alpha) \quad n \geq 6 \text{ and } \sum_{i=2}^n m_i - 2 \leq m_1 \leq \sum_{i=2}^n m_i + 1.$$

We have the inequalities $\sum_{i=2}^n m_i \geq 5$ and $m_1 \geq 3$.

- When $m_1 = 3$, then we see $G = K_{3,1,1,1,1,1}$, and L is a spanning subgraph of G .
- When $m_1 \geq 4$, we can regard $V(L) = \{v_{1i}, v_{21}, v_{31}, v_{41}, v_{51}\}$ ($1 \leq i \leq 4$). If G also satisfies the inequality $\sum_{i=2}^n m_i - 1 \leq m_1 \leq \sum_{i=2}^n m_i + 1$, clearly there exists a Hamilton path $\Delta_{G'}$ of G' which passes alternately the vertices of V'_1 and the others.

If $m_1 = \sum_{i=2}^n m_i - 2$, we have a Hamilton path of $G' \Delta_{G'} = v_* v_{61} v_{15} v_* \cdots v_{1m_1} v_*$ ($v_* = v_{22}$ ($n = 6$), $v_* = v_{71}$ ($n \geq 7$)), where $v_* \notin V'_1$ and after the vertex v_{15} , this path passes alternately the vertices of V'_1 and the others.

$$(\beta) \quad n \geq 7 \text{ and } \sum_{i=2}^n m_i - 4 \leq m_1 \leq \sum_{i=2}^n m_i - 3.$$

We have the inequalities $\sum_{i=2}^n m_i \geq 6$. When $m_1 = 2$, L is a spanning subgraph of G . When $m_1 \geq 3$, we can regard $V(L) = \{v_{1i}, v_{j1}\}$ where $1 \leq i \leq 3$, $2 \leq j \leq 6$.

Then we have a Hamilton path of $G' \Delta_{G'} = v_* v_{71} v_{14} v_* \cdots v_{1m_1} v_*$ ($v_* = v_{22}, v_{81}$ or ϕ) where $v_* \notin V'_1$ and after the vertex v_{14} , this path passes alternately the vertices of V'_1 and the others.

Case II. $n \leq 5$

- (1) $m_2 \geq 4$

We can regard L as $K_{4,4} - e$, and we can take the set of vertices of L as follows.

(α) If G satisfies $m_3 \leq m_1 - 4$, we regard $V(L) = \{v_{1i}, v_{2j}\}$ ($1 \leq i, j \leq 4$). Then we have $m'_k = m_k - 4$ ($k = 1, 2$), $m'_l = m_l$ ($3 \leq l \leq 5$).

(β) If G satisfies $m_1 \geq m_3 \geq m_1 - 3$, we have three cases as follows.

- G satisfies $m_1 = m_3$.

When $m_1 - 2 \geq m_4$, we regard the vertices $V(L) = \{v_{1i}, v_{2i}, v_{3j}\}$ ($i = 1, 2, 1 \leq j \leq 4$). Then we have $m'_k = m_1 - 2$ ($k = 1, 2$), $m'_3 = m_1 - 4$, $m'_k = m_k$ ($k = 4, 5$).

When $m_1 - 1 \leq m_4$, we regard the set of vertices $V(L) = \{v_{2i}, v_{3j}, v_{41}\}$ ($1 \leq i \leq 4, 1 \leq j \leq 3$). Then we have $m'_k = m_k$ ($k = 1, 5$), $m'_2 = m_1 - 4$,

$$m'_3 = m_1 - 3, m'_4 = m_4 - 1.$$

- G satisfies (a) $m_3 = m_1 - 1$, or (b) $m_3 = m_1 - 2$ and $m_4 \geq 1$.

We may regard the set of vertices $V(L) = \{v_{11}, v_{2i}, v_{3j}\}$ or $\{v_{km}, v_{2i}\}$ where i, j, k, m satisfies the conditions ($1 \leq i \leq 4, 1 \leq j \leq 3, k = 1, 3, m = 1, 2$).

- G satisfies (a) $m_3 = m_1 - 2$ and $m_4 = 0$ or (b) $m_3 = m_1 - 3$.

We may regard the set of vertices $V(L) = \{v_{1i}, v_{2j}, v_{31}\}$ ($1 \leq i \leq 3, 1 \leq j \leq 4$).

Then we have the inequalities $m'_1 \geq m'_l$ ($2 \leq l \leq 5$), and $m'_1 \leq \sum_{i=2}^5 m'_i + 1$. Therefore G' is a pseudo Hamilton graph.

(2) $m_2 \leq 3$

(α) It is clear for the case of $m_1 \leq 2$.

(β) If G satisfies $m_1 \geq 7$, then we have $m_1 - 4 \geq m_2$. We can regard L as $K_{4,4} - e$ where the only four elements of $V(L)$ are contained in V_1 . Then we have the inequalities $m'_1 \geq m'_i$ ($2 \leq i \leq 5$), and $m'_1 \leq \sum_{i=2}^5 m'_i + 1$. Hence G' is a pseudo Hamilton graph.

(γ) If G satisfies $3 \leq m_1 \leq 6$, we have two cases as follows.

- $m_4 \geq 2$

We can regard L as $K_{4,4} - e$ where $V(L) = \{v_{ij}\}, \{v_{1i}, v_{2j}, v_{31}, v_{41}\}, \{v_{1i}, v_{11}\}$ or $\{v_{1h}, v_{2j}, v_{n1}\}$ ($1 \leq i \leq 4, j = 1$ or $2, 2 \leq l \leq 5, 1 \leq h \leq 3$ and $3 \leq n \leq 5$). Then we have the inequalities $m'_k \leq 2$ ($1 \leq k \leq 5$), and $m'_1 \leq \sum_{i=2}^5 m'_i + 1$. Hence G' is a pseudo Hamilton graph.

- $m_4 \leq 1$

(a) If G satisfies $\sum_{i=2}^5 m_i \geq 4$ and $4 \leq m_1 \leq 6$. We can regard L as $K_{4,4} - e$, and the set of vertices $V(L)$ contains v_{1i} and v_{j1} ($1 \leq i \leq 4, j = 2$ or 3).

Then we have the inequality $m'_l \leq 2$ ($1 \leq l \leq 5$), and we see G' is a pseudo Hamilton graph.

(b) If G satisfies $\sum_{i=2}^5 m_i \geq 4$, and $m_1 = 3$. If G is neither $K_{3,2,2}$ nor $K_{3,3,3}$, we can regard L as $K_{3,3,1}$, and the set of vertices $V(L)$ contains v_{1i} and v_{41} ($1 \leq i \leq 3$) or v_{ji} and v_{31} ($j = 1, 2$). It is easy to see that there is a Hamilton path $\Delta_{G'}$ of G' .

(c) If G satisfies $\sum_{i=2}^5 m_i \leq 3$, there are exactly five graphs $K_{4,1,1,1}, K_{4,2,1}, K_{3,1,1,1}, K_{3,2,1}, K_{3,1,1}$. We know these graphs are strongly disconnected.

Therefore, this completes the proof of Theorem 1.4 $\square$

By Theorem 1.4, we classify the complete n -partite graphs $G = K_{m_1, \dots, m_n}$ ($m_1 \geq m_2 \geq \dots \geq m_n$) which are pseudo Hamilton graphs, in Table A.

Table A.

	G is strongly discatenable and completely H -path book presentable.	G is neither strongly discatenable nor completely H -path book presentable.
I.	$K_1, K_{1,1}, K_{1,1,1}$ $K_{1,1,1,1}, K_{1,1,1,1,1}$ $K_2, K_{2,2}, K_{2,2,2}$ $K_{2,1}, K_{2,1,1}, K_{2,1,1,1}$ $K_{2,1,1,1,1}, K_{2,2,1}$ $K_{2,2,1,1}, K_{2,2,2,1}$ $K_{3,3}, K_{3,2}, K_{4,3}$	$\underbrace{K_{1,1,\dots,1}}_n = K_n \quad (n \geq 6)$ $\underbrace{K_{2,1,\dots,1}}_n \quad (n \geq 6)$ $\underbrace{K_{2,2,1,\dots,1}}_n \quad (n \geq 5)$ $\underbrace{K_{2,2,2,1,\dots,1}}_n \quad (n \geq 5)$ $\underbrace{K_{2,2,2,2,1,\dots,1}}_n \quad (n \geq 5)$ $\underbrace{K_{2,2,\dots,2}}_n \quad (n \geq 4)$ $K_{m,m} \quad (m \geq 4)$ $K_{m,m-1} \quad (m \geq 5)$ $K_{m,m,1} \quad (m \geq 3)$
II.	$K_{3,1,1}, K_{3,2,1}, K_{3,2,2}$ $K_{3,1,1,1}$ $K_{4,2,1}, K_{4,1,1,1}$	The remainder graphs which satisfy (*) $(*) \quad m_1 \leq \sum_{i=2}^n m_i + 1$

I. G is H -path extendable.II. G is not H -path extendable.

6. Five relations and transformations of graphs

At first, we define five transformations and relations between two graphs ([B]). The transformations denoted by T_i induce the relations between the two graphs G and H denoted by R_i as follows, where $0 \leq i \leq 4$ and both G and H are simple and pseudo Hamilton graphs.

- (0) GR_0H, HR'_0G if and only if H is obtained from G by deleting an edge or an isolated vertex of G .
- (1) GR_1H, HR'_1G if and only if H is obtained from G by contracting an edge e of G where e has an endvertex of valency at most two.
- (2) GR_2H, HR'_2G if and only if H is obtained from G by contracting an edge e of G where both endvertices of e have valencies at least three.
- (3) GR_3H, HR'_3G if and only if G has a vertex P with exactly three neighbours Q_1, Q_2 and Q_3 which are pairwise non-adjacent, and H is obtained from G by deleting P and adding the edges $\overline{Q_1Q_2}, \overline{Q_2Q_3}$ and $\overline{Q_3Q_1}$.
The operation transforming G into H is called a star-triangle-transformation.
(Sometimes it is called $Y\Delta$ transformation, and the transformation of R'_3 is called ΔY transformation.)
- (4) GR_4H, HR'_4G if and only if G has two adjacent vertices P_1 and P_2 of valency three without common neighbours and the neighbours of P_1 and P_2 are pairwise non-adjacent, and H is obtained from G by subdividing the edge $\overline{P_1P_2}$ by a new vertex and then applying the star-triangle-transformation to both P_1 and P_2 .

DEFINITION 6.1. Let F denote the family of graphs satisfying the property $P(F)$. The family F (or the property $P(F)$) is said to be closed under the transformations $\{T_i\}$ or $\{T'_i\}$ if and only if for all $G \in F$, if GR_iH then $H \in F$ or for all $H \in F$, if HR'_iG then $G \in F$ respectively.

Böhme has obtained the following propositions for the transformations T_i and T'_i , where $0 \leq i \leq 4$, about the strong discatenability.

PROPOSITION 6.2 ([B]). Let F be the family of graphs satisfying the property: strong discatenability. F is closed under the transformations $T_0, T'_0, T_1, T'_1, T_2, T_3, T'_3$ and T_4 . $\square$

Then, we studied the families of graphs which are satisfying the properties H -path extendability and completely H -path book presentability.

PROPOSITION 6.3. Let F be a family of graphs which are pseudo Hamilton graphs and satisfying the property: H -path extendability. Then F is closed under the transformations T'_0 and T_1 , and F is not closed under the transformations $T_0, T'_1, T_2, T'_2, T_3, T'_3, T_4$ and T'_4 .

We need the fundamental lemma as follows.

LEMMA 6.4. Let G be a pseudo Hamilton, H -path extendable graph, where G do not have free vertices. If G has a vertex with valency two, then G is a cycle graph, or one of the four graphs in Fig. 9 (a), (b), (c) and (d).

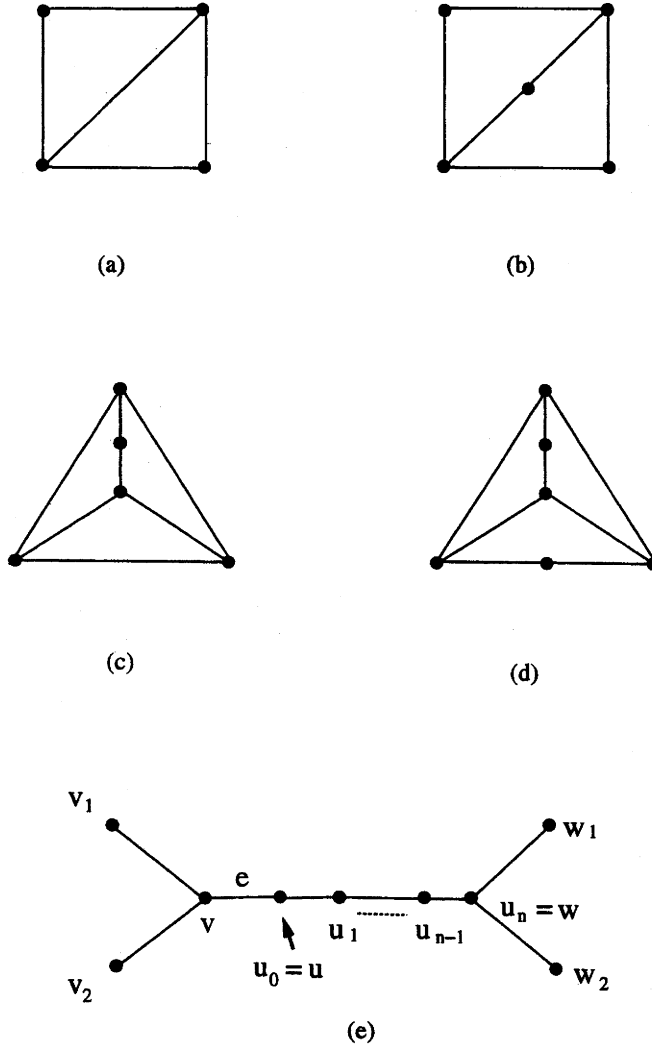


Fig. 9.

PROOF. If all vertices of G have valencies two, then G is a cycle graph.

If G has a cut vertex or a cut edge, then G is not H -path extendable. We may assume that G does not contain cut vertices and cut edges, and contains an edge e whose endpoints are u and v such that u has valency two, and v has valency at least three. If v is only one vertex with valency more than two in G , then G is homeomorphic to a bouquet and v is a cut vertex. So we may assume that G contains a vertex w which satisfies the conditions $(\star)$, as follows (Fig. 9 (e)).

(*) There exists a sequence of vertices $\{u_i\}$ ($0 \leq i \leq n$, $1 \leq n$), $u_0, u_1, u_2, \dots, u_{n-1}, u_n$ ($u_0 = u$, $u_n = w$), such that the vertex u_i has valency two for each i ($1 \leq i \leq n-1$) and u_n has valency more than two, and $u_j u_{j+1} \in E(G)$ ($0 \leq j \leq n-1$).

Since the edge e is not a cut edge, there exist at least two paths l_1 and l_2 with the endvertices v and w . G is a simple graph and v and w have valencies more than two, then G contains the vertices $v_i \in N(v)$ and $w_j \in N(w)$ such that $v_i, w_j \neq u$ ($1 \leq i, j \leq 2$), where $N(x)$ is a set of vertices which are neighbours of the vertex x . Let l_1, l_2 be paths which contain the vertices u and $\{v_1, w_1\}$ respectively. We have two cases (1) and (2) as follows.

(1) Two vertices v and w are adjacent.

Let $v = w_1$, and $w = v_1$. If (i) $v_2 \neq w_2$, or (ii) $v_2 = w_2$, and, v or w have valencies more than three, there is no Hamilton path that contains the paths $v_2 v w w_2$, $v_3 v w w_2$, and $w_3 w v v_2$ respectively, where $v_3 \in N(v)$, $w_3 \in N(w)$. So G is not H -path extendable, hence $v_2 = w_2$ and v and w have not valencies not more than three.

If u and w are non-adjacent, there is no Hamilton path that contains the path $u v w u_{n-1}$ and G is not H -path extendable. So u and w are adjacent.

If $v_2 = w_2$ has valency more than two, then v and w have valencies more than three, so $v_2 = w_2$ has valency two.

Hence we have an H -path extendable graph G in Fig. 9 (a).

(2) Two vertices v and w are non-adjacent.

- If v and w do not have same neighbour, there is no Hamilton path in G which contains a path $v_2 v v_1 \dots w_1 w w_2$ where $v v_1 \dots w_1 w = l_2$. So G is not H -path extendable.
- If v and w have same neighbour, let $v_1 = w_1$.

By the same arguments as the case (1), replacing the edge vw by a path $vv_1(=w_1)w$, we conclude that u and w are adjacent, and $v_2 = w_2$, and v and w have valencies not more than three.

If the vertex $v_1(=w_1)$ has valency more than three, there are at least two paths, say l_3 and l_4 , which do not contain the vertices v and w because the vertices v and w have valencies not more than three. We may assume that l_3 is not an edge, then there is no Hamilton path that contains a path $vv_1 \dots v_2 w u$ where $v_1 \dots v_2 = l_4$ and G is not H -path extendable. Hence the vertex $v_1(=w_1)$ does not have valency more than three.

Simiraly $v_2(=w_2)$ does not have valency more than three.

When v_1 and v_2 have valencies two, we have an H -path extendable graph in Fig. 9 (b).

When v_1 has valency three, v_2 has also valency three because v_1 is not a cut vertex.

Let the path $v_1 \cdots v_2 = l_5$, where l_5 does not contain both v and w . If l_5 contains two edges v_1a and bv_2 where $a \neq b$, and a and b are vertices on l_5 , there is no Hamilton path which contains the path av_1wv_2b . It follows that l_5 is an edge v_1v_2 , or l_5 consists of two edges v_1a and av_2 of which a has valency two.

Hence we have two H -path extendable graphs in Fig. 9 (c) and (d) as required. This completes the proof of Lemma 6.4 $\square$

PROOF OF PROPOSITION 6.3.

It is clear on the transformation T'_0 .

For the case of T_1 , it is easy to see by Lemma 6.4.

For the transformations T_0 and T'_1 , let G be $K_{3,2}$ and G_1 (Fig. 2) respectively. For T_2 and T_3 , let G be $K_{4,3}$. For T'_2 , let H be $K_{5,4}$. We know the graphs $K_{3,2}$,

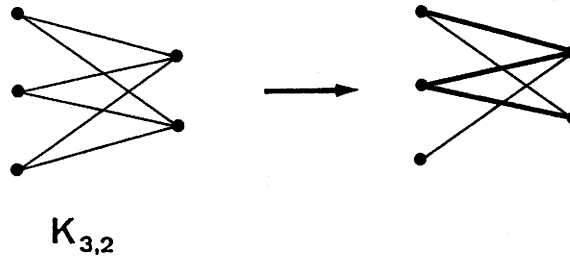


Fig. 10.

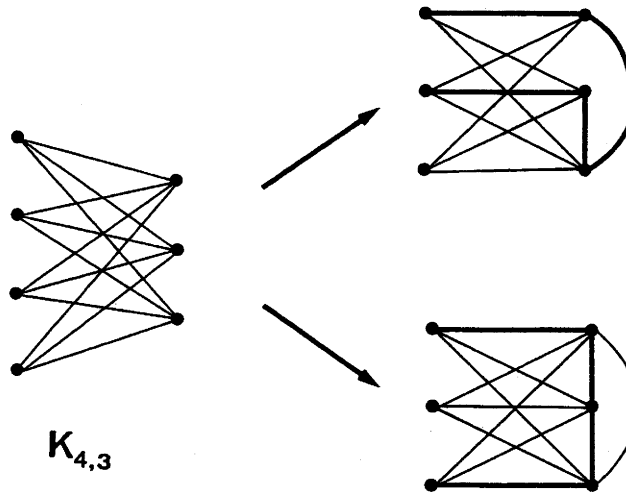


Fig. 11.

K_6 , $K_{4,3}$ and $K_{5,4}$ are H -path extendable, but the graphs which obtained under the transformations are not H -path extendable (Fig. 10, 11, 12). For T'_3 , let H be $K_{3,3,1}$, then the graph G_8 is obtained by the transformation. From Theorem 1.3, G is H -path extendable. But we see that H is not H -path extendable (Fig. 13). Hence we conclude that F is not closed under the transformation T'_3 .

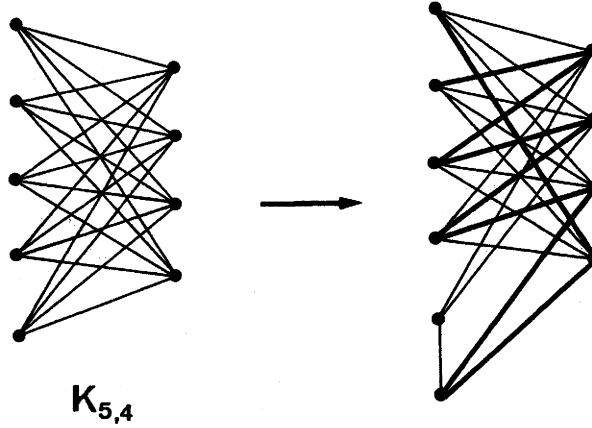


Fig. 12.

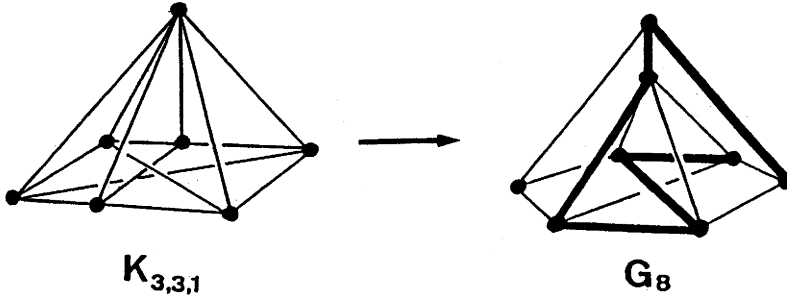


Fig. 13.

The transformation T_4 is the composition of T'_1 and T_3 , and T'_4 is the composition of T'_3 and T_1 . So F is not closed under T_4 and T'_4 . $\square$

Next we discuss the property: completely H -path book presentable.

PROPOSITION 6.5. *Let F be a family of the graphs which are pseudo Hamilton and completely H -path book presentable. Then F is closed under the transformations T_0 and T'_1 , and F is not closed under the transformations T'_0 , T_1 , T_2 , T'_2 and T'_4 .*

PROOF. The case of T_0 is trivial.

We show the cases of T'_0 , T_2 and T'_2 by examples. (Fig. 14 and 15). Easily we see that the graphs $K_{4,4} - e$ and $K_{4,4}$ are obtained by the sequence of transformations from H_i ($1 \leq i \leq 3$), and each H_i is strongly discatenable. Then the graphs H_i are completely H -path book presentable.

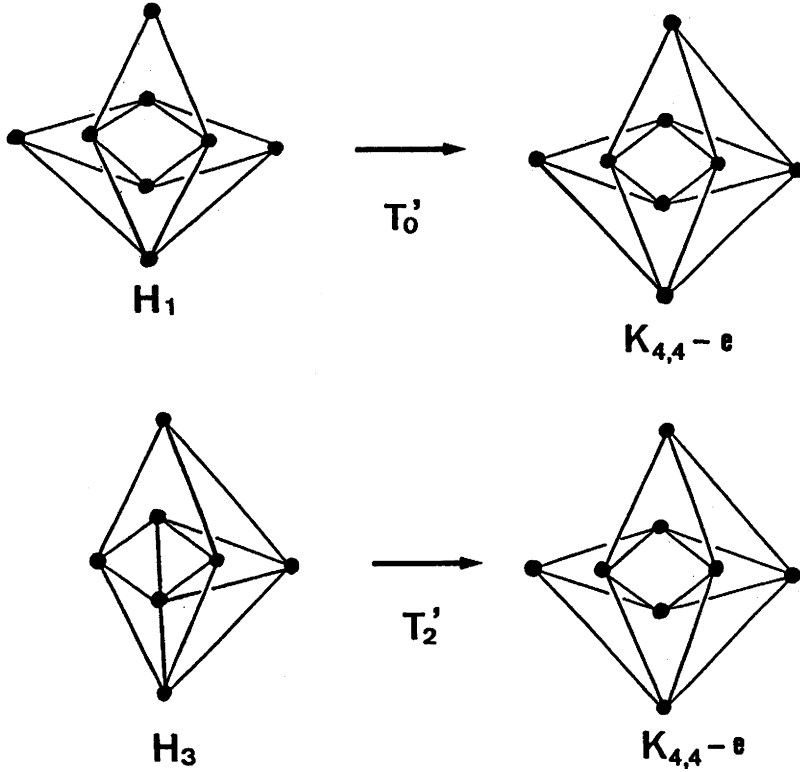


Fig. 14.

For T'_1 , the number of the Hamilton paths of G is less than or equal to the number of the Hamilton paths of H . Hence, if H is completely H -path book presentable, then G is completely H -path book presentable. Then F is closed under the transformation T'_1 .

For T_1 , we remember the graph G_1 in Fig. 2. We know the graph G_1 is completely H -path book presentable, and the complete graph on six vertices K_6 is not completely H -path book presentable. The difference between G_1 and K_6 is the six vertices with valency two. Here, we contract these six vertices of G_1 , one after another, then we obtain K_6 at last. Hence F is not closed under the transformation T_1 , and so F is not closed under the transformation T'_4 . $\square$

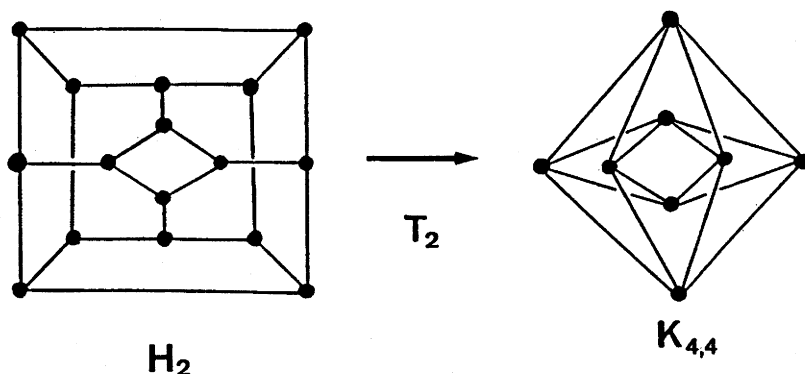


Fig. 15.

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