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Murakami, Hitoshi

Ohtsuki, Tomotada

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KOBE UNIVERSITY

MURAKAMI, H. and OHTSUKI, T.
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QUANTUM $Sp(n)$ INVARIANT OF LINKS VIA AN INVARIANT OF COLORED PLANAR GRAPHS*

By Hitoshi MURAKAMI and Tomotada OHTSUKI

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For each simple Lie group G, there exists the quantum G invariant of colored links; we mean by color of a link an assignment of representations of G to components of the link. See, for example, [12, 13]. In the present paper, we construct quantum $Sp(n)$ invariant of links with a single color corresponding to the vector representation of $Sp(n)$ in a combinatorial way, using invariants of colored planar graphs. A similar approach was discussed for the quantum $SU(n)$ invariant in a previous paper (joint with S. Yamada) [11].

A rigorous construction of the quantum $Sp(n)$ invariant was originally obtained as follows. We attach a representation of the quantum group $U_q(sp_n)$ to each component of a link, and we fix a link diagram of the link. We associate a crossing of the diagram to an intertwiner of $R_1 \otimes R_2$ to $R_2 \otimes R_1$, where R_1 and R_2 are representations attached to the two components meeting the crossing. By composing the all intertwiners and by taking traces where strings are closed, we obtain a value as the result; it is the quantum $Sp(n)$ invariant of the link. In this paper, we consider a link attached the representation of $U_q(sp_n)$ obtained by deforming the vector representation of $Sp(n)$, see Remark 1.1 for a relation between the above definition and our construction of the invariant in this paper.

The above definition of quantum invariants has various backgrounds such as conformal field theory, and the backgrounds guarantee the existence of the invariants. However, it might be difficult to give a direct elementary proof of the existence of the invariants because of complexity of calculations along the backgrounds (i.e. to give a direct proof of the isotopy invariance of the value) and to realize the existence of the invariants clearly in a combinatorial viewpoint.

On the other hand, it is well known that the quantum $Sp(n)$ invariant of links is equal (up to normalization) to Kauffman polynomial [4], which is characterized by a skein relation, i.e., we have an alternative definition of the quantum $Sp(n)$ invariant $P^{Sp(n)}(L) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of an oriented link L by the following three

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formulas

$$(0.1)\quad \begin{cases} P^{Sp(n)}(L) = (-q^{n+1/2})^{-w(D)}\langle D\rangle^{Sp(n)}, \\ \langle D_+\rangle^{Sp(n)} - \langle D_-\rangle^{Sp(n)} = (q^{1/2} - q^{-1/2})(\langle D_0\rangle^{Sp(n)} - \langle D_\infty\rangle^{Sp(n)}), \\ \langle O\rangle^{Sp(n)} = -[2n+1] + 1, \end{cases}$$

where q is an indeterminate, we put $[m] = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$, O is the trivial link diagram of the trivial knot, D is an unoriented link diagram of L (forgetting the orientation of L), $w(D)$ is a writhe of D (before forgetting the orientation), and D_+, D_-, D_0 and D_∞ are four diagrams as shown in Figure 1; the four diagrams are identical in the exterior of the parts shown in the figure. Further, we suppose that $\langle D\rangle^{Sp(n)}$ is a regular isotopy invariant. Then, the value of $P^{Sp(n)}(L)$ is uniquely determined by this characterization.

By the above characterization, we can define the quantum $Sp(n)$ invariant of links in an elementary and combinatorial way. However, if we calculate the value of the quantum $Sp(n)$ invariant of links by the above characterization, it is not easy to show the uniqueness of the value (i.e. the fact that the value of the invariant is well defined, not depending on a choice of the procedure of calculating the value of $P^{Sp(n)}(L)$), since the definition is, indeed elementary, but not constructive.

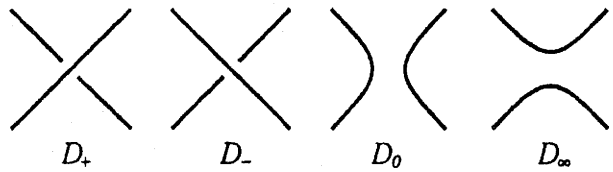


Fig. 1. Four link diagrams for the skein relation

Our aim in the present paper is to give an elementary and constructive definition of the quantum $Sp(n)$ invariant of links, and to give a direct proof of the isotopy invariance of it depending only on our definition. Our approach is also motivated by the success of the linear skein theory for the quantum $SU(2)$ invariant [8, 9]; the linear skein theory is also developed for rank 2 Lie algebras in [6, 7]. See also [10] for quantum $Sp(2)$ invariants of 3-manifolds via linear skein theory.

In Section 1, we give invariants of colored planar graphs, and prove properties of the invariants of graphs in Section 2. In Section 3, we define an invariant of links, prove the isotopy invariance of it, and show that it coincides with the quantum $Sp(n)$ invariant of links.

1. Invariant of graphs

In this section we define an invariant of colored unoriented planar trivalent graphs.

Throughout this paper, we fix an integer $n \geq 2$, and put $\mathcal{N} = \{-n, -n+1, \ldots, -1, 1, 2, \ldots, n\}$. Let q be an indeterminate and we put $[m] = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$.

Let G be an unoriented planar trivalent graph, each of whose edges has one of three colors; we express the colors by single, double or triple lines in pictures. At each vertex, the number of lines of one of the three incident edges must be the sum of numbers of lines of the other two edges. Then our graph can be divided into pieces depicted in Figure 2, after deforming it by isotopy of the plane if necessary. Alternatively, we can characterize the graphs we consider by the property that they can be obtained by gluing pieces in Figure 2. Further we suppose that our graph has no double line such that there are triple lines connected to the double line at each end of the double line, see Figure 3, though this is a technical assumption. In the following of this paper, a *graph* denotes such a graph.

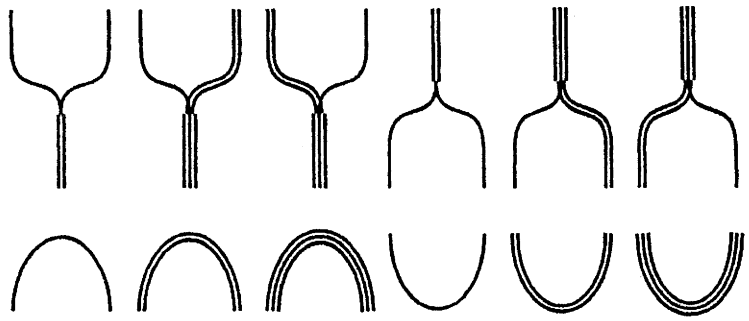


Fig. 2. The elementary pieces of our graphs.
We consider only graphs obtained by gluing these pieces.

For a given graph G, a *stable* σ is an assignment of a pair of a subset of $\mathcal{N} \cup \{0\}$ and a local orientation to each edge of G in the following way; here, we mean by a local orientation an orientation of the edge. We assign a subset $\sigma(e)$ of $\mathcal{N} \cup \{0\}$ to an edge e as

$$\begin{aligned}
&\sigma(e) \in \{A \subset \mathcal{N} \mid \#A = 1\}, && \text{if } e \text{ is a single edge,}\\
&\sigma(e) \in \{A \subset \mathcal{N} \mid \#A = 2\} \cup \{0\}, && \text{if } e \text{ is a double edge,}\\
&\sigma(e) \in \{A \subset \mathcal{N} \mid \#A = 3\} \cup \{\{i, 0\} \mid i \in \mathcal{N}\}, && \text{if } e \text{ is a triple edge.}
\end{aligned}$$

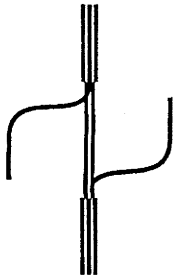


Fig. 3. We do not allow such a double line like in this picture.

We identify an assignment to an edge with the assignment obtained by changing the signs of the elements of $\mathcal{N}$ and by reversing the local orientation of the edge. We further suppose that, for a given state, the set of elements of $\mathcal{N}$ should be preserved at each vertex, where we orient all the edges near the crossing in Figure 2 upward. Here we regard 0 as $\{i, -i\}$ if necessary, when considering it. We show an example of a state in Figure 4.

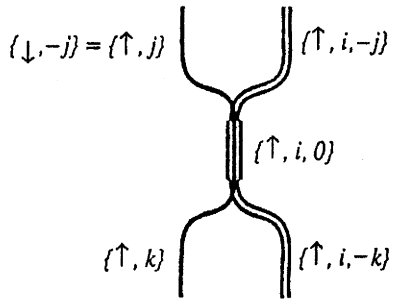


Fig. 4. An admissible state

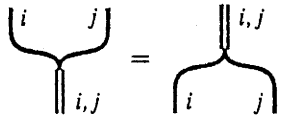
For a graph G, we define our invariant $\langle G \rangle \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$ (note that it is not trivial that $\langle G \rangle$ belongs to $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ by the definition, though it is proved by using lemmas obtained below.) by

$$\langle G \rangle = \sum_{\text{states}} (-1)^{\#\{0\}} \prod_{\text{pieces}} \quad \text{(the weight of a piece under the state)} \tag{1.1}$$

where the sum runs over all possible states and the product runs over all pieces obtained by dividing G into elementary pieces shown in Figure 2 and we mean by $\#\{0\}$ the number of 0 given by the state to the edges of G, and we give the weights for the elementary pieces as follows. In the pictures, each local orientation

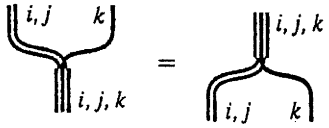
is upward, and we use the notation ε_a defined by $\varepsilon_a = 1$ if $a > 0, 0$ if $a = 0, -1$ if $a < 0$.

(1.2)



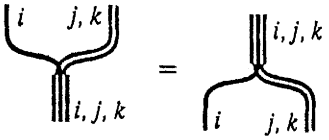
$$= \varepsilon_{i-j}\sqrt{[n]}q^{\varepsilon_{i-j}/4}$$

(1.3)



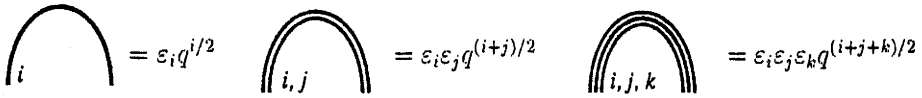
$$= \varepsilon_{i-k}\varepsilon_{j-k}\sqrt{[n-1]}q^{(\varepsilon_{i-k}+\varepsilon_{j-k})/4}$$

(1.4)

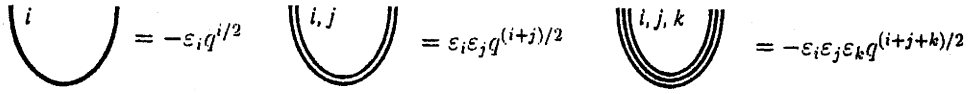


$$= \varepsilon_{i-j}\varepsilon_{i-k}\sqrt{[n-1]}q^{(\varepsilon_{i-j}+\varepsilon_{i-k})/4}$$

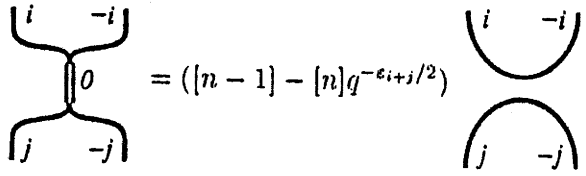
(1.5)



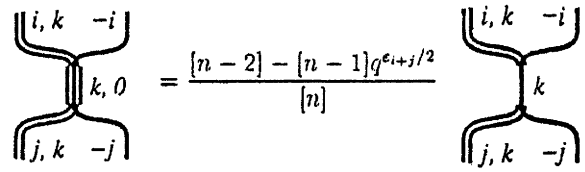
(1.6)



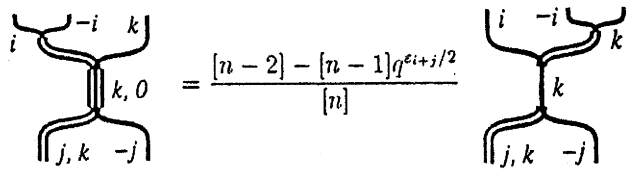
(1.7)



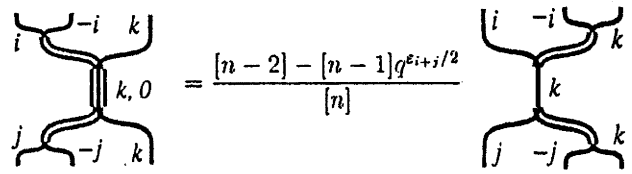
(1.8)



(1.9)



(1.10)



As for formulas (1.7) to (1.10), the value of weight of the left hand side of each formula is defined by the value of right hand side, which is already defined. Further, note that, as for the weights of pieces including a triple line, there are more cases than the three cases in (1.8), (1.9) and (1.10). The weights for the other cases are obtained by extending the three cases in the way how they satisfy Lemma 2.8 below, though we do not give all cases here.

REMARK 1.1. A background of the above definition is as follows. Let V_1 be the vector representation of the Lie algebra $sp(n)$. We have the natural action of $sp(n)$ on the exterior product $\Lambda^2 V_1$; it decomposes into a direct sum of irreducible representations as $\Lambda^2 V_1 = V_2 \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation. Similarly, we have $\Lambda^3 V_1 = V_3 \oplus V_1$ for some irreducible representation V_3; see, for example, [2] for these calculations. Let V_1', V_2' and V_3' be the representations of $U_q(sp_n)$ obtained by deforming the representations V_1, V_2 and V_3.

In this section, we denoted edges to which V_1', V_2' and V_3' are attached by single, double and triple edges, respectively. Further, we can identify

$$\text{the set of basis of } V_1' \text{ with } \{A \subset \mathcal{N} \mid \#A = 1\},$$
$$\text{the set of basis of } V_2' \text{ with } \{A \subset \mathcal{N} \mid \#A = 2\} \cup \{0\},$$
$$\text{the set of basis of } V_3' \text{ with } \{A \subset \mathcal{N} \mid \#A = 3\} \cup \{\{i, 0\} \mid i \in \mathcal{N}\}.$$

Here, to be precise, we regard the state 0 as a "negative" state, i.e., the set $\{A \subset \mathcal{N} \mid \#A = 2\}$ and $\{0\}$ is the set of basis of $\Lambda^2 V_1$ and $\mathbb{C}$ and we regard

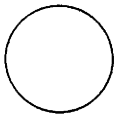
V_2 as $\Lambda^2 V_1 \ominus \mathbb{C}$. In fact, this is the reason why we need the term $(-1)^{\#\{0\}}$ in (1.1). Furthermore, we correspond an intertwiner of V_2' to $V_1' \otimes V_1'$ to, say, the left vertex in (1.2). The weights we defined in this section are the entries of the matrices of such intertwiners with respect to the basis given above.

By this approach, we can reconstruct intertwiners for the quantum $Sp(n)$ invariant given in [12].

2. Properties of $\langle G \rangle$

In this section we prove some properties of the invariant $\langle G \rangle$ of a graph G, in order to define an invariant of links in the next section. For simplicity we often express the value of $\langle G \rangle$ simply by a picture of G.

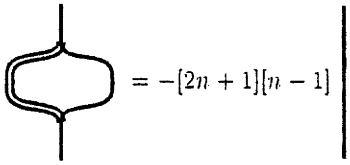
LEMMA 2.1.



$$= -[2n+1] + 1$$

PROOF. We divide this picture into two elementary pieces; upper half one and lower one. Since the weights of these pieces are $\varepsilon_i q^{i/2}$ and $-\varepsilon_i q^{i/2}$, the value of the left hand side is $\sum_{i \in \mathcal{N}} (-q^i)$, which is equal to the right hand side. □

LEMMA 2.2.

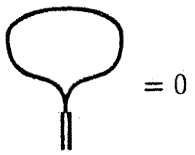


$$= -[2n+1][n-1]$$

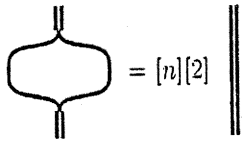
PROOF. We sum up the weights over all states fixing the state i of the upper end and j the lower end in the left hand side. Then we obtain the value to be $-[2n+1][n-1]$ if $i = j, 0$ if $i \neq j$. This implies the required formula. □

We can check the following lemmas by similar elementary calculations as above, i.e., we fix any state at each end of both sides of a formula, and sum up the product of the weights over all internal states, and then we can check the values of both sides of the formula are equal for arbitrary fixed states at the ends of the graphs in both sides of the formula. The detailed proofs are left to the readers. See [11] for similar proofs.

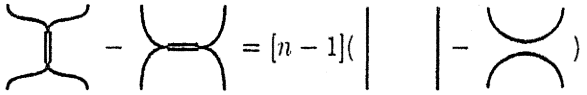
LEMMA 2.3.



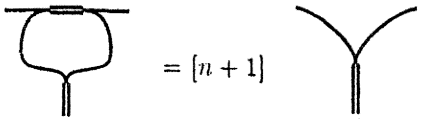
LEMMA 2.4.



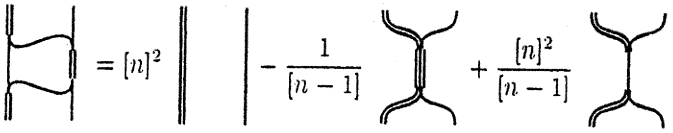
LEMMA 2.5.



LEMMA 2.6.



LEMMA 2.7. (1)

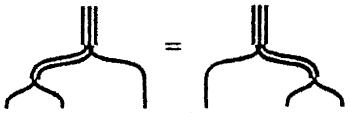


(2) *The formula obtained from the above formula by reflecting left and right.*

LEMMA 2.8. (1)



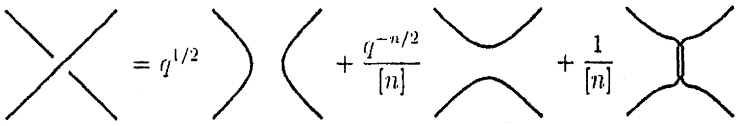
(2)



3. Invariant of links

In this section, we define an invariant of links using the invariant of planar graphs defined in Section 1, and show that it coincides with the quantum $Sp(n)$ invariant of links.

By the following formula, we extend the invariant of planar graphs defined in Section 1 to obtain the invariant of unoriented link diagrams; we denote it by $\langle D \rangle$ for a link diagram D.



Here, the meaning of the formula is as follows. For a given link diagram D with one crossing, we obtain three planar graphs by replacing a crossing of D with three pictures in the right hand side of the formula. Lines of D become single edges of the graphs. We define the value D to be a linear sum of the values of the invariants of the three graphs defined in Section 1 with the coefficients written in the formula. Further, if D has m crossings, we obtain 3^m graphs by replacing the crossings as above, and define the value of $\langle D \rangle$ to be a linear sum of the invariants of 3^m graphs in the same way.

THEOREM 3.1. *For an oriental link L,*

$$(-q^{n+1/2})^{-w(D)}\langle D \rangle$$

is an isotopy invariant of L, where D is a link diagram of L (forgetting the orientation), and $w(D)$ is the writhe of D, i.e. the number of the positive crossings

of D minus the number of the negative crossings of D. We denote the invariant by $P(L)$.

By Lemma 2.5, we obtain the same formula as the second formula in (0,1) for our invariant $\langle D\rangle$ instead of $\langle D\rangle^{Sp(n)}$. Moreover, we have $\langle O\rangle = \langle O\rangle^{Sp(n)}$ by Lemma 2.1 for the trivial diagram O of the trivial knot. Hence we have $\langle D\rangle = \langle D\rangle^{Sp(n)}$ for any link diagram D. Therefore, we have

THEOREM 3.2. *The invariant $P(L)$ coincides with the quantum $Sp(n)$ invariant.*

PROOF. [Proof of Theorem 3.1] It is sufficient to show invariance of $(-q^{n+1/2})^{-w(D)}\langle D\rangle$ under Reidemeister moves I, II and III (see, for example, [1]); we show their pictures in Figure 5 and denote them by RI, RII and RIII in the following.

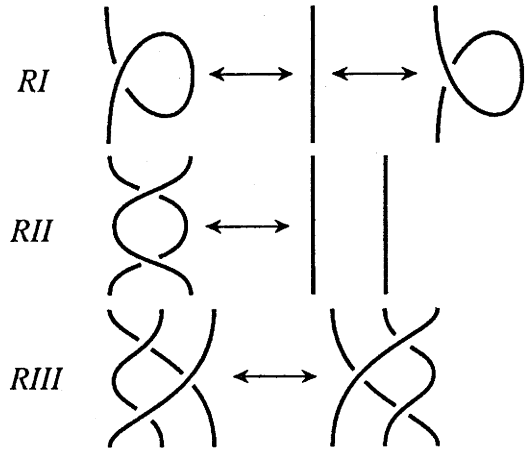
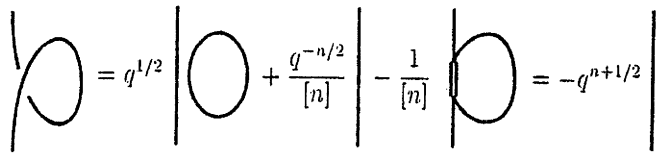


Fig. 5. Reidemeister moves I, II and III

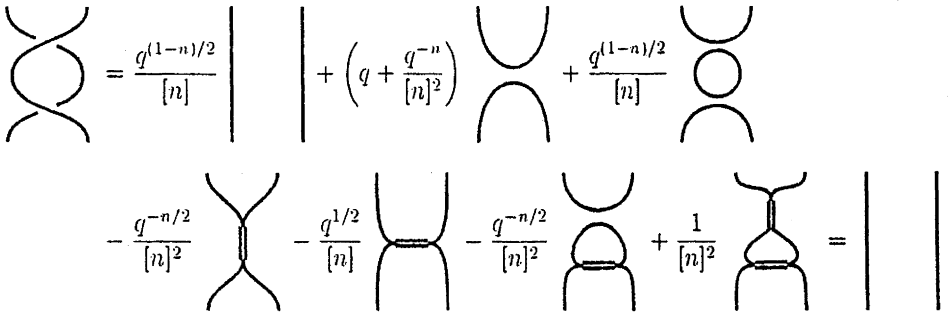
Firstly, we show the invariance under RI as follows. We have



where we obtain the first equality by expanding the crossing in the left hand side, and obtain the second equality by Lemma 2.1 and 2.2. Since the coefficient $-q^{n+1/2}$ in the last term of the formula cancels with the change of the writhe, we

obtain the invariance under RI which removes a positive crossing. Similarly, we obtain the invariance under RI which removes a negative crossing; in this case, we have the coefficient $-q^{-n-1/2}$ in the last term.

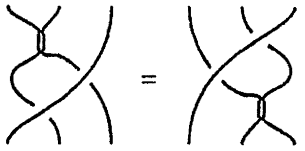
Secondly, we show the invariance under RII by the following formula



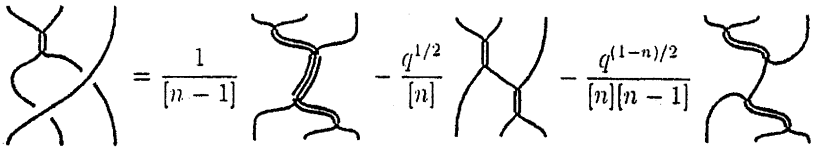
where we obtain the first equality by expanding the two crossings, and we obtain the second equality by using Lemmas 2.1, 2.2, 2.3, 2.5 and 2.6.

Lastly, we show the invariance under RIII as follows. We expand the upper left crossing in the let hand side of the move RIII shown in Figure 5 and expand the lower right crossing in the right hand side. Further, by using the move RII which is proved above, we obtain the following formula; it is sufficient to show the formula for proving invariance under RIII.

(3.1)



We expand the remaining two crossing in the left hand side, and use lemmas in Section 2. Then we can deform the left hand side of the above formula as follows.



This formula implies that the left hand side of (3.1) is equal to a formula which is symmetric under π rotation. Hence, we obtain (3.1), completing the proof. □

REMARK 3.3. By putting $n = 1$, we have the quantum $Sp(1)$ invariant, i.e. the quantum $SU(2)$ invariant. It is well known that the invariant coincides with the Jones polynomial [3, 5]. In this case, one may follow our arguments here with some modifications.

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Hitoshi Murakami
Department of Mathematics
School of Science and Engineering
Waseda University
Ohkubo, Shinjuku-ku, Tokyo 169
Japan
E-mail: hitoshi@uguisu.co.jp

Tomotada Ohtsuki
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152
Japan
E-mail: tomotada@is.titech.ac.jp