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EXAMPLE OF HYPERBOLIC KNOT WHICH DO NOT ADMIT DEPTH 1 FOLIATION

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1. Introduction

Let K be a knot in the 3-sphere S^3 . The exterior of K , denoted by $E(K)$, is the closure of the complement of a regular neighborhood of K . In [4], [7], D. Gabai showed that for any knot K , $E(K)$ admits a taut foliation of finite depth. Then in [1], J. Cantwell and L. Conlon defined the C^q -depth of the knot K as follows:

DEFINITION. Let $0 \leq q \leq \infty$. Then K is of C^q -depth at most k if $E(K)$ admits a taut foliation $\mathcal{F}(K)$ of class C^q , transverse to $\partial E(K)$ and of depth at most k . If K is of C^q -depth at most k , and not C^q -depth at most $k-1$, then K is of C^q -depth k . If K is not of C^q -depth at most k for any integer k , the K has infinite C^q -depth.

In [4], [6], Gabai showed that every knot K has finite C^0 -depth. In [1], Cantwell-Conlon proved that, for any positive integer k , there exist infinitely many knots which have C^2 -depth $> k$. Then [3], they asked:

QUESTION. Is every simple knot either fibered or depth one?

The purpose of this paper is to give a negative answer to this question. In fact, we show:

THEOREM. *Let K be a knot with a Seifert surface S as in Figure 1.1. Then K has the following properties.*

1. *Every minimal genus Seifert surface for K is isotopic to S ,*
2. *K is a hyperbolic knot, and*
3. *The C^0 -depth of K is 2.*

2. Preliminaries

Throughout this paper, we work in the smooth category. All submanifolds are in general position unless otherwise specified. For a subcomplex H of a com-

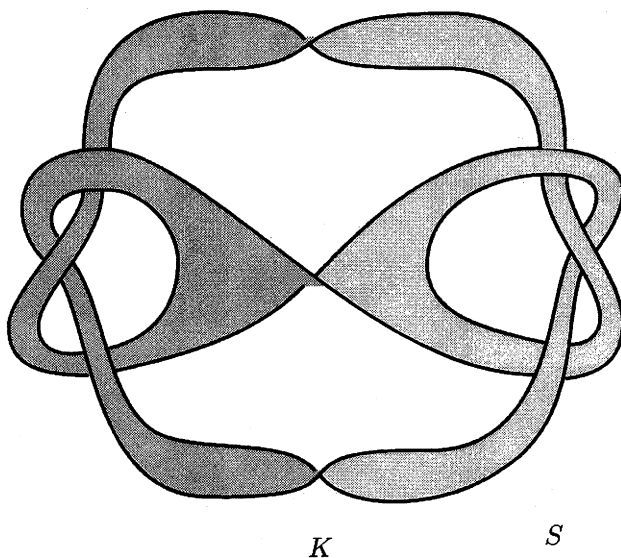


Fig. 1.1.

plex K , $N(H, K)$ denotes a regular neighborhood of H in K . When K is well understood, we often abbreviate $N(H, K)$ to $N(H)$. Let N be a manifold embedded in a manifold M with $\dim N = \dim M$. Then $\text{Fr}_M N$ denotes the frontier of N in M . For the definitions of standard terms in 3-dimensional topology, we refer to [8], and [9].

Then we recall the definition of *sutured manifold* [4]. A sutured manifold (M, γ) is a compact 3-manifold M together with a set γ of mutually disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. In this paper, we mainly treat the case of $T(\gamma) = \emptyset$. The core curves of $A(\gamma)$, denoted by $s(\gamma)$, are the *sutures*. Every component of $R(\gamma) = \partial M - \text{Int } \gamma$ is oriented, and $R_+(\gamma)$ ($R_-(\gamma)$ respectively) denotes the union of the components whose normal vectors points out of (into respectively) M . Moreover the orientations of $R(\gamma)$ must be coherent with respect to $s(\gamma)$.

Let S be a Seifert surface for K . Then we may suppose that $S \cap E(K)$ is obtained from S by removing a regular neighborhood of ∂S , and, hence $S \cap E(K)$ is homeomorphic to S . Then we abbreviate $S \cap E(K)$ by denoting S .

Let $N_S = N(S; E(K))$, and $\delta_S = N_S \cap \partial E(K)$. The pair (N_S, δ_S) is called a *product sutured manifold* obtained from S . Let $M_S = \text{cl}(E(K) - N_S)$, $\gamma_S = M_S \cap \partial E(K)$. (M_S, γ_S) is a *complementary sutured manifold* for S .

Let M be a 3-manifold. A surface S properly embedded in M is called *essential* if it is incompressible and not parallel to a subsurface of ∂M . A knot K is called *simple* if $E(K)$ does not contain an essential torus.

3. Proof of conclusion 1 of Theorem

Let K, S be as in section 1. Let $M = M_S$, $\gamma = \gamma_s$, and $s = s(\gamma_S)$. Let P be a disk with two holes properly embedded in M as in Figure 3.1. Let H_1, H_2 be the closures of the components of $M - P$, and $Q_1 = H_1 \cap \partial M$, and $Q_2 = H_2 \cap \partial M$. We note that Q_i is a disk with two holes embedded in ∂H_i . Then let $s_1 = s \cap H_1$, and $s_2 = s \cap H_2$.

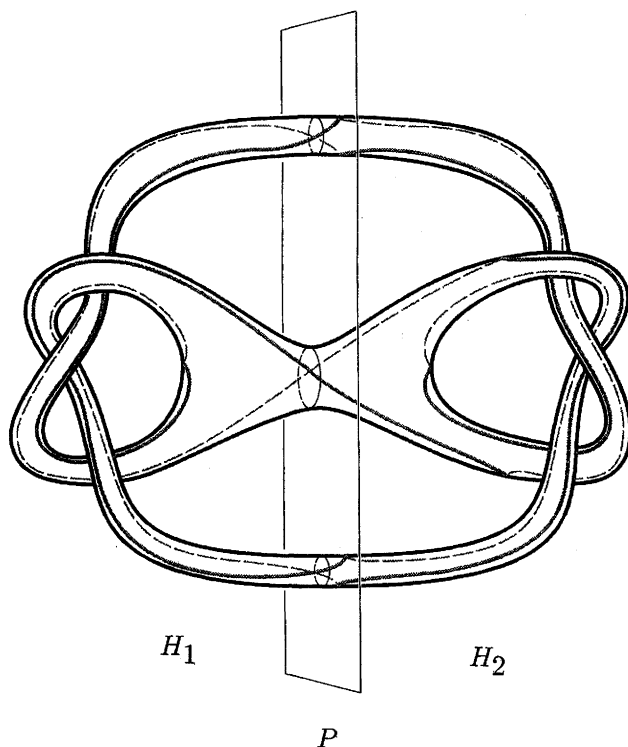


Fig. 3.1.

Let F be a minimal genus Seifert surface for K such that $S \cap F = K$. We show that F is isotopic to S . This fact together with Theorem 3.1 of [10] or Proposition 5 of [13] shows that the minimal genus Seifert surfaces for K are mutually isotopic, which gives the conclusion 1 of Theorem.

LEMMA 3.1. *Let D be an essential disk properly embedded in H_i ($i = 1, 2$). Then $\#\{\partial D \cap \partial Q_i\} \geq 6$.*

PROOF. We may suppose that $\#\{\partial D \cap \partial Q\}$ is minimal in the isotopy class of ∂D . Let $D_{i,1}, D_{i,2}, D_{i,3}$ be essential disks properly embedded in H_i as in

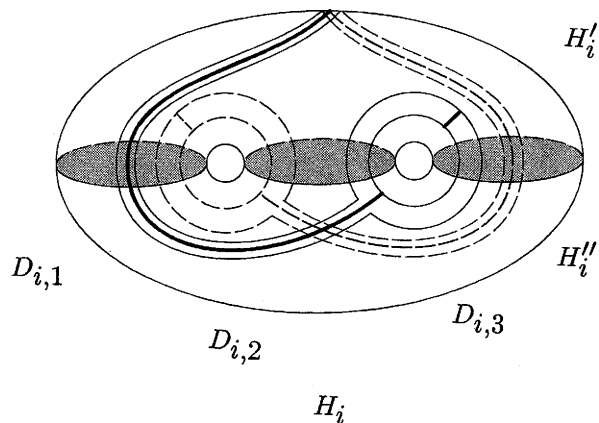


Fig. 3.2.

Figure 3.2. Then let H'_i (H''_i respectively) be the closure of the component of $H - (D_{i,1} \cup D_{i,2} \cup D_{i,3})$ such that $H'_i \cap Q_i$ consists of three rectangles (two hexagons respectively).

By using standard innermost disk arguments, we may suppose that no component of $D \cap (D_{i,1} \cup D_{i,2} \cup D_{i,3})$ is a circle. Let α ($\subset D$) be an outermost component of $D \cap (D_{i,1} \cup D_{i,2} \cup D_{i,3})$, and Δ the closure of a component of $D - \alpha$ such that $\text{Int } \Delta \cap (D_{i,1} \cup D_{i,2} \cup D_{i,3}) = \emptyset$. Let $\beta = \Delta \cap \partial H_i$. Then $\#\{\beta \cap \partial Q_i\} \geq 2$ (see Figure 3.3). Since D has at least two Δ 's as above, this shows that $\#\{\partial D \cap \partial Q_i\} \geq 4$.

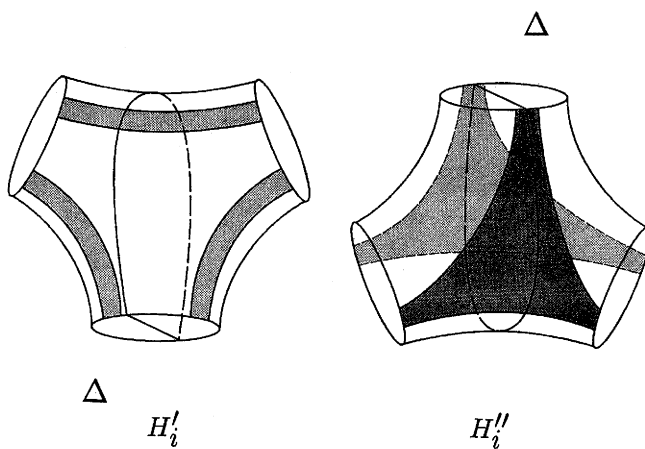


Fig. 3.3.

Assume that $\#\{\partial D \cap \partial Q_i\} = 4$. By the above arguments, we see that there are exactly two outermost components in $D \cap (D_{i,1} \cup D_{i,2} \cup D_{i,3})$. Let n be the number of the components of $D \cap (D_{i,1} \cup D_{i,2} \cup D_{i,3})$, and $\Delta_1, \dots, \Delta_{n+1}$ the closures of the components of $D - (D \cap (D_{i,1} \cup D_{i,2} \cup D_{i,3}))$ as in Figure 3.4.

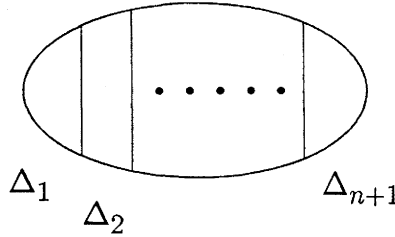


Fig. 3.4.

Suppose that $n = 1$. Then, by changing the suffix of Δ if necessary, we may suppose that $\Delta_1 \subset H'_i$, and $\Delta_2 \subset H''_i$. Then $\partial(\Delta_1 \cap \partial H_i)$ is contained in $\partial H_i - Q_i$, and $\partial(\Delta_2 \cap \partial H_i)$ is contained in Q_i (see Figure 3.3), a contradiction.

Suppose that $n > 1$. Since the argument is symmetric with respect to Q_i and $\text{cl}(\partial H_i - Q_i)$, we may suppose that $\Delta_1 \subset H'_i$. Then Δ_2 is a rectangle such that two edges of Δ_2 is contained in $D_{i,1} \cup D_{i,2} \cup D_{i,3}$ and both of the rest two edges are contained in Q_i or $\partial H - Q_i$. Since $\#\{(\Delta_1 \cap \partial H_i) \cap \partial Q_i\} = 2$, we see that the endpoints of the arc $\Delta_1 \cap \partial H_i$ are contained in mutually different components of $H''_i \cap (\partial H_i - Q_i)$. But by the minimality of $\#\{\partial D \cap \partial Q_i\}$, it is easy to see that this is impossible.

Let $F_1 = F \cap H_1$, $F_2 = F \cap H_2$. By Lemma 3.1, we see that no component of F_1 is a bi-gon (:disk B such that $\#\{\partial B \cap \partial Q_i\} = 2$), or 4-gon (:disk B such that $\#\{\partial B \cap \partial Q_i\} = 4$). Moreover we note that $\partial F_i \cap \gamma (= \partial F_i \cap \partial M)$ consists of three arcs, say $\gamma_{i,1}$, $\gamma_{i,2}$, $\gamma_{i,3}$. It is easy to see that this fact together with Lemma 3.1 shows that F_1, F_2 are disks.

LEMMA 3.2. *Let D be a separating essential disk properly embedded in H_i such that $D \cap Q_i (= \partial D \cap Q_i) = \gamma_{i,1} \cup \gamma_{i,2} \cup \gamma_{i,3}$. Then D is properly isotopic in the proper isotopy class $(H_i, \partial H_i, Q_i)$ to the disk G as in Figure 3.5.*

PROOF. We note that D is a hexagon such that three edges of D are $\gamma_{i,1}$, $\gamma_{i,2}$, $\gamma_{i,3}$. Then we suppose that; (1) $D_{i,2} \cap (\gamma_{i,1} \cup \gamma_{i,2} \cup \gamma_{i,3}) = \emptyset$, and (2) $\#\{D \cap D_{i,2}\}$ is minimal in the proper isotopy class in $(H_i, \partial H_i, Q_i)$. By the assumption (2) we see that no component of $D \cap D_{i,2}$ is a circle. Suppose that there is an arc component α of $D \cap D_{i,2}$ such that $\partial \alpha$ is contained in a component of $\partial D - (\gamma_{i,1} \cup \gamma_{i,2} \cup \gamma_{i,3})$, say β . Then α and a subarc of β bounds a disk Δ in D . Here we may suppose that α is outermost, i.e. $\text{Int } \Delta \cap D_{i,2} = \emptyset$. Then

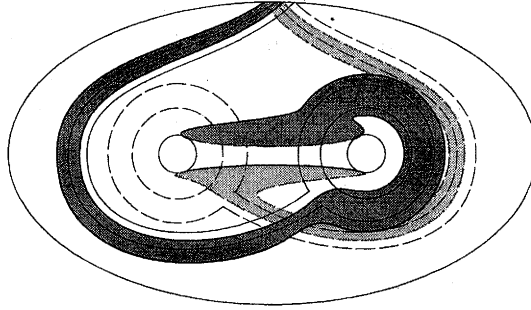


Fig. 3.5.

it is easy to see that we can remove α by using an isotopy as in Figure 3.6 (i), contradicting the assumption.

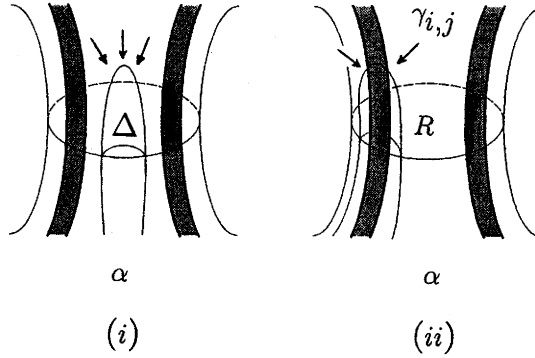


Fig. 3.6.

Suppose that there is an arc component α such that $\partial\alpha$ is contained in pairwise different components of $\partial D - (\gamma_{i,1} \cup \gamma_{i,2} \cup \gamma_{i,3})$. Then α is parallel to some $\gamma_{i,j}$, say. Let R be the square region between α and $\gamma_{i,j}$. Since R is a disk, we easily see that $j = 1$ or 3 , and R is as in Figure 3.6 (ii). Then we can remove α by an isotopy which makes $\gamma_{i,j}$ to go through $\partial D_{i,2}$ as in Figure 3.6 (ii), contradicting the assumption. Hence we see that $D \cap D_{i,2} = \emptyset$. Then let T be the solid torus obtained from H_i by cutting along $D_{i,2}$, and $D_{i,2}^+$, $D_{i,2}^-$ copies of $D_{i,2}$ in ∂T . Since D is separating in H_i we see that D is boundary parallel in T . Then let D^* be the disk in ∂T such that $\partial D^* = \partial D$. Since D is essential, and separating in H_i , we see that $D_{i,2}^+$, $D_{i,2}^- \subset D^*$. It is easy to see that this shows that D is isotopic to G .

Completion of Proof. Let F_i be the hexagon as above. Assume that F_i is an essential disk in H_i . Since F and S represent the same homology class, we see that F_i is separating in H_i . Hence, by Lemma 3.2, we see that F_i corresponds to G in Figure 3.5. But, by a direct observation (see Figure 3.7), we see that $F_1 \cap P$ and $F_2 \cap P$ cannot be matched to make F . Here we note that the curves in Figure 3.7 represent the boundaries of G .

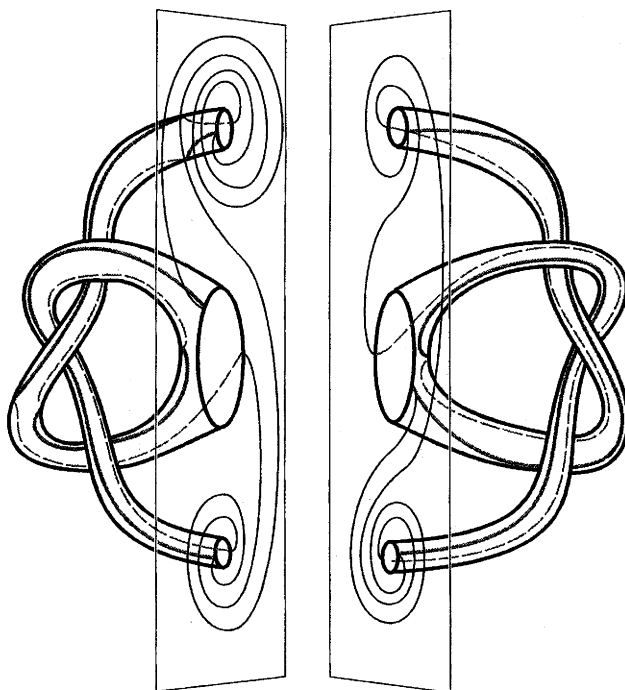


Fig. 3.7.

Hence each F_i is inessential (i.e. parallel to a disk in ∂H_i), and this shows that F_i is parallel to a copy of $\text{Fr}_{E(K)}N(S, E(K)) \cap H_i$ in H_i . And this shows that F is parallel to S .

4. Proof of conclusion 2 of Theorem

LEMMA 4.1. *Let K be a genus 1 knot such that $E(K)$ contains an essential torus T . Then there is a minimal genus Seifert surface S for K such that $S \cap T = \emptyset$.*

PROOF. Let U, E be the closures of the components of $S^3 - T$ such that

$K \subset U$. Then we note that U is a solid torus, and E is homeomorphic to the exterior of a nontrivial knot K' . Let S be a minimal genus Seifert surface for K such that $\# \{S \cap T\}$ is minimal among all minimal genus Seifert surfaces for K .

Assume that $\# \{S \cap T\} > 0$.

Claim 1. Every incompressible annulus properly embedded in U (E resp.) is separating in U (E resp.).

PROOF. First, we note that $H_1(U, \mathbf{Z}) \cong H_2(U, \partial U, \mathbf{Z}) (\cong \mathbf{Z})$, and a simple closed curve ℓ in ∂U represents a generator of $H_1(U, \mathbf{Z})$. Let A be a non-separating annulus properly embedded in U . Since A is non-separating, we see that A represents a generator of $H_2(U, \partial U, \mathbf{Z})$. This shows that the algebraic intersection number of ℓ and ∂A is 1. Hence one component of ∂A is contractible in ∂U , and this shows that A is compressible in U , a contradiction.

The arguments above works for E , and this completes the proof.

By the minimality of $\# \{S \cap T\}$, and the incompressibility of T in E , and the irreducibility of E , we see that each component of $S \cap T$ is essential in S and T . Then by using Euler characteristic considerations, we have either; every component of $S \cap E$ is an annulus; $S \cap E$ consists of a torus with one hole, and a number of (possibly empty) annulus; or $S \cap E$ consists of a disk with two holes, and a number of (possibly empty) annulus. Then we have:

Claim 2. $S \cap E$ does not contain an annulus.

PROOF. Assume that $S \cap T$ contains an annulus component. Since $S \cap T$ contains at most one component which is not an annulus, we see, by Claim 1, that there is an outermost annulus A_1 in $S \cap E$, i.e. there is a component G of $E - A_1$ such that $G \cap S = \emptyset$. Then let $A^* = \text{cl}(G \cap \partial E)$, and $S^* = (S - A_1) \cup A^*$. We note that S^* is a minimal genus Seifert surface for K and by moving S^* by a tiny isotopy, we have $\# \{S^* \cap T\} = \# \{S \cap T\} - 2$, a contradiction.

By Claim 2, we have the following two cases.

Case 1. $S \cap E$ consists of a torus with one hole.

We first note that $S \cap U$ consists of an annulus in this case. Then we note that $\partial(S \cap E)$ is a longitude of the nontrivial knot K_1 . These facts show that K is isotopic in U to the core curve of the solid torus U , contradicting the fact that T is essential in $E(K)$.

Case 2. $S \cap E$ consists of a disk with two holes.

In this case $S \cap U$ consists of two annuli A_1, A_2 such that $K \subset \partial A_1$. Then there is a component of $U - A_2$, say G , such that $G \cap S = \emptyset$. Then let $A^* =$

$\text{cl}(G \cap \partial U)$, and $S^* = (S - A_1) \cup A^*$. We note that S^* is a minimal genus Seifert surface for K and by moving S^* by a tiny isotopy, we have $\#\{S^* \cap T\} = \#\{S \cap T\} - 2$, a contradiction.

This completes the proof of Lemma 4.1.

LEMMA 4.2. *Let E be the closure of the complement of the genus 2 handlebody as in Figure 4.1. Then E does not contain an incompressible torus.*

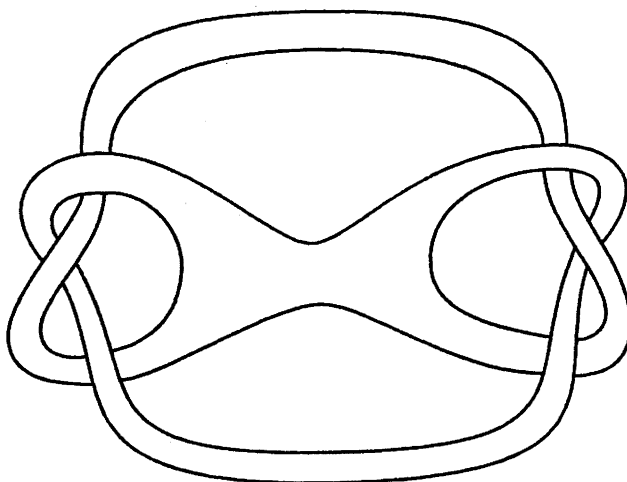


Fig. 4.1.

PROOF. For the proof of this, we consider a tangle (B, t) as in Figure 4.2. We immediately see from Figures 4.1, and 4.2, that it is enough to show that the exterior of t in B does not contain an incompressible torus for the proof of Lemma 4.2.

Assume that the exterior of t in B contains an incompressible torus T . Let N be the 2-fold branched cover of B along t , and $p : N \rightarrow B$ the covering projection. Then we see that N is a Seifert fibered space with orbit manifold a disk and two exceptional fibers. It is known that N does not contain an incompressible torus [9]. Hence we see that $p^{-1}(T)$ is a compressible torus in N . Then by the equivariant loop theorem ([11]), there exists a compressing disk $\tilde{D}$ for $p^{-1}(T)$ such that $p(\tilde{D}) \cap \tilde{D} = \emptyset$ or $p(\tilde{D}) = \tilde{D}$. Since T is incompressible, we see that $p(\tilde{D}) = \tilde{D}$. Here we note that the fixed point set of $p|_{\tilde{D}} : \tilde{D} \rightarrow \tilde{D}$ consists of one point in $\text{Int } \tilde{D}$. Hence $D = p(\tilde{D})$ is a compressing disk for T in B such that $D \cap t$ consists of a point. This shows that T is obtained from a 2-sphere by adding a knotted tube inside, and a component of t goes through the tube once. But this

is a contradiction, since the components of t are unknotted.

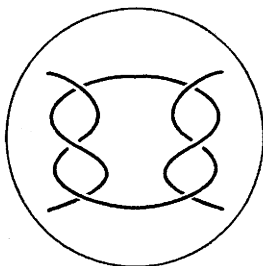


Fig. 4.2.

Now the conclusion 1 of Theorem together with Lemmas 4.1 and 4.2 shows that K is a simple knot.

5. Proof of conclusion 3 of Theorem

Let M , s be as in section 3, and $\widehat{M}$ the manifold obtained from M by attaching a 2-handle along s . Then $\partial\widehat{M}$ consists of two tori, say T^+ , T^- , where T^ϵ corresponds to $R_\epsilon(M)$. By Theorem 1.3 of [2], it is enough to show the following proposition for the proof of the fact that the depth of K is greater than one.

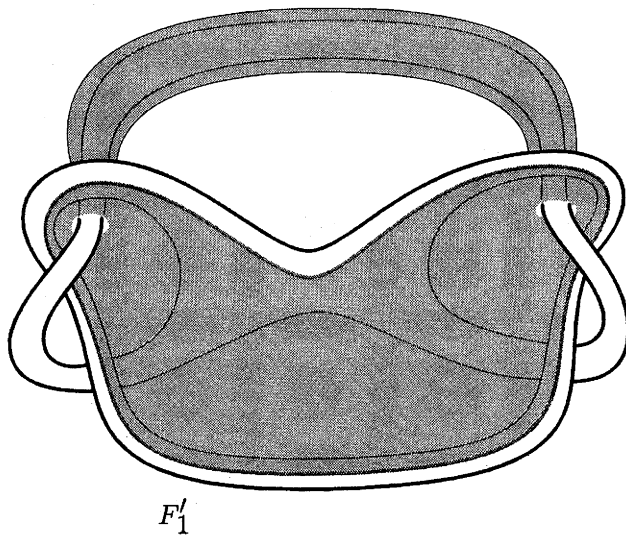


Fig. 5.1.

PROPOSITION 5.1. $\widehat{M}$ is not homeomorphic to the total space of a surface bundle over S^1 .

PROOF. Let F'_1 be a genus 1 surface properly embedded in M as in Figure 5.1. We note that $\partial F'_1$ is isotopic in ∂M to a simple closed curve disjoint from s . By using this isotopy, we can obtain a genus 1 surface F_1 properly embedded in $(M, \partial M - s)$. We can obtain another genus one surface F_2 properly embedded in $(M, \partial M - s)$ by rotating F_1 in Figure 5.1 180 degree in x -axis.

It is easy to see that $H_2(\widehat{M}, \partial\widehat{M}; \mathbf{R}) \cong \mathbf{R}^2 = [F_1] \oplus [F_2] = \langle \alpha_1 \rangle \oplus \langle \alpha_2 \rangle$, where F_1, F_2 are regarded as surfaces properly embedded in $(\widehat{M}, \partial\widehat{M})$. Then we consider the annulus A' properly embedded in M as in Figure 5.2. We note that $\partial A'$ is isotopic to $\partial F_1 \cup \partial F_2$ in ∂M . By using this isotopy, we can obtain an annulus A properly embedded in $(M, \partial M - s)$.

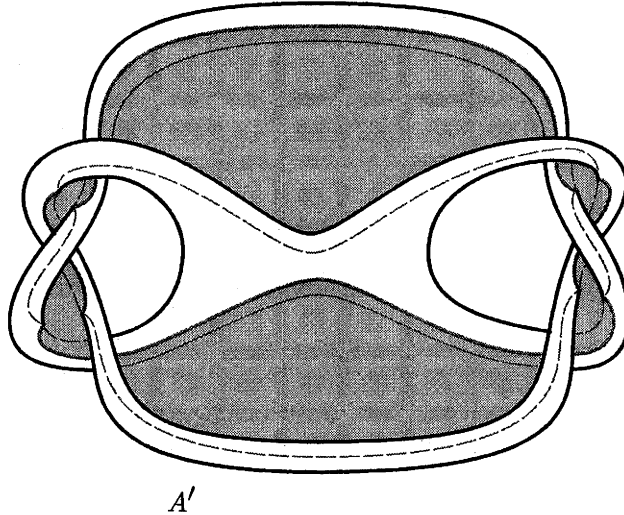


Fig. 5.2.

Let x be the Thurston norm on $H_2(\widehat{M}, \partial\widehat{M}; \mathbf{R})$. Since $\alpha_i = [F_i]$, we see that $x(\alpha_i) \leq 1$. Since K is a non-trivial knot, $x(\alpha_i) \neq 0$. Hence we have $x(\alpha_i) = 1$. Moreover, by using Poincaré duality, we see that $[A] = [F_1] + [F_2] \in H_2(\widehat{M}, \partial\widehat{M}; \mathbf{R})$ provided that F_i has an appropriate orientation, where A is regarded as a surface properly embedded in $(\widehat{M}, \partial\widehat{M})$. This shows that $x(\alpha_1 + \alpha_2) = 0$. Hence the unit ball in $\mathbf{R}^2$ with respect to the norm x is an infinite region bounded by two lines as in Figure 5.3.

We note that, for the proof of Proposition 5.1, we have to show that every rational point in $H_2(\widehat{M}, \partial\widehat{M}; \mathbf{R}) \cong \mathbf{R}^2$ cannot be represented by a fiber of a sur-

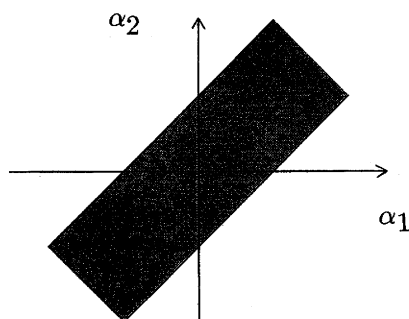


Fig. 5.3.

face bundle structure of $\widehat{M}$. However, by Thurston [14], it is enough to show that at least one rational point, for example α_1 in these lines cannot be represented by a fiber.

Assume that α_1 is represented by a fiber F . Let T^ε be the component of $\partial\widehat{M}$ such that $\partial F_1 \cap T^\varepsilon = \emptyset$. Hence $\partial F_1 \cap T^\varepsilon$ represents 0 in $H_1(T^\varepsilon, \mathbf{R})$. However this is impossible, since F is a fiber.

This completes the proof of Proposition 5.1.

PROPOSITION 5.2. *The sutured manifold (M, γ) admits a depth 2 foliation.*

PROOF. Let A the annulus properly embedded in M in the proof of Proposition 5.1. It is elementary to show that, by applying sutured manifold decomposition on (M, γ) along A , we obtain a sutured manifold (M', γ') as in Figure 5.4 where M' corresponds to the outside of the genus two surface. It is easy to

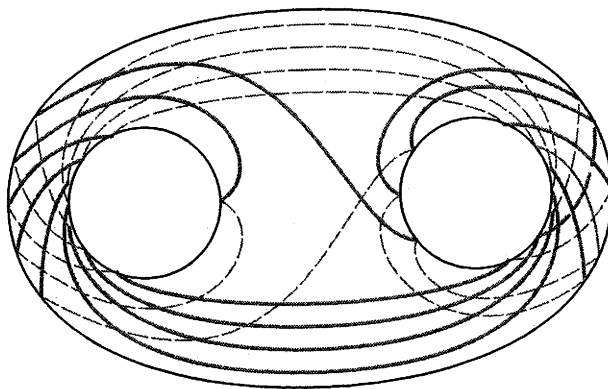


Fig. 5.4.

verify that this sutured manifold is completely disk decomposable [5]. Hence, by Theorem 3.14 of [5], the sutured manifold (M', γ') admits a foliation of depth 1. Then by pulling back this foliation as in [4], it is shown that the sutured manifold (M, γ) admits a foliation of depth 2.

Propositions 5.1 and 5.2 complete the proof of conclusion 3 of Theorem.

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