



On some q -confluent hypergeometric systems associated with the quantum grassmannians of type $(2, n)$

Nakatani, Minobu

(Citation)

Kobe Journal of Mathematics, 14(1):63-89

(Issue Date)

1997-06

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0003648>

(URL)

<https://hdl.handle.net/20.500.14094/E0003648>



ON SOME q -CONFLUENT HYPERGEOMETRIC SYSTEMS ASSOCIATED WITH THE QUANTUM GRASSMANNIANS OF TYPE $(2, n)$

Minobu NAKATANI

(Received January 20, 1997)

(Revised May 29, 1997)

Introduction

It has been known that the contiguity relations for Gauss hypergeometric function ${}_2F_1$ give a representation of the Lie algebra $\mathfrak{gl}(4)$, while it is recently that the algebraic structure behind the contiguity relations for Heine's series ${}_2\varphi_1$, one q -analogue of Gauss hypergeometric function, was clarified. In 1992, E. Horikawa [H] showed that the contiguity relations for Heine's series ${}_2\varphi_1$ give a representation of the quantum algebra $U_q(\mathfrak{gl}(4))$, a quantum analogue of the universal enveloping algebra of $\mathfrak{gl}(4)$. We recall the definition of $U_q(\mathfrak{gl}(n))$ later.

While the hypergeometric functions of several variables are realized as solutions of the system of differential equations related to Grassmannian manifolds, M. Noumi [Nm] defined a system of q -difference equations by using the representation of $U_q(\mathfrak{gl}(n))$ on the quantum Grassmannian that is a q -analogue of the Grassmannian manifold $G_{2,n}$. He showed that q -hypergeometric function φ_D , which is a q -analogue of Lauricella's hypergeometric function F_D , can be realized as a solution of the system and that the contiguity relations for φ_D give a representation of $U_q(\mathfrak{gl}(n))$. Furthermore, the author and Noumi [NN] defined a q -difference system by using representation of $U_q(\mathfrak{gl}(n))$ and realized a q -analogue of the hypergeometric function related to $G_{k,n}$ as a solution of this system.

One of the next steps is to discuss the confluent cases. E. Date and E. Horikuchi [DH] have already showed that the contiguity relations for the q -confluent hypergeometric function of several variables, including the q -Kummer function, give a representation of a q -deformed Lie algebra.

In this paper, we discuss the q -confluent hypergeometric functions, including the q -Kummer and the q -Bessel function from a different point of view.

In the classical case, we can realize the confluent hypergeometric function as a solution of the system of differential equations by using a representation of the universal enveloping algebra $U(\mathfrak{gl}(n))$. In the present case, we introduce two

types of *q*-confluent hypergeometric system by using a representation of $U_q(\mathfrak{gl}(n))$ on the polynomial ring, and we realize *q*-confluent hypergeometric functions as solutions of these systems. They are given by the series

$$\begin{aligned} & \varphi^{(1)} \left(\begin{matrix} \alpha; \beta_5, \dots, \beta_n \\ \gamma \end{matrix}, q; y \right) \\ &= \sum_{\nu_j \geq 0} \frac{(\alpha; q)_{|\nu|} (\beta_5; q)_{\nu_5} \cdots (\beta_n; q)_{\nu_n} (-1)^{\nu_2}}{(\gamma; q)_{|\nu|} (q; q)_{\nu_2} (q; q)_{\nu_5} \cdots (q; q)_{\nu_n}} q^{\nu_2(\nu_2-1)/2} y_2^{\nu_2} y_5^{\nu_5} \cdots y_n^{\nu_n} \end{aligned}$$

and

$$\begin{aligned} & \varphi^{(2)} \left(\begin{matrix} \beta_5, \dots, \beta_n \\ \gamma \end{matrix}, q; z \right) \\ &= \sum_{\nu_j \geq 0} \frac{(\beta_5; q)_{\nu_5} \cdots (\beta_n; q)_{\nu_n}}{(\gamma; q)_{|\nu|} (q; q)_{\nu_2} (q; q)_{\nu_5} \cdots (q; q)_{\nu_n}} z_4^{\nu_4} z_5^{\nu_5} \cdots z_n^{\nu_n}, \end{aligned}$$

where $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$. See Section 4 for the notations.

The plan of the contents is as follows. In Section 1, we give a brief description of the confluent hypergeometric systems in the classical case, associated with the Grassmannian. In Section 2, we recall the quantum algebra $U_q(\mathfrak{gl}(n))$ and introduce a *noncommutative* algebra that is a left $U_q(\mathfrak{gl}(n))$ -module. In Section 3, we translate the action of $U_q(\mathfrak{gl}(n))$ on the noncommutative algebra into the action on a *commutative* polynomial ring. In Section 4, we define the system of *q*-difference equations by use of the action of $U_q(\mathfrak{gl}(n))$. Moreover, we realize *q*-confluent hypergeometric functions as solutions of this system. In Section 5, we compute the contiguity relations for those *q*-hypergeometric function, and in the last section 6, we define *q*-confluent hypergeometric systems.

The author would like to thank Professors Etsuro Date, Masatoshi Noumi, and Takeshi Sasaki for many useful suggestions, and Professor Hironobu Kimura for his lectures.

In what follows, we fix a complex number q with $0 < |q| < 1$.

1. Review of the confluent hypergeometric system

In this section, we recall the definition of the confluent hypergeometric system briefly. Refer to [KHT1–3] for detail.

For a fixed integer n , we set $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)_{\alpha_i \in \mathbb{N}}$, which is a partition for n ; that is, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$ and $|\alpha| = \alpha_1 + \cdots + \alpha_l = n$. For example, there are 5 types of partition of 4; $(1, 1, 1, 1)$, $(2, 1, 1)$, $(2, 2)$, $(3, 1)$, and (4) .

We associate to each partition $\alpha = (\alpha_1, \dots, \alpha_l)$ the following maximal abelian

subgroup H_α of $G = GL(n, \mathbb{C})$:

$$H_\alpha = \left\{ \begin{pmatrix} h^{(1)} & & & \\ & h^{(2)} & & \\ & & \ddots & \\ & & & h^{(l)} \end{pmatrix}; h^{(i)} \in J(\alpha_i) \right\}, \quad (1.1)$$

where $J(m)$ is the Jordan group of size m :

$$J(m) = \left\{ h = \begin{pmatrix} h_1 & h_2 & \cdots & h_m \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_2 \\ & & & h_1 \end{pmatrix}; h_1 \neq 0, h_i \in \mathbb{C} \right\} \subset GL(m). \quad (1.2)$$

For later use, we fix three types of partition of n :

$$\alpha^{(0)} = (\overbrace{1, \dots, 1}^n), \quad \alpha^{(1)} = (2, \overbrace{1, \dots, 1}^{n-2}), \quad \alpha^{(2)} = (2, 2, \overbrace{1, \dots, 1}^{n-4}). \quad (1.3)$$

For each of these three partitions, H_α is given as follows:

$$\begin{aligned} H_{\alpha^{(0)}} &= \left\{ h = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix}; h_i \in \mathbb{C}^\times \right\}, \\ H_{\alpha^{(1)}} &= \left\{ h = \begin{pmatrix} h_1 & h_2 & & \\ & h_1 & & \\ & & h_3 & \\ & & & \ddots \\ & & & & h_n \end{pmatrix}; \begin{matrix} h_1, h_3, \dots, h_n \in \mathbb{C}^\times \\ h_2 \in \mathbb{C} \end{matrix} \right\}, \\ H_{\alpha^{(2)}} &= \left\{ h = \begin{pmatrix} h_1 & h_2 & & & \\ & h_1 & & & \\ & & h_3 & h_4 & \\ & & & h_3 & \\ & & & & h_5 \\ & & & & & \ddots \\ & & & & & & h_n \end{pmatrix}; \begin{matrix} h_1, h_3, h_5, \dots, h_n \in \mathbb{C}^\times \\ h_2, h_4 \in \mathbb{C} \end{matrix} \right\}. \end{aligned}$$

We denote by $\tilde{H}_\alpha$ the universal covering group of H_α and by $\chi_\alpha : \tilde{H}_\alpha \rightarrow \mathbb{C}$ the character of H_α parametrized by $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. For instance,

$$\chi_{\alpha^{(0)}}(h; \lambda) = h_1^{\lambda_1} h_2^{\lambda_2} \cdots h_n^{\lambda_n} \quad (h \in H_{\alpha^{(0)}}), \quad (1.4)$$

$$\chi_{\alpha^{(1)}}(h; \lambda) = h_1^{\lambda_1} \exp\left(\lambda_2 \frac{h_2}{h_1}\right) h_3^{\lambda_3} \cdots h_n^{\lambda_n} \quad (h \in H_{\alpha^{(1)}}), \quad (1.5)$$

and

$$\begin{aligned} \chi_{\alpha^{(2)}}(h; \lambda) &= h_1^{\lambda_1} \exp\left(\lambda_2 \frac{h_2}{h_1}\right) h_3^{\lambda_3} \exp\left(\lambda_4 \frac{h_4}{h_3}\right) h_5^{\lambda_5} \cdots h_n^{\lambda_n} \\ &\quad (h \in H_{\alpha^{(2)}}). \end{aligned} \quad (1.6)$$

We next recall the definition of the confluent hypergeometric system associated with $G_{2,n}$. Let $M(2, n)$ be the space of $2 \times n$ -matrices, an element of which is denoted by $T = (t_{rj})_{1 \leq r \leq 2, 1 \leq j \leq n}$. The *general confluent hypergeometric system* $E(\alpha; \lambda)$ of differential equations on the coordinates $\{t_{rj}\}$ is the system consisting of the equations

$$\Phi(gT) = \det(g)^{-1} \Phi(T) \quad (g \in GL(2)), \quad (1.7)$$

$$\Phi(Th) = \Phi(T) \chi_{\alpha}(h, \lambda) \quad (h \in \tilde{H}_{\alpha}), \quad (1.8)$$

$$\left(\frac{\partial^2}{\partial t_{1i} \partial t_{2j}} - \frac{\partial^2}{\partial t_{2i} \partial t_{1j}} \right) \Phi(T) = 0 \quad (1 \leq i < j \leq n). \quad (1.9)$$

For the conditions (1.7) and (1.8) to be compatible, it is necessary that $\lambda_1 + \cdots + \lambda_n = -2$ for $\alpha = \alpha^{(0)}$, $\lambda_1 + \lambda_3 + \cdots + \lambda_n = -2$ for $\alpha = \alpha^{(1)}$, and $\lambda_1 + \lambda_3 + \lambda_5 + \cdots + \lambda_n = -2$ for $\alpha = \alpha^{(2)}$. This is because $(cI_k)T = T(cI_n)$ for any $c \in \mathbb{C}^{\times}$ and identity matrices $I_k \in GL(k)$ and $I_n \in GL(n)$.

We can realize the Lauricella hypergeometric function F_D as a solution of the system $E(\alpha^{(0)}; \lambda)$. The system for $M(k, n)$, the space of $k \times n$ -matrices, can be defined similarly.

Let us look for special solutions of the systems $E(\alpha^{(1)}, \lambda)$ and $E(\alpha^{(2)}, \lambda)$. We decompose $2 \times n$ matrix T as follows:

$$T = \begin{pmatrix} t_{11} & t_{13} \\ t_{21} & t_{23} \end{pmatrix} \times T', \quad (1.10)$$

$$T' = \begin{pmatrix} 1 & x_2^1 & 0 & x_4^1 & \cdots & x_n^1 \\ 0 & x_2^2 & 1 & x_4^2 & \cdots & x_n^2 \end{pmatrix}, \quad (1.11)$$

where $x_j^1 = -\xi_{13}^{-1} \xi_{3j}$, $x_j^2 = \xi_{13}^{-1} \xi_{1j}$, and $\{\xi_{ij}\}$ are the Plücker coordinates of $G_{2,n}$ defined by

$$\xi_{ij} = t_{1i} t_{2j} - t_{2i} t_{1j}. \quad (1.12)$$

The Plücker coordinates satisfy the relation

$$\xi_{ij} = -\xi_{ji} \quad (1.13)$$

and the Plücker relations

$$\xi_{\alpha\beta_1}\xi_{\beta_2\beta_3} - \xi_{\alpha\beta_2}\xi_{\beta_1\beta_3} + \xi_{\alpha\beta_3}\xi_{\beta_1\beta_2} = 0 \quad (1.14)$$

for $1 \leq \alpha \leq n$ and $1 \leq \beta_1 < \beta_2 < \beta_3 \leq n$.

We further decompose T' in two ways:

$$\begin{aligned} T' &= \begin{pmatrix} 1 & 0 \\ 0 & -\frac{x_4^2}{x_4^1} \end{pmatrix} T_{(1)} \begin{pmatrix} 1 & x_2^1 & & & & \\ & 1 & & & & \\ & & -\frac{x_4^1}{x_4^2} & & & \\ & & & x_4^1 & & \\ & & & & \ddots & \\ & & & & & x_n^1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & x_2^2 \end{pmatrix} T_{(2)} \begin{pmatrix} 1 & x_2^1 & & & & \\ & 1 & & & & \\ & & \frac{1}{x_2^2} & \frac{x_4^2}{x_2^2} & & \\ & & & \frac{1}{x_2^2} & & \\ & & & & \frac{1}{x_2^2} & \\ & & & & & \frac{x_5^2}{x_2^2} \\ & & & & & & \ddots \\ & & & & & & & \frac{x_n^2}{x_2^2} \end{pmatrix}, \end{aligned} \quad (1.15)$$

where

$$T_{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & y_2 & 1 & -1 & y_5 & \cdots & y_n \end{pmatrix}, \quad (1.16)$$

and

$$T_{(2)} = \begin{pmatrix} 1 & 0 & 0 & z_4 & z_5 & \cdots & z_n \\ 0 & 1 & 1 & 0 & 1 & \cdots & 1 \end{pmatrix}. \quad (1.17)$$

Here $y_2 = -\frac{x_4^1 x_2^2}{x_4^2}$, $y_j = -\frac{x_4^1 x_j^2}{x_4^2 x_j^1}$ ($j \geq 5$) and $z_4 = x_4^1 x_2^2$, $z_j = -\frac{x_j^1 x_2^2}{x_j^2}$ ($j \geq 5$).

Relative to the decomposition (1.10), $\Phi(T)$ can be written in the form

$$\Phi(T) = \xi_{13}^{-1} G(x) \quad (1.18)$$

for some function $G(x)$ in the coordinates $x = (x_2^1, x_2^2, x_4^1, x_4^2, \dots, x_n^1, x_n^2)$ because of (1.7).

Relative to the decomposition (1.15), when $\lambda_2 = 1$ for the partition $\alpha^{(1)}$, $G(x)$ can be written in the form

$$G^{(1)}(x) = (x_4^1)^{\lambda_3 + \lambda_4 + 1} (x_4^2)^{-\lambda_3 - 1} \exp(x_2^1) \prod_{j \geq 5} (x_j^1)^{\lambda_j} F^{(1)}(y) \quad (1.19)$$

for some function $F^{(1)}(y)$ of $(n-3)$ variables $y = (y_2, y_5, \dots, y_n)$. A candidate for $F^{(1)}$ is defined as follows:

$$F^{(1)}(y) = \sum_{\nu_j \geq 0} \frac{(\lambda_3 + 1)_{|\nu|} (-\lambda_5)_{\nu_5} \cdots (-\lambda_n)_{\nu_n}}{(\lambda_3 + \lambda_4 + 2)_{|\nu|} (1)_{\nu_2} (1)_{\nu_5} \cdots (1)_{\nu_n}} y_2^{\nu_2} y_5^{\nu_5} \cdots y_n^{\nu_n}, \quad (1.20)$$

where

$$(a)_n = \begin{cases} 1 & (n = 0) \\ a(a+1) \cdots (a+n-1) & (n \geq 1). \end{cases}$$

Then it is seen that $\Phi(T) = \xi_{13}^{-1} G^{(1)}(x)$ is a solution of the system $E(\alpha^{(1)}; \lambda)$. For the partition $\alpha^{(2)}$, when $\lambda_2 = \lambda_4 = 1$, $G(x)$ can be written in the form

$$G^{(2)}(x) = (x_2^2)^{\lambda_1 + 1} \exp(x_2^1) \exp(x_4^2) \prod_{j \geq 5} (x_j^2)^{\lambda_j} F^{(2)}(z), \quad (1.21)$$

for some function $F^{(2)}$ of $(n-3)$ variables $z = (z_4, z_5, \dots, z_n)$. We define the function $F^{(2)}$ by

$$F^{(2)}(z) = \sum_{\nu_j \geq 0} \frac{(-\lambda_5)_{\nu_5} \cdots (-\lambda_n)_{\nu_n}}{(\lambda_1 + 2)_{|\nu|} (1)_{\nu_4} (1)_{\nu_5} \cdots (1)_{\nu_n}} z_4^{\nu_4} z_5^{\nu_5} \cdots z_n^{\nu_n}. \quad (1.22)$$

Then it is known that $\Phi(T) = \xi_{13}^{-1} G^{(2)}(x)$ is a solution of the system $E(\alpha^{(2)}; \lambda)$.

In particular, when $n = 4$, the function $F^{(1)}(y_2) = {}_1F_1\left(\begin{smallmatrix} a \\ c \end{smallmatrix}; y_2\right)$ is the Kummer function and

$$\begin{aligned} J_v(t) &= \sum_{m \geq 0} \frac{(-1)^m (t/2)^{v+2m}}{m! \Gamma(v+m+1)} = \frac{(t/2)^v}{\Gamma(v+1)} {}_0F_1\left(\begin{smallmatrix} - \\ v+1 \end{smallmatrix}; -\frac{t^2}{4}\right) \\ &= (-y_4)^{-\frac{1-c}{2}} \frac{1}{\Gamma(c)} F^{(2)}(z_4), \end{aligned} \quad (1.23)$$

is the Bessel function. Therefore, the Kummer and the Bessel functions are realized as the solutions of the confluent hypergeometric system.

We next recall the relation between the confluent hypergeometric system and Lie algebra. Let us put

$$E_{ij} = \sum_{r=1}^2 t_{ri} \frac{\partial}{\partial t_{rj}}. \quad (1.24)$$

Then $\{E_{ij}\}_{1 \leq i, j \leq n}$ generate a Lie algebra isomorphic to $\mathfrak{gl}(n)$, and we can verify that

$$(E_{ij} + \delta_{ij})E_{kl} - E_{kj}E_{il} = (t_{1i}t_{2k} - t_{2i}t_{1k}) \left(\frac{\partial^2}{\partial t_{1j} \partial t_{2l}} - \frac{\partial^2}{\partial t_{2j} \partial t_{1l}} \right) \quad (1.25)$$

for each $k \neq j$. In terms of these operators, we can rewrite the differential equation (1.8) as follows:

$$\begin{aligned} (E_{11} + E_{22})\Phi(T) &= \lambda_1 \Phi(T), \\ E_{12}\Phi(T) &= \Phi(T), \\ E_{jj}\Phi(T) &= \lambda_j \Phi(T) \quad (j \geq 3). \end{aligned} \quad (1.26.1)$$

The function $F^{(1)}$ is a solution of the system of the differential equations (1.26.1) and the equation

$$\{(E_{ij} + \delta_{ij})E_{kl} - E_{kj}E_{il}\}\Phi(T) = 0. \quad (1.27)$$

For the partition $\alpha^{(2)}$, we can rewrite the differential equation (1.8) as follows.

$$\begin{aligned} (E_{11} + E_{22})\Phi(T) &= \lambda_1 \Phi(T), \\ E_{12}\Phi(T) &= \Phi(T), \\ (E_{33} + E_{44})\Phi(T) &= \lambda_3 \Phi(T), \\ E_{34}\Phi(T) &= \Phi(T), \\ E_{jj}\Phi(T) &= \lambda_j \Phi(T) \quad (j \geq 5), \end{aligned} \quad (1.26.2)$$

Then the function $F^{(2)}$ is a solution of the system of the differential equations (1.26.2), and

$$\{(E_{ij} + \delta_{ij})E_{kl} - E_{kj}E_{il}\}\Phi(T) = 0. \quad (1.27)$$

Namely, the confluent hypergeometric function is realized as a solution of the system of differential equations defined by the actions of $\mathfrak{gl}(n)$. The main idea

of this paper is to define the system q -difference equations defined by the action of $U_q(\mathfrak{gl}(n))$ instead of $\mathfrak{gl}(n)$, and realize q -confluent hypergeometric function, including the q -Kummer and the q -Bessel function, as a solution of this system. We here recall that the q -Kummer function is defined by the series

$${}_1\varphi_1\left(\begin{matrix} a \\ c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (-1)^n q^{n(n-1)/2}}{(c; q)_n (q; q)_n} z^n,$$

and the q -Bessel function is defined by the series

$$J_\nu(z; q) = z^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} {}_2\varphi_1\left(\begin{matrix} 0, 0 \\ q^{\nu+1} \end{matrix}; q, -z^2\right).$$

We will see that both are the solutions of the system of q -difference equations to be defined later.

2. Quantum Grassmannians

In this section, we consider the quantum analogue of the coordinate ring $A(M(2, n))$. It has a structure of left $U_q(\mathfrak{gl}(n))$ -module. Refer to [Nm], [Nk], and [NN].

We first recall the definition of $U_q(\mathfrak{gl}(n))$ briefly. The algebra $U_q(\mathfrak{gl}(n))$, a quantum analogue of the universal enveloping algebra $U(\mathfrak{gl}(n))$, is the Hopf algebra generated by e_j, f_j ($1 \leq j \leq n-1$) and $q^{\pm \epsilon_j}$ ($1 \leq j \leq n$). We use the notation $q^h = q^{a_1 \epsilon_1} \cdots q^{a_n \epsilon_n}$ for any linear combination $h = a_1 \epsilon_1 + \cdots + a_n \epsilon_n$ with integral coefficients. The relations for the generators are given by

$$\begin{aligned} q^0 &= 1, \quad q^h q^{h'} = q^{h+h'}, \\ q^h e_j q^{-h} &= q^{(h, \epsilon_j - \epsilon_{j+1})} e_j, \quad q^h f_j q^{-h} = q^{(h, -\epsilon_j + \epsilon_{j+1})} f_j \quad (1 \leq j \leq n-1), \\ e_i f_j - f_j e_i &= \delta_{i,j} \frac{q^{\epsilon_i - \epsilon_{i+1}} - q^{-\epsilon_i + \epsilon_{i+1}}}{q - q^{-1}} \quad (1 \leq i, j \leq n-1), \\ e_i e_j &= e_j e_i, \quad f_i f_j = f_j f_i \quad (|i-j| \geq 2), \\ e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0 \quad (|i-j| = 1), \\ f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 &= 0 \quad (|i-j| = 1), \end{aligned} \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the canonical symmetric bilinear form such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. The *coproduct* Δ , the *counit* ε , and the *antipode* S for the generators of

$U_q(\mathfrak{gl}(n))$ are given as

$$\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_j) &= e_j \otimes 1 + q^{\epsilon_j - \epsilon_{j+1}} \otimes e_j, \\
\Delta(f_j) &= f_j \otimes q^{-\epsilon_j + \epsilon_{j+1}} + 1 \otimes f_j, \\
\varepsilon(q^h) &= 1, \quad \varepsilon(e_j) = \varepsilon(f_j) = 0, \\
S(q^h) &= q^{-h}, \quad S(e_j) = -q^{-\epsilon_j + \epsilon_{j+1}} e_j, \quad S(f_j) = -f_j q^{\epsilon_j - \epsilon_{j+1}},
\end{aligned} \tag{2.2}$$

for $1 \leq j \leq n-1$.

Secondly, we denote the algebra which is generated by the noncommutative elements t_{rj} ($1 \leq r \leq 2$; $1 \leq j \leq n$) by $A_q(M(2, n))$. The commutation relations among the generators are listed as follows:

$$\begin{aligned}
t_{1i}t_{1j} &= qt_{1j}t_{1i}, \quad t_{1i}t_{2i} = qt_{2i}t_{1i}, \quad t_{1j}t_{2j} = qt_{2j}t_{1j}, \quad t_{2i}t_{2j} = qt_{2j}t_{2i}, \\
t_{1j}t_{2i} &= t_{2i}t_{1j}, \quad t_{1i}t_{2j} - t_{2j}t_{1i} = (q - q^{-1})t_{1j}t_{2i},
\end{aligned} \tag{2.3}$$

for $1 \leq i < j \leq n$. This commutation relation is often called the $Mat_q(2)$ -relations.

The algebra $A_q(M(2, n))$ has a structure of left $U_q(\mathfrak{gl}(n))$ -module which is determined by the action of generators q^{ϵ_i} , e_j , f_j ($1 \leq i \leq n$; $1 \leq j \leq n-1$) on $\{t_{rj}\}$ by the formulas

$$q^{\epsilon_i} \cdot t_{rj} = q^{\delta_{ij}} t_{rj}, \quad e_k \cdot t_{rj} = \delta_{k+1,j} t_{rk}, \quad f_k \cdot t_{rj} = \delta_{k,j} t_{rk+1}. \tag{2.4}$$

This action of $U_q(\mathfrak{gl}(n))$ extends to the whole $A_q(M(2, n))$ so that

- (a) $a \cdot 1 = \varepsilon(a)1$ for all $a \in U_q(\mathfrak{gl}(n))$;
- (b) if $a \in U_q(\mathfrak{gl}(n))$ and $\Delta(a) = \sum_i a'_i \otimes a''_i$, then

$$a \cdot (\varphi\psi) = \sum_i (a'_i \cdot \varphi) (a''_i \cdot \psi) \quad \text{for all } \varphi, \psi \in A_q(M(2, n)).$$

We next define the *quantum minor determinants* (or *quantum minors*) of degree 2, which are analogues of the Plücker coordinates, as follows:

$$\xi_{ij} = t_{1i}t_{2j} - qt_{2i}t_{1j} \quad (1 \leq i, j \leq n). \tag{2.5}$$

By the definition (2.5), we can easily check that

$$\xi_{ii} = 0$$

for $1 \leq i \leq n$, and

$$\xi_{ij} = (-q)^{-1} \xi_{ji}$$

for $i < j$. Analogously to the Plücker relations (1.11), these quantum minors satisfy the relations

$$\xi_{\alpha\beta_1}\xi_{\beta_2\beta_3} - q\xi_{\alpha\beta_2}\xi_{\beta_1\beta_3} + q^2\xi_{\alpha\beta_3}\xi_{\beta_1\beta_2} = 0 \quad (2.6)$$

for $1 \leq \alpha \leq n$ and $1 \leq \beta_1 < \beta_2 < \beta_3 \leq n$. By these relations we can see that the commutation relations among the quantum minors are the following:

$$\begin{aligned} \xi_{ri}\xi_{rj} &= q\xi_{rj}\xi_{ri} \quad (1 \leq r \leq n; 1 \leq i < j \leq n), \\ \xi_{ri}\xi_{si} &= q\xi_{si}\xi_{ri} \quad (1 \leq r < s \leq n; 1 \leq i \leq n), \end{aligned}$$

and, for $1 \leq r < i < s < j \leq n$,

$$\begin{aligned} \xi_{ri}\xi_{sj} &= q^2\xi_{sj}\xi_{ri}, \\ \xi_{rj}\xi_{is} &= \xi_{is}\xi_{ri}, \\ \xi_{rs}\xi_{ij} - \xi_{ij}\xi_{rs} &= (q - q^{-1})\xi_{rj}\xi_{is}. \end{aligned}$$

We denote by $\mathcal{A}$ the noncommutative subalgebra in $A_q(M(2, n))$, that is generated by the quantum minors of degree 2. It is a left $U_q(\mathfrak{gl}(n))$ -submodule of $A_q(M(2, n))$. In fact, the action of the generators of $U_q(\mathfrak{gl}(n))$ on the quantum minors is determined by

$$\begin{aligned} q^{\epsilon_r} \cdot \xi_{ij} &= q^{\delta_{ri} + \delta_{rj}} \xi_{ij}, \\ e_l \cdot \xi_{ij} &= \delta_{l+1i} \xi_{lj} + \delta_{l+1j} q^{(\epsilon_l - \epsilon_{l+1}, \epsilon_i)} \xi_{il}, \\ f_l \cdot \xi_{ij} &= \delta_{li} q^{-(\epsilon_l - \epsilon_{l+1}, \epsilon_j)} \xi_{l+1j} + \delta_{lj} q^{(\epsilon_l - \epsilon_{l+1}, \epsilon_i)} \xi_{l+1i}. \end{aligned} \quad (2.7)$$

In the following, the algebra $\mathcal{A}[\xi_{13}^{-1}]$ plays an important role. We call this algebra the *localization* of $\mathcal{A}$ relative to ξ_{13} . By the relation (2.6), we have

PROPOSITION 2.1. *The localization $\mathcal{A}[\xi_{13}^{-1}]$ is a left $U_q(\mathfrak{gl}(n))$ -module, and is generated by the quantum minors $\{\xi_{13}^{\pm 1}, \xi_{12}, \xi_{32}, \xi_{14}, \xi_{34}, \dots, \xi_{1n}, \xi_{3n}\}$.*

Now, for each matrix $\nu = (\nu_{rj})_{r=1,3; j=2,4,\dots,n}$ of nonnegative integers, we set

$$\xi^\nu = \xi_{3n}^{\nu_{3n}} \dots \xi_{34}^{\nu_{34}} \xi_{32}^{\nu_{32}} \xi_{1n}^{\nu_{1n}} \dots \xi_{14}^{\nu_{14}} \xi_{12}^{\nu_{12}}. \quad (2.8)$$

Then we have

$$\mathcal{A}[\xi_{13}^{-1}] = \bigoplus_{\mu \in \mathbb{Z}, \nu \in \mathbb{Z}_{\geq 0}} \mathbb{C} \xi^\nu \xi_{13}^\mu. \quad (2.9)$$

We define a $U_q(\mathfrak{gl}(n))$ -submodule of $\mathcal{A}[\xi_{13}^{-1}]$ by putting

$$\mathcal{S}_{2,n} = \bigoplus_{\nu \in (\mathbf{Z}_{\geq 0})^{2 \times (n-2)}} \mathbf{C} \xi^\nu \xi_{13}^{-|\nu|-1}, \quad (2.10)$$

where $|\nu| = \sum \nu_{rj}$. We denote the action of $a \in U_q(\mathfrak{gl}(n))$ on the element $s \in \mathcal{S}_{2,n}$ by $\bar{\rho}(a)s$.

3. The action of $U_q(\mathfrak{gl}(n))$ on the polynomial ring

In this section, we translate the action of $U_q(\mathfrak{gl}(n))$ on the noncommutative algebra $\mathcal{S}_{2,n}$ into the action on a commutative polynomial ring in terms of the q -difference operators.

Let $\mathbf{C}[x]$ be the commutative polynomial ring, in variables $x = (x_2^1, x_2^2, x_4^1, x_4^2, \dots, x_n^1, x_n^2)$, and, for each matrix $\nu = (\nu_{rj})_{r=1,3;j=2,4,\dots,n}$, put

$$x^\nu = x_2^{1\nu_{32}} x_4^{1\nu_{34}} \cdots x_n^{1\nu_{3n}} x_2^{2\nu_{12}} x_4^{2\nu_{14}} \cdots x_n^{2\nu_{1n}}. \quad (3.1)$$

We next fix arbitrarily a function $\phi : M(2, n-2; \mathbf{Z}_{\geq 0}) \rightarrow \mathbf{Z}$ of the form

$$\phi(\nu) = \sum_{r,s=1,3; i,j=2,4,\dots,n} \alpha_{r,s,i,j} \nu_{r,i} \nu_{s,j} + \beta_{r,j} \nu_{r,j} + c, \quad (3.2)$$

where $\nu = (\nu_{rj})_{r=1,3;j=2,4,\dots,n}$ is a $2 \times (n-2)$ -matrix. We define the isomorphism of vector space

$$\eta_\phi : \mathbf{C}[x] \rightarrow \mathcal{S}_{2,n}$$

by putting

$$\eta_\phi(x^\nu) = \xi^\nu \xi_{13}^{-|\nu|-1} q^{\phi(\nu)} \quad (3.3)$$

for all matrix $\nu \in M(2, n-2; \mathbf{Z}_{\geq 0})$. Then we can regard $\mathbf{C}[x]$ a left $U_q(\mathfrak{gl}(n))$ -module; namely,

$$\rho_\phi(a) = \eta_\phi^{-1} \circ \rho(a) \circ \eta_\phi \quad (3.4)$$

for all $a \in U_q(\mathfrak{gl}(n))$.

Let us denote by $q^{\theta_{r,i}}$ the q -shift operator in the variable x_i^r :

$$q^{\theta_{r,i}} f(x_2^1, \dots, x_n^2) = f(x_2^1, \dots, q x_1^r, \dots, x_n^2). \quad (3.5)$$

Then the action of generators of $U_q(\mathfrak{gl}(n))$ in the case of $\phi(\nu) \equiv 0$ is given as follows:

$$\begin{aligned}
\rho_0(q^{\epsilon_1}) &= q^{-1-\theta_{1,2}-\theta_{1,4\dots n}}, \\
\rho_0(q^{\epsilon_3}) &= q^{-1-\theta_{2,2}-\theta_{2,4\dots n}}, \\
\rho_0(q^{\epsilon_j}) &= q^{\theta_{1,j}+\theta_{2,j}} \quad (j = 2, 4, \dots, n), \\
(q - q^{-1})\rho_0(e_1) &= \frac{1}{x_2^2}(1 - q^{2\theta_{1,2}})q^{-\theta_{1,2}-\theta_{2,2}+\theta_{2,4\dots n}+1}, \\
(q - q^{-1})\rho_0(f_1) &= \sum_{j=4}^n \frac{x_2^2 x_j^1}{x_j^2} (q^{2\theta_{2,j}} - 1) q^{2(\theta_{1,2}+\theta_{1,4\dots j-1})+\theta_{1,j\dots n}+2(\theta_{2,2})+\theta_{2,j+1\dots n}+2} \\
&\quad - x_2^1 (1 - q^{2(\theta_{1,2}+\theta_{1,4\dots n}+\theta_{2,2})+2}) q^{-\theta_{1,2}-\theta_{1,4\dots n}-\theta_{2,4\dots n}-2}, \\
(q - q^{-1})\rho_0(e_2) &= x_2^2 (1 - q^{2(\theta_{1,2}+\theta_{2,2}+\theta_{2,4\dots n})+2}) q^{\theta_{2,2}+\theta_{1,4\dots n}+1} \\
&\quad - \sum_{j=4}^n \frac{x_2^1 x_j^2}{x_j^1} (q^{2\theta_{1,j}} - 1) q^{-\theta_{1,2}-\theta_{1,j\dots n}+\theta_{2,2}-\theta_{2,4\dots j}-2}, \\
(q - q^{-1})\rho_0(f_2) &= \frac{1}{x_2^2} (q^{2\theta_{2,2}} - 1) q^{-\theta_{1,2}-\theta_{1,4\dots n}-3\theta_{2,2}-\theta_{2,4\dots n}}, \\
(q - q^{-1})\rho_0(e_3) &= -\frac{1}{x_4^2} (1 - q^{2\theta_{2,4}}) q^{\theta_{1,2}+\theta_{1,5\dots n}-\theta_{2,2}-\theta_{2,4}}, \\
(q - q^{-1})\rho_0(f_3) &= x_4^2 (1 - q^{2(\theta_{1,4}+\theta_{2,2}+\theta_{2,4\dots n})+2}) q^{-\theta_{1,2}-\theta_{1,4\dots n}-\theta_{2,4\dots n}-1} \\
&\quad + \frac{x_4^1 x_2^2}{x_2^1} (1 - q^{2\theta_{1,2}}) q^{\theta_{1,4\dots n}+2(\theta_{2,2}+\theta_{2,4})+\theta_{2,5\dots n}+3} \\
&\quad + \sum_{j \geq 5} \frac{x_4^1 x_j^2}{x_j^1} (1 - q^{2\theta_{1,j}}) q^{-(\theta_{1,2}+\theta_{1,j\dots n})+2\theta_{2,2}+\theta_{2,4}+\theta_{2,j+1\dots n}}, \\
(q - q^{-1})\rho_0(e_j)_{j \geq 4} &= \frac{x_j^1}{x_{j+1}^1} (q^{\theta_{1,j+1}} - q^{-\theta_{1,j+1}}) + \frac{x_j^2}{x_{j+1}^2} (q^{\theta_{2,j+1}} - q^{-\theta_{2,j+1}}) q^{\theta_{1,j}-\theta_{1,j+1}}, \\
(q - q^{-1})\rho_0(f_j)_{j \geq 4} &= \frac{x_{j+1}^1}{x_j^1} (q^{\theta_{1,j}} - q^{-\theta_{1,j}}) q^{\theta_{2,j+1}-\theta_{2,j}} + \frac{x_{j+1}^2}{x_j^2} (q^{\theta_{2,j}} - q^{-\theta_{2,j}}),
\end{aligned}$$

where $\theta_{r,i\dots j} = \theta_{r,i} + \theta_{r,i+1} + \dots + \theta_{r,j}$. When ϕ' and ϕ'' are functions of degree ≤ 2 , we can easily see that

$$\rho_{\phi'}(a) = q^{-(\phi' - \phi'')(\theta)} \circ \rho_{\phi''}(a) \circ q^{(\phi' - \phi'')(\theta)} \quad (3.6)$$

for all $a \in U_q(\mathfrak{gl}(n))$ and each matrices $\theta = (\theta_{r,j})_{r=1,2; j=2,4,\dots,n}$. Thus we can

translate the action of $U_q(\mathfrak{gl}(n))$ on the noncommutative algebra to the action on the polynomial ring $\mathbb{C}[x]$.

4. The q -confluent hypergeometric functions

In this section, we introduce some q -difference operators corresponding to the differential operator (1.25), and define two types of systems of q -difference equations by using the action of these elements. We further realize q -confluent hypergeometric functions as solutions of these systems.

As a q -analogue of the differential operator (1.25) in the case of $i = j$, $k = l = j + 1$; namely,

$$(E_{jj} + 1)E_{j+1j+1} - E_{j+1j}E_{jj+1},$$

we consider the elements C_j in $U_q(\mathfrak{gl}(n))$ defined by

$$(q - q^{-1})^2 C_j = (q^{1+\epsilon_j} - q^{-1-\epsilon_j})(q^{\epsilon_{j+1}} - q^{-\epsilon_{j+1}}) - (q - q^{-1})^2 f_j e_j \quad (4.1)$$

for $1 \leq j \leq n - 1$. Each of them is the Casimir elements of the subalgebra of $U_q(\mathfrak{gl}(n))$ generated by $\{q^{\epsilon_j}, q^{\epsilon_{j+1}}, e_j, f_j\}$, isomorphic to $U_q(\mathfrak{gl}(2))$. We shall also define later the q -difference operators corresponding to order differential operators of (1.25).

Now, in view of (3.6) and the formulas at the end of the previous section, we define

$$\begin{aligned} \phi_0(\nu) &= \sum_{j=4}^{n-1} \nu_j^3 (\nu_{j+1}^1 + \cdots + \nu_n^1) + 2\nu_2^1 (\nu_4^3 + \cdots + \nu_n^3) \\ &\quad - (\nu_2^3 + 1)(\nu_4^1 + \cdots + \nu_n^1 + 1). \end{aligned} \quad (4.2)$$

For $\phi = \phi_0$, we can see that the actions of $\{C_j\}_{1 \leq j \leq n-1}$ are given as follows:

$$\begin{aligned} &(q - q^{-1})^2 \rho_{\phi_0}(C_1) \\ &= \sum_{j=4}^n q^{\theta_{1,4} + \cdots + \theta_{1,j-1} - \theta_{1,j} - \cdots - \theta_{1,n} - \theta_{2,2}} \\ &\quad \times \left\{ \frac{x_j^1 x_2^2}{x_2^1 x_j^2} (1 - q^{2\theta_{1,2}})(1 - q^{2\theta_{2,j}}) - (1 - q^{2\theta_{2,2}})(1 - q^{2\theta_{2,j}}) \right\}, \\ &(q - q^{-1})^2 \rho_{\phi_0}(C_2) \\ &= \sum_{j=4}^n q^{-\theta_{2,2} + \cdots + \theta_{2,j} + \theta_{2,j+1} - \cdots - \theta_{2,n}} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{x_2^1 x_j^2}{x_2^2 x_j^1} (1 - q^{2\theta_{1,j}})(1 - q^{2\theta_{2,2}}) - (1 - q^{2\theta_{1,2}})(1 - q^{2\theta_{2,j}}) \right\}, \\
& (q - q^{-1})^2 \rho_{\phi_0}(C_3) \\
& = q^{-\theta_{1,4} - \theta_{2,2} - \theta_{2,5} \dots n} \\
& \times \left\{ \frac{x_4^1 x_2^2}{x_2^1 x_4^2} (1 - q^{2\theta_{2,2}})(1 - q^{2\theta_{2,4}}) - (1 - q^{2\theta_{1,4}})(1 - q^{2\theta_{2,2}}) \right\}, \\
& + \sum_{j=5}^n q^{-\theta_{1,4} + \theta_{2,2} - \theta_{2,5} \dots j + \theta_{2,j+1} \dots n} \\
& \times \left\{ \frac{x_4^1 x_j^2}{x_j^1 x_4^2} (1 - q^{2\theta_{1,j}})(1 - q^{2\theta_{2,4}}) - (1 - q^{2\theta_{1,4}})(1 - q^{2\theta_{2,j}}) \right\}, \\
& (q - q^{-1})^2 \rho_{\phi_0}(C_j)_{j \geq 4} \\
& = q^{-\theta_{1,j} - \theta_{1,j+1} - \theta_{2,j} + \theta_{2,j+1}} \\
& \times \left\{ -\frac{x_{j+1}^1 x_j^2}{x_{j+1}^2 x_j^1} (1 - q^{2\theta_{1,j}})(1 - q^{2\theta_{2,j+1}}) + (1 - q^{2\theta_{1,j+1}})(1 - q^{2\theta_{2,j}}) \right\} \\
& + q^{-1 - \theta_{1,j} - \theta_{1,j+1} - \theta_{2,j} - \theta_{2,j+1}} \\
& \times \left\{ -\frac{x_j^1 x_{j+1}^2}{x_j^2 x_{j+1}^1} (1 - q^{2\theta_{1,j+1}})(1 - q^{2\theta_{2,j}}) + (1 - q^{2\theta_{1,j}})(1 - q^{2\theta_{2,j+1}}) \right\}.
\end{aligned}$$

Note that all the actions of Casimir elements include q -difference operators which we can regard as analogues of Laplacian such as

$$\left\{ -\frac{x_j^1 x_{j+1}^2}{x_j^2 x_{j+1}^1} (1 - q^{2\theta_{1,j+1}})(1 - q^{2\theta_{2,j}}) + (1 - q^{2\theta_{1,j}})(1 - q^{2\theta_{2,j+1}}) \right\}.$$

This fact will turn out to show the advantage of our choice of ϕ_0 . We introduce the following two systems of q -difference equations, on the commutative polynomial ring $\mathbf{C}[x]$:

$\overline{E}_1(\lambda, \phi)$:

$$\rho_\phi(q^{\epsilon_1 + \epsilon_2})G(x) = q^{\lambda_1}G(x), \tag{4.3}$$

$$\rho_\phi(q^{\epsilon_j})G(x) = q^{\lambda_j}G(x) \quad (j = 3, \dots, n),$$

$$\rho_\phi(q^{-\epsilon_1}e_1)G(x) = q^{-\lambda_1}G(x), \tag{4.4}$$

$$\rho_\phi(C_j)G(x) = 0, \tag{4.5}$$

$\overline{E}_2(\lambda, \phi)$:

$$\begin{aligned}\rho_\phi(q^{\epsilon_1+\epsilon_2})G(x) &= q^{\lambda_1}G(x), \\ \rho_\phi(q^{\epsilon_3+\epsilon_4})G(x) &= q^{\lambda_3}G(x), \\ \rho_\phi(q^{\epsilon_j})G(x) &= q^{\lambda_j}G(x) \quad (j = 5, \dots, n),\end{aligned}\tag{4.6}$$

$$\begin{aligned}\rho_\phi(q^{-\epsilon_1}e_1)G(x) &= q^{-\lambda_1}G(x), \\ \rho_\phi(q^{-\epsilon_3}e_3)G(x) &= q^{\lambda_5+\dots+\lambda_n}G(x),\end{aligned}\tag{4.7}$$

$$\rho_\phi(C_j)G(x) = 0.\tag{4.5}$$

See (1.26.1), (1.26.2), and (1.27) for this description. Both systems are parametrized by $\lambda = (\lambda_1, \dots, \lambda_n)$. The construction of solutions of these systems can be done by referring to (1.20) and (1.22). Namely, we define two q -hypergeometric functions. One is $\varphi^{(1)}$ given by

$$\begin{aligned}\varphi^{(1)}\left(\begin{matrix} \alpha; \beta_5, \dots, \beta_n \\ \gamma \end{matrix}, q; y\right) \\ = \sum_{\nu_j \geq 0} \frac{(\alpha; q)_{|\nu|} (\beta_5; q)_{\nu_5} \dots (\beta_n; q)_{\nu_n} (-1)^{\nu_2}}{(\gamma; q)_{|\nu|} (q; q)_{\nu_2} (q; q)_{\nu_5} \dots (q; q)_{\nu_n}} q^{\nu_2(\nu_2-1)/2} y_2^{\nu_2} y_5^{\nu_5} \dots y_n^{\nu_n},\end{aligned}\tag{4.8}$$

in the coordinates $y = (y_2, y_5, \dots, y_n)$. Recall that $(a; q)_n = (1-a)(1-aq) \dots (1-aq^{n-1})$. The other is

$$\begin{aligned}\varphi^{(2)}\left(\begin{matrix} \beta_5, \dots, \beta_n \\ \gamma \end{matrix}, q; z\right) \\ = \sum_{\nu_j \geq 0} \frac{(\beta_5; q)_{\nu_5} \dots (\beta_n; q)_{\nu_n}}{(\gamma; q)_{|\nu|} (q; q)_{\nu_2} (q; q)_{\nu_5} \dots (q; q)_{\nu_n}} z_4^{\nu_4} z_5^{\nu_5} \dots z_n^{\nu_n},\end{aligned}\tag{4.9}$$

in the coordinates $z = (z_4, \dots, z_n)$. We can easily check that they have the formal limits

$$\begin{aligned}\lim_{q \rightarrow 1} \varphi^{(1)}\left(\begin{matrix} q^a; q^{b_5}, \dots, q^{b_n} \\ q^c \end{matrix}, q; (q-1)y_2, y_5, \dots, y_n\right) \\ = F^{(1)}\left(\begin{matrix} a; b_5, \dots, b_n \\ c \end{matrix}; y\right),\end{aligned}\tag{4.10}$$

and

$$\begin{aligned} \lim_{q \rightarrow 1} \varphi^{(2)} \left(\begin{matrix} q^{b_5}, \dots, q^{b_n} \\ q^c \end{matrix}, q; (1-q)^2 z_4, (1-q) z_5, \dots, (1-q) z_n \right) \\ = F^{(2)} \left(\begin{matrix} b_5, \dots, b_n \\ c \end{matrix}; z \right). \end{aligned} \quad (4.11)$$

We next define the q -analogue of the exponential function as follows:

$$e_q(x) = \frac{1}{((1-q)x; q)_\infty}, \quad (4.12)$$

where $(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n$. It satisfies the q -difference equation $\frac{1}{x} \frac{(1-q^x)}{(1-q)} u = u$, which is a q -analogue of the differential equation $du/dx = u$.

We can now define the analogues of (1.19) and (1.21); we put

$$\begin{aligned} G^{(1)}(x; \lambda) &= x_4^{1^{\lambda_3+\lambda_4+1}} x_4^{2^{-\lambda_3-1}} e_{q^2}(x_1) \prod_{j \geq 5} (x_j^1)^{\lambda_j} \\ &\times \varphi^{(1)} \left(\begin{matrix} q^{2\lambda_3+2}, q^{-2\lambda_5}, \dots, q^{-2\lambda_n} \\ q^{2\lambda_3+2\lambda_4+4} \end{matrix}; q^2, (1-q^2)y_2, y_5, \dots, y_n \right), \end{aligned} \quad (4.13)$$

where $y_2 = -\frac{x_1^1 x_3}{x_2^2}$, $y_j = -\frac{x_j^2 x_4^1}{x_j^1 x_4^2}$ ($j \geq 5$), and

$$\begin{aligned} G^{(2)}(x; \lambda) &= x_2^{2\lambda_1+1} e_{q^2}(x_2^1) e_{q^2}(x_4^2) \prod_{j \geq 5} (x_j^2)^{\lambda_j} \\ &\times \varphi^{(2)} \left(\begin{matrix} q^{-2\lambda_5}, \dots, q^{-2\lambda_n} \\ q^{2\lambda_3+4} \end{matrix}; q^2, (1-q^2)^2 z_4, (1-q^2) z_5, \dots, (1-q^2) z_n \right), \end{aligned} \quad (4.14)$$

where $z_4 = x_2^2 x_4^1$, $z_j = \frac{x_j^1 x_2^2}{x_j^2}$,

$$\phi_1(\nu) = \phi_0(\nu) - \nu_4^1(\nu_4^1 + 1) - \sum_{j \geq 5} \nu_j^3(\nu_j^3 + 1) - \nu_2^1(\nu_2^1 - 1), \quad (4.15)$$

and

$$\phi_2(\nu) = \phi_0(\nu) - \sum_{j \geq 5} \nu_j^1(\nu_j^1 + 1). \quad (4.16)$$

The result of this section is the following theorem that can be seen by a straightforward computation.

THEOREM 4.1. The functions $G^{(1)}$ and $G^{(2)}$ satisfy $\overline{E}_1(\lambda, \phi_1)$ and $\overline{E}_2(\lambda, \phi_2)$, respectively.

Let us check the relation of the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ with the q -Kummer and the q -Bessel function. Generally, q -hypergeometric series ${}_r\varphi_s$ is defined as follows:

$$\begin{aligned} {}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x \right) & \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n (q; a)_n} ((-1)^n q^{n(n-1)/2})^{s-r+1} x^n. \end{aligned} \quad (4.17)$$

The q -Kummer function is a special case of this series:

$${}_1\varphi_1 \left(\begin{matrix} a \\ c \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (-1)^n q^{n(n-1)/2}}{(c; q)_n (q; q)_n} z^n, \quad (4.18)$$

Then, it is easy to see

$$\varphi^{(1)} \left(\begin{matrix} a \\ c \end{matrix}; q, y_2 \right) = {}_1\varphi_1 \left(\begin{matrix} a \\ c \end{matrix}; q, y_2 \right). \quad (4.19)$$

As for q -Bessel functions, we know several definitions. Here we treat a q -Bessel function which was determined in [Ja]

$$J_v(x; q) = z^v \frac{(q^{v+1}; q)_{\infty}}{(q; q)_{\infty}} {}_2\varphi_1 \left(\begin{matrix} 0, 0 \\ q^{v+1} \end{matrix}; q, -x^2 \right). \quad (4.20)$$

See [GR, Ja, K]. Similarly to the q -Kummer function, we have

$$\varphi^{(2)} \left(\begin{matrix} - \\ c \end{matrix}; q, z_4 \right) = C z_4^{-(\lambda_1+1)/2} J_{\lambda_1+1}((z_4)^{1/2}; q). \quad (4.21)$$

Here $C = \frac{(q^2; q^2)_{\infty}}{(q^{2\lambda_3+2}; q^2)_{\infty}}$. These relations show that we have realized the q -Kummer and the q -Bessel functions as solutions of the systems of q -difference equations defined above.

5. The contiguity relations

We call linear relation among several q -hypergeometric functions whose parameters are shifted the *contiguity relations*. In this section, we compute such relations for q -confluent hypergeometric function. These relations will be used in the next section to define q -confluent hypergeometric systems.

For the sake of simplicity, we write

$$\varphi_\lambda^{(1)} = \varphi^{(1)} \left(\begin{matrix} q^{2\lambda_3+2}, q^{-2\lambda_5}, \dots, q^{-2\lambda_n} \\ q^{2\lambda_3+2\lambda_4+4} \end{matrix}; q^2, (1-q^2)y_2, y_5, \dots, y_n \right) \quad (5.1)$$

and

$$\varphi_\lambda^{(2)} = \varphi^{(2)} \left(\begin{matrix} q^{-2\lambda_5}, \dots, q^{-2\lambda_n} \\ q^{2\lambda_3+4} \end{matrix}; q^2, (1-q^2)^2 z_4, (1-q^2)z_5, \dots, (1-q^2)z_n \right). \quad (5.2)$$

For the element $a \in U_q(\mathfrak{gl}(n))$, we define the operators $\pi_{1\lambda}(a)$ and $\pi_{2\lambda}(a)$ for $a \in U_q(\mathfrak{gl}(n))$ by the following identities.

$$\rho_{\phi_1}(a)G^{(1)} = (x_4^1)^{\lambda'_3+\lambda'_4+1}(x_4^2)^{-\lambda'_3-1}e_{q^2} \prod_{j \geq 5} (x_j^1)^{\lambda'_j} \pi_{1\lambda}(a) \varphi_{\lambda'}^{(1)} \quad (5.3)$$

and

$$\rho_{\phi_2}(a)G^{(2)} = (x_2^2)^{\lambda'_1+1}e_{q^2}(x_2^1)e_{q^2}(x_4^2) \prod_{j \geq 5} (x_j^2)^{\lambda'_j} \pi_{2\lambda}(a) \varphi_{\lambda'}^{(2)}. \quad (5.4)$$

To state the contiguity relations we need the q -analogue of the root vectors E_{ij} of $\mathfrak{gl}(n)$; they are the elements $\widehat{E}_{ij} \in U_q(\mathfrak{gl}(n))$ defined inductively by the following formulas.

$$\begin{aligned} \widehat{E}_{jj+1} &= e_j, & \widehat{E}_{j+1j} &= f_j, \\ \widehat{E}_{ij} &= \widehat{E}_{ik}\widehat{E}_{kj} - q^{\pm 1}\widehat{E}_{kj}\widehat{E}_{ik} & (i \leq k \leq j). \end{aligned}$$

Now we have next two propositions.

PROPOSITION 5.1. *The following is the list of contiguity relations for the q -hypergeometric function $\varphi_\lambda^{(1)}$:*

$$\begin{aligned} \pi_{1\lambda}(q^{\epsilon_1+\epsilon_2})\varphi_\lambda^{(1)} &= q^{\lambda_1}\varphi_\lambda^{(1)} \\ \pi_{1\lambda}(q^{\epsilon_j})\varphi_\lambda^{(1)} &= q^{\lambda_j}\varphi_\lambda^{(1)} \quad (j \geq 3) \\ \pi_{1\lambda}(q^{-\epsilon_1}\widehat{E}_{12})\varphi_\lambda^{(1)} &= q^{-\lambda_1-1}\varphi_\lambda^{(1)} \\ \pi_{1\lambda}(q^{-\epsilon_1}\widehat{E}_{13})\varphi_\lambda^{(1)} &= q^{-\lambda_3+\lambda_4+2}[\lambda_3+\lambda_4+1]\varphi_{\lambda+\alpha_{1,3}}^{(1)} \\ \pi_{1\lambda}(q^{-\epsilon_1}\widehat{E}_{14})\varphi_\lambda^{(1)} &= -q^{\lambda_4+\dots+\lambda_n+1}[\lambda_3+\lambda_4+1]\varphi_{\lambda+\alpha_{1,4}}^{(1)} \\ \pi_{1\lambda}(q^{-\epsilon_1}\widehat{E}_{1j})\varphi_\lambda^{(1)} &= q^{-\lambda_3-\lambda_j+\lambda_{j+1}+\dots+\lambda_n}[\lambda_j]\varphi_{\lambda+\alpha_{1,j}}^{(1)} \quad (j \geq 5) \end{aligned}$$

$$\begin{aligned}
\pi_{1\lambda}(q^{\epsilon_2}\widehat{E}_{32})\varphi_\lambda^{(1)} &= q^{2\lambda_3+\lambda_5+\dots+\lambda_n}\frac{[\lambda_3+1]}{[\lambda_3+\lambda_4+2]}\varphi_{\lambda+\alpha_{3,1}}^{(1)} \\
\pi_{1\lambda}(q^{\epsilon_2}\widehat{E}_{42})\varphi_\lambda^{(1)} &= -q^{\lambda_3-2}\frac{[\lambda_4+1]}{[\lambda_3+\lambda_4+2]}\varphi_{\lambda+\alpha_{4,1}}^{(1)} \\
\pi_{1\lambda}(q^{\epsilon_2}\widehat{E}_{j2})\varphi_\lambda^{(1)} &= -q^{2\lambda_3-\tau_{3,j}+2\lambda_j+1}\varphi_{\lambda+\alpha_{j,1}}^{(1)} \quad (j \geq 5) \\
\pi_{1\lambda}(\widehat{E}_{34})\varphi_\lambda^{(1)} &= q^{2\lambda_3+\lambda_5+\dots+\lambda_n+1}[\lambda_3+1]\varphi_{\lambda+\alpha_{3,4}}^{(1)} \\
\pi_{1\lambda}(\widehat{E}_{3j})\varphi_\lambda^{(1)} &= q^{\lambda_3-\lambda_4-\lambda_j+\lambda_{j+1}+\dots+\lambda_n-1}\frac{[\lambda_3+1][\lambda_j]}{[\lambda_3+\lambda_4+2]}\varphi_{\lambda+\alpha_{3,j}}^{(1)} \quad (j \geq 5) \\
\pi_{1\lambda}(\widehat{E}_{43})\varphi_\lambda^{(1)} &= q^{-2\lambda_3-\lambda_5-\dots-\lambda_n+1}[\lambda_4+1]\varphi_{\lambda+\alpha_{4,3}}^{(1)} \\
\pi_{1\lambda}(\widehat{E}_{j3})\varphi_\lambda^{(1)} &= q^{-\lambda_3+\lambda_4+\lambda_j-\lambda_{j+1}-\dots-\lambda_n}[\lambda_3+\lambda_4+1]\varphi_{\lambda+\alpha_{j,3}}^{(1)} \quad (j \geq 5) \\
\pi_{1\lambda}(\widehat{E}_{4j})\varphi_\lambda^{(1)} &= q^{-\lambda_3+\tau_{4,j}-2\lambda_j-1}\frac{[\lambda_4+1][\lambda_j]}{[\lambda_3+\lambda_4+2]}\varphi_{\lambda+\alpha_{4,j}}^{(1)} \quad (j \geq 5) \\
\pi_{1\lambda}(\widehat{E}_{j4})\varphi_\lambda^{(1)} &= q^{\lambda_3-\tau_{4,j}+2\lambda_j+3}[\lambda_3+\lambda_4+1]\varphi_{\lambda+\alpha_{j,4}}^{(1)} \quad (j \geq 5) \\
\pi_{1\lambda}(\widehat{E}_{ij})\varphi_\lambda^{(1)} &= q^{2\lambda_i-2\lambda_j+\tau_{i,j}+2}[\lambda_j]\varphi_{\lambda+\alpha_{i,j}}^{(1)} \quad (5 \leq i < j) \\
\pi_{1\lambda}(\widehat{E}_{ji})\varphi_\lambda^{(1)} &= q^{-2\lambda_i+2\lambda_j-\tau_{i,j}+2}[\lambda_i]\varphi_{\lambda+\alpha_{j,i}}^{(1)} \quad (5 \leq i < j),
\end{aligned}$$

where

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

Here $\alpha_{i,j}$ and $\tau_{i,j}$ are determined by

$$\lambda + \alpha_{i,j} = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots, \lambda_n)$$

for $1 \leq i \neq j \leq n$, and

$$\tau_{i,j} = \begin{cases} -\lambda_{i+1} - \dots - \lambda_{j-1} & (j - i > 1) \\ 0 & (j - i = 1) \end{cases}$$

for $1 \leq i < j \leq n$.

PROPOSITION 5.2. *The following is the list of contiguity relations for the q -hypergeometric function $\varphi_\lambda^{(2)}$:*

$$\pi_{2\lambda}(q^{\epsilon_1+\epsilon_2})\varphi_\lambda^{(2)} = q^{\lambda_1}\varphi_\lambda^{(2)}$$

$$\begin{aligned}
\pi_{2\lambda}(q^{\epsilon_3+\epsilon_4})\varphi_\lambda^{(2)} &= q^{\lambda_3}\varphi_\lambda^{(2)} \\
\pi_{2\lambda}(q^{\epsilon_j})\varphi_\lambda^{(2)} &= q^{\lambda_j}\varphi_\lambda^{(2)} \quad (j \geq 5) \\
\pi_{2\lambda}(q^{-\epsilon_1}\widehat{E}_{12})\varphi_\lambda^{(2)} &= q^{-\lambda_1-1}\varphi_\lambda^{(2)} \\
\pi_{2\lambda}(q^{-\epsilon_3}\widehat{E}_{34})\varphi_\lambda^{(2)} &= q^{\lambda_5+\dots+\lambda_n}\varphi_\lambda^{(2)} \\
\pi_{2\lambda}(q^{-\epsilon_1-\epsilon_4}\widehat{E}_{14})\varphi_\lambda^{(2)} &= -q^{2\lambda_5+\dots+2\lambda_n}\frac{1}{[\lambda_1+2]}\varphi_{\lambda+\alpha_{1,3}}^{(2)} \\
\pi_{2\lambda}(q^{-\epsilon_1-\epsilon_4}\widehat{E}_{1j})\varphi_\lambda^{(2)} &= -q^{\tau_{4,j}+2\lambda_{j+1}+\dots+2\lambda_n}\frac{[\lambda_j]}{[\lambda_1+2]}\varphi_{\lambda+\alpha_{1,j}}^{(2)} \quad (j \geq 5) \\
\pi_{2\lambda}(q^{-\epsilon_3}\widehat{E}_{3j})\varphi_\lambda^{(2)} &= q^{-\lambda_j+\lambda_{j+1}+\dots+\lambda_n}[\lambda_j]\varphi_{\lambda+\alpha_{3,j}}^{(2)} \quad (j \geq 5) \\
\pi_{2\lambda}(q^{\epsilon_2+\epsilon_4}\widehat{E}_{32})\varphi_\lambda^{(2)} &= q^{\lambda_3+2}[\lambda_1+1]\varphi_{\lambda+\alpha_{3,1}}^{(2)} \\
\pi_{2\lambda}(q^{\epsilon_2+\epsilon_4}\widehat{E}_{j2})\varphi_\lambda^{(2)} &= q^{-\lambda_1+\tau_{4,j}-2\lambda_{j+1}-\dots-2\lambda_n+1}[\lambda_1+1]\varphi_{\lambda+\alpha_{j,1}}^{(2)} \quad (j \geq 5) \\
\pi_{2\lambda}(q^{\epsilon_4}\widehat{E}_{j4})\varphi_\lambda^{(2)} &= -q^{-\tau_{4,j}+2\lambda_j+1}\varphi_{\lambda+\alpha_{j,3}}^{(2)} \quad (j \geq 5) \\
\pi_{2\lambda}(\widehat{E}_{ij})\varphi_\lambda^{(2)} &= q^{2\lambda_i-2\lambda_j+\tau_{i,j}+2}[\lambda_j]\varphi_{\lambda+\alpha_{i,j}}^{(2)} \quad (5 \leq i < j) \\
\pi_{2\lambda}(\widehat{E}_{ji})\varphi_\lambda^{(2)} &= q^{-2\lambda_i+2\lambda_j-\tau_{i,j}+2}[\lambda_i]\varphi_{\lambda+\alpha_{j,i}}^{(2)} \quad (5 \leq i < j).
\end{aligned}$$

Now we define two subalgebras U_1 and U_2 of the algebra $U_q(\mathfrak{gl}(n))$ by

$$\begin{aligned}
U_1 &= \mathbf{C}\{q^{\epsilon_j}, e_j, f_j\} \quad (j \geq 3) \\
U_2 &= \mathbf{C}\{q^{\epsilon_j}, e_j, f_j\} \quad (j \geq 5).
\end{aligned}$$

They are isomorphic to $U_q(\mathfrak{gl}(n-2))$ and $U_q(\mathfrak{gl}(n-4))$, respectively. Then the propositions above particularly imply the following.

COROLLARY 5.3. *The mapping $\pi_{1\lambda}$ (resp. $\pi_{2\lambda}$) gives a representation of the algebra U_1 (resp. U_2) on the space of functions $\{\varphi_\lambda^1; \lambda \in \lambda_0 + \mathbf{Z}\}$ (resp. $\{\varphi_\lambda^2; \lambda \in \lambda_0 + \mathbf{Z}\}$) for each fixed λ_0 .*

We next examine the algebraic structure of the set of root vectors that are not in the subalgebras U_1 or U_2 . Define subsets of $U_q(\mathfrak{gl}(n))$ by

$$\begin{aligned}
H^1 &= \mathbf{C}\{q^{\epsilon_1+\epsilon_2}, q^{-\epsilon_1}\widehat{E}_{12}\}, \\
\Lambda_1^1 &= \mathbf{C}\{q^{-\epsilon_1}\widehat{E}_{1j}\}_{3 \leq j \leq n}, \\
\Lambda_2^1 &= \mathbf{C}\{q^{\epsilon_2}\widehat{E}_{j2}\}_{3 \leq j \leq n}.
\end{aligned}$$

and put

$$\begin{aligned} X_{1j} &= q^{-\epsilon_1} \widehat{E}_{1j} & (2 \leq j \leq n), \\ X_{j2} &= q^{\epsilon_2} \widehat{E}_{j2} & (3 \leq j \leq n), \\ X_{ij} &= \widehat{E}_{ij} & (3 \leq i \neq j \leq n). \end{aligned}$$

Then we can see the following commutation relations:

$$\begin{aligned} H^1 &\text{ is abelian,} \\ [X_{1i}, X_{1j}]_{q^{-1}} &= 0 & (X_{1j} \in \Lambda_1^1, 3 \leq i < j \leq n), \\ [X_{i2}, X_{j2}]_q &= 0 & (X_{j2} \in \Lambda_2^1, 3 \leq i < j \leq n), \\ U_1 &\text{ is isomorphic to } U_q(\mathfrak{gl}(n-2)), \end{aligned}$$

where

$$[A, B]_q = AB - qBA, \quad [A, B]_{q^{-1}} = AB - q^{-1}BA.$$

In the following we use the notation

$$[A, B]_{q^2} = AB - q^2BA.$$

The contiguity relations in Proposition 5.1 imply the commutation relations among these subsets. We give a list of them as follows: we list, in the left side, subsets concerned and, in the right side, the explicit commutation relations.

$$\begin{aligned} H^1, \text{ other: } & \pi_{1\lambda}([q^{\epsilon_1+\epsilon_2}, X_{1j}]_q) = 0 & (X_{1j} \in \Lambda_1^1) \\ & \pi_{1\lambda}([X_{12}, X_{1j}]_{q^{-1}}) = 0 & (X_{1j} \in \Lambda_1^1) \\ & \pi_{1\lambda}([q^{\epsilon_1+\epsilon_2}, X_{j2}]_{q^{-1}}) = 0 & (X_{j2} \in \Lambda_2^1) \\ & \pi_{1\lambda}([X_{12}, X_{j2}]_q) = 0 & (X_{j2} \in \Lambda_2^1) \\ & \pi_{1\lambda}([H^1, U_1]) = 0 \\ \Lambda_1^1, \Lambda_2^1 : & \pi_{1\lambda}([X_{1i}, X_{j2}]_{q^2}) = 0 & (3 \leq i < j \leq n) \\ & \pi_{1\lambda}([X_{1j}, X_{j2}]) = q^{-\lambda_1+\lambda_j-1} \varphi_\lambda^{(1)} & (3 \leq j \leq n) \\ & \pi_{1\lambda}([X_{1j}, X_{i2}]_q) = 0 & (3 \leq i < j \leq n) \end{aligned}$$

$$\begin{aligned}
\Lambda_1^1, U_1 : \quad & \pi_{1\lambda}([X_{1r}, X_{ij}]) = 0 & (r < (i, j) \text{ or } (i, j) < r) \\
& \pi_{1\lambda}([X_{1i}, X_{ij}]_q) = \pi_{1\lambda}(X_{1j}) & (3 \leq i < j \leq n) \\
& \pi_{1\lambda}([X_{1r}, X_{ij}]_{q^{-2}}) = 0 & (i < r < j) \\
& \pi_{1\lambda}([X_{1r}, X_{ij}]_{q^2}) = 0 & (i > r > j) \\
& \pi_{1\lambda}([X_{1j}, X_{ij}]_{q^{-1}}) = 0 & (X_{j2} \in \Lambda_2^1) \\
\\
\Lambda_1^2, U_1 : \quad & \pi_{1\lambda}([X_{ij}, X_{r2}]) = 0 & (r < i, j \text{ or } r > i, j) \\
& \pi_{1\lambda}([X_{ji}, X_{i2}]_{q^{-1}}) = \pi_{1\lambda}(X_{j2}) & (3 \leq i < j \leq n) \\
& \pi_{1\lambda}([X_{ij}, X_{r2}]_{q^2}) = 0 & (i > r > j) \\
& \pi_{1\lambda}([X_{ji}, X_{j2}]_q) = 0 & (3 \leq i < j \leq n)
\end{aligned}$$

This list shows us that the contiguity relations give a kind of representation of the set $H^1 \oplus \Lambda_1^1 \oplus \Lambda_2^1 \oplus U_1$ through the mapping $\pi_{1\lambda}$. We, however, remark that this is not a true representation because we used several kinds of bracket operations that depend on the elements inside the bracket, except the case $q = 1$.

A similar reasoning for the second case is possible. Define the subsets

$$\begin{aligned}
H_1^2 &= \mathbf{C}\{q^{\epsilon_1+\epsilon_2}, q^{-\epsilon_1} \widehat{E}_{12}\}, \\
H_3^2 &= \mathbf{C}\{q^{\epsilon_3+\epsilon_4}, q^{-\epsilon_3} \widehat{E}_{34}\}, \\
\Lambda_1^2 &= \mathbf{C}\{q^{-\epsilon_1+\epsilon_4} \widehat{E}_{1j}, \}_{4 \leq j \leq n}, \\
\Lambda_2^2 &= \mathbf{C}\{q^{\epsilon_2+\epsilon_4} \widehat{E}_{j2}, \}_{j=3,5,\dots,n}, \\
\Lambda_3^2 &= \mathbf{C}\{q^{-\epsilon_3} \widehat{E}_{3j}, \}_{4 \leq j \leq n}, \\
\Lambda_4^2 &= \mathbf{C}\{q^{\epsilon_4} \widehat{E}_{j4}, \}_{5 \leq j \leq n},
\end{aligned}$$

and put

$$\begin{aligned}
Y_{12} &= q^{-\epsilon_1} \widehat{E}_{12}, \\
Y_{1j} &= q^{-\epsilon_1-\epsilon_4} \widehat{E}_{1j} & (4 \leq j \leq n), \\
Y_{3j} &= q^{-\epsilon_3} \widehat{E}_{3j} & (4 \leq j \leq n), \\
Y_{j2} &= q^{\epsilon_2+\epsilon_4} \widehat{E}_{j2} & (3 \leq j \leq n), \\
Y_{j4} &= q^{\epsilon_4} \widehat{E}_{j4} & (5 \leq j \leq n), \\
Y_{ij} &= \widehat{E}_{ij} & (3 \leq i \neq j \leq n).
\end{aligned}$$

The commutation relations are

$$\begin{aligned}
& H_r^2 \text{ is abelian for } r = 1, 3, \\
& [Y_{ri}, Y_{rj}]_{q^{-1}} = 0 \quad (Y_{rj} \in \Lambda_r^2, r = 1, 3; 3 \leq i < j \leq n), \\
& [Y_{ir}, Y_{jr}]_q = 0 \quad (Y_{jr} \in \Lambda_r^2, r = 2, 4; 3 \leq i < j \leq n), \\
& U_2 \text{ is isomorphic to } U_q(\mathfrak{gl}(n-4)).
\end{aligned}$$

The contiguity relations now in Proposition 5.2 imply the commutation relations as follows:

$$\begin{aligned}
H_1^2, \text{ other: } & \pi_{2\lambda}([H_1^2, H_3^2]) = 0 \\
& \pi_{2\lambda}([q^{\epsilon_1+\epsilon_2}, Y_{1j}]_q) = 0 \quad (Y_{1j} \in \Lambda_1^2) \\
& \pi_{2\lambda}([Y_{12}, Y_{1j}]_{q^{-1}}) = 0 \quad (Y_{1j} \in \Lambda_1^2) \\
& \pi_{2\lambda}([q^{\epsilon_1+\epsilon_2}, Y_{j2}]_{q^{-1}}) = 0 \quad (Y_{j2} \in \Lambda_2^2) \\
& \pi_{2\lambda}([Y_{12}, Y_{j2}]_q) = 0 \quad (Y_{j2} \in \Lambda_2^2) \\
& \pi_{2\lambda}([H_1^2, \Lambda_r^2]) = 0 \quad (r = 3, 4) \\
& \pi_{2\lambda}([H_1^2, U_2]) = 0 \\
\\
H_3^2, \Lambda_1^2 : & \pi_{2\lambda}([q^{\epsilon_3+\epsilon_4}, Y_{14}]_{q^{-1}}) = 0 \\
& \pi_{2\lambda}([Y_{34}, Y_{14}]) = 0 \\
& \pi_{2\lambda}([q^{\epsilon_3+\epsilon_4}, Y_{1j}]) = 0 \quad (5 \leq j \leq n) \\
& \pi_{2\lambda}([Y_{34}, Y_{1j}]_{q^{-1}}) = 0 \quad (5 \leq j \leq n) \\
\\
H_3^2, \Lambda_2^2 : & \pi_{2\lambda}([q^{\epsilon_3+\epsilon_4}, Y_{32}]_q) = 0 \\
& \pi_{2\lambda}([Y_{34}, Y_{32}]) = 0 \\
& \pi_{2\lambda}([q^{\epsilon_3+\epsilon_4}, Y_{j2}]) = 0 \quad (5 \leq j \leq n) \\
& \pi_{2\lambda}([Y_{34}, Y_{j2}]_q) = 0 \quad (5 \leq j \leq n) \\
\\
H_3^2, \Lambda_3^2 : & \pi_{2\lambda}([q^{\epsilon_3+\epsilon_4}, Y_{3j}]_q) = 0 \quad (Y_{3j} \in \Lambda_3^2) \\
& \pi_{2\lambda}([Y_{34}, Y_{3j}]_{q^{-1}}) = 0 \quad (Y_{3j} \in \Lambda_3^2) \\
\\
H_3^2, \Lambda_4^2 : & \pi_{2\lambda}([q^{\epsilon_3+\epsilon_4}, Y_{j4}]_{q^{-1}}) = 0 \quad (Y_{j4} \in \Lambda_4^2) \\
& \pi_{2\lambda}([Y_{34}, Y_{j4}]_q) = 0 \quad (Y_{j4} \in \Lambda_4^2) \\
\\
H_3^2, U_2 : & \pi_{2\lambda}([H_3^2, U_2]) = 0
\end{aligned}$$

$$\begin{aligned}
\Lambda_1^2, \Lambda_3^2 : \quad & \pi_{2\lambda}([Y_{1i}, Y_{3j}]_{q^{-2}}) = 0 & (i < j) \\
& \pi_{2\lambda}([Y_{1j}, Y_{3j}]_{q^{-1}}) = 0 \\
& \pi_{2\lambda}([Y_{1j}, Y_{3i}]) = 0 & (i < j) \\
\\
\Lambda_1^2, \Lambda_4^2 : \quad & \pi_{2\lambda}([Y_{1i}, Y_{j4}]_q) = 0 & (i \neq j) \\
& \pi_{2\lambda}([Y_{1j}, Y_{j4}]_q) = q^{2\lambda_5 + \dots + 2\lambda_n - \lambda_j + 1} \frac{1}{[\lambda_1 + 1]} \varphi_{\lambda + \alpha_{1,3}}^{(2)} & (5 \leq j \leq n) \\
\\
\Lambda_2^2, \Lambda_3^2 : \quad & \pi_{2\lambda}([Y_{i2}, Y_{3j}]_{q^2}) = 0 & (i < j) \\
& \pi_{2\lambda}([Y_{j2}, Y_{3j}]) = q^{\lambda_3 \lambda_j + 2} [\lambda_1 + 1] \varphi_{\lambda + \alpha_{3,1}}^{(2)} & (5 \leq j \leq n) \\
& \pi_{2\lambda}([Y_{j2}, Y_{3i}]) = 0 & (i < j) \\
\\
\Lambda_2^2, \Lambda_4^2 : \quad & \pi_{2\lambda}([Y_{i2}, Y_{j4}]_{q^{-2}}) = 0 & (i < j) \\
& \pi_{2\lambda}([Y_{1j}, Y_{3j}]_{q^{-2}}) = 0 \\
& \pi_{2\lambda}([Y_{1j}, Y_{3i}]_{q^{-1}}) = 0 & (i < j)
\end{aligned}$$

In addition, the triplet $\{\Lambda_1^2, \Lambda_2^2, U_2\}$ has a similar structure as the triplet $\{\Lambda_1^1, \Lambda_2^1, U_1\}$ and this is the case also for the triplets $\{\Lambda_3^2, \Lambda_4^2, U_2\}$ and $\{\Lambda_1^1, \Lambda_2^1, U_1\}$.

The list above implies a similar remark as in the former case; namely, we have a kind of representation of the set $H_1^2 \oplus H_3^2 \oplus \Lambda_1^2 \oplus \Lambda_2^2 \oplus \Lambda_3^2 \oplus \Lambda_4^2 \oplus U_2$ through the mapping $\pi_{2\lambda}$.

6. The q -confluent hypergeometric systems

In this section we are going to define the final form of q -confluent hypergeometric systems by use of the contiguity relations and by use of L -operators.

We first introduce the L -operators, which we can regard as a q -analogue of (1.25), by the following formulas:

$$L_{ij}^+ = \begin{cases} q^{\epsilon_i} & (i = j) \\ (q - q^{-1}) \widehat{E}_{ji} q^{\epsilon_j} & (i < j) \\ 0 & (i > j) \end{cases} \quad (6.1)$$

and

$$L_{ij}^- = \begin{cases} q^{-\epsilon_i} & (i = j) \\ -(q - q^{-1}) q^{-\epsilon_i} \widehat{E}_{ji} & (i > j) \\ 0 & (i < j). \end{cases} \quad (6.2)$$

For each scalar μ , called the spectral parameter, we put

$$L_{ij}(\mu) = \mu L_{ij}^+ - \mu^{-1} L_{ij}^-. \quad (6.3)$$

Refer to [Ji, NUW]. It is a straightforward task to rewrite the contiguity relations in terms of L -operators, which however we do not write down here.

We next define, instead of the differential operator (2.27), the elements of $U_q(\mathfrak{gl}(n))$ by

$$\Delta_{\gamma\delta}^{\alpha\beta} = L_{\alpha\beta}(q^{\delta_{\alpha,\beta}})L_{\gamma\delta}(1) - L_{\alpha\delta}(1)L_{\gamma\beta}(1) \quad (6.4)$$

for each $\alpha \neq \delta$. We have in particular

$$\Delta_{j+1j+1}^{jj} = (q - q^{-1})^2 C_j.$$

Now we can propose the final forms of q -difference systems by adding equations for the operators $\Delta_{\gamma\delta}^{\alpha\beta}$ to the systems $\bar{E}_1(\lambda, \phi)$ and $\bar{E}_2(\lambda, \phi)$:

$E_1(\lambda, \phi)$:

$$\begin{aligned} \rho_\phi(q^{\epsilon_1+\epsilon_2})G(x) &= q^{\lambda_1}G(x), \\ \rho_\phi(q^{\epsilon_j})G(x) &= q^{\lambda_j}G(x) \quad (j = 3, 4). \end{aligned} \quad (4.3)$$

$$\rho_\phi(q^{-\epsilon_1}e_1)G(x) = q^{-\lambda_1}G(x), \quad (4.4)$$

$$\rho_\phi(\Delta_{\gamma\delta}^{\alpha\beta})G(x) = 0. \quad (6.5)$$

$E_2(\lambda, \phi)$:

$$\begin{aligned} \rho_\phi(q^{\epsilon_1+\epsilon_2})G(x) &= q^{\lambda_1}G(x), \\ \rho_\phi(q^{\epsilon_3+\epsilon_4})G(x) &= q^{\lambda_3}G(x), \end{aligned} \quad (4.6)$$

$$\rho_\phi(q^{\epsilon_j})G(x) = q^{\lambda_j}G(x),$$

$$\begin{aligned} \rho_\phi(q^{-\epsilon_1}e_1)G(x) &= q^{-\lambda_1}G(x), \\ \rho_\phi(q^{-\epsilon_3}e_3)G(x) &= q^{\lambda_3+\dots+\lambda_n}G(x), \end{aligned} \quad (4.7)$$

$$\rho_\phi(\Delta_{\gamma\delta}^{\alpha\beta})G(x) = 0. \quad (6.5)$$

The contiguity relations imply the following theorem.

THEOREM 6.1. *The functions $G^{(1)}$ and $G^{(2)}$ satisfy $E_1(\lambda, \phi_1)$ and $E_2(\lambda, \phi_2)$, respectively.*

PROOF. We can check this theorem directly with contiguity relations. For example, if $5 \leq i < j \leq n$, we have

$$\begin{aligned}\rho_{\phi_r}(L_{ii}(q)L_{jj}(1))G^{(r)} &= (q^{1+\theta_i} - q^{-1-\theta_i})(q^{\theta_j} - q^{-\theta_j})G^{(r)} \\ &= (q^{1+\lambda_i} - q^{-1-\lambda_i})(q^{\lambda_j} - q^{-\lambda_j})G^{(r)}, \\ \rho_{\phi_r}(L_{ij}(1)L_{ji}(1))G^{(r)} &= (q - q^{-1})^2 \rho_{\phi_r}(\hat{E}_{ji}\hat{E}_{ij})G^{(r)} \\ &= (q^{1+\lambda_i} - q^{-1-\lambda_i})(q^{\lambda_j} - q^{-\lambda_j})G^{(r)}.\end{aligned}$$

Here $r = 1, 2$. Hence we have

$$\rho_{\phi_r}(\Delta_{jj}^{ii})G^{(r)} = 0 \quad (r = 1, 2).$$

DEFINITION 6.2. We call the system $E_1(\lambda, \phi_1)$ the *q-confluent hypergeometric system of Kummer type*, and $E_2(\lambda, \phi_2)$ the *q-confluent hypergeometric system of Bessel type*.

REMARK 6.3. Relative to the Grassmannian $G_{2,4}$, we can realize the Hermite function and the Airy function as the solutions of the confluent hypergeometric systems corresponding to the partitions (3, 1) and (4) of number 4, respectively. However, the problem of defining the *q*-confluent hypergeometric systems for this case is still open.

References

- [DH] E. Date and E. Horiuchi: On contiguity relations of Jackson's basic hypergeometric series $\gamma_1(a; b; c; x, y, 1/2)$ and its generalizations, Preprint, 1995.
- [G] I.M. Gelfand: General theory of hypergeometric functions, Dokl. Akad. Nauk SSSR, **288** (1986), 14–18.
- [GG] I.M. Gelfand and M.I. Graev: Hypergeometric functions related to the Grassmannian $G_{3,6}$, Math. Sbornik, **180** (1989), 3–38.
- [GR] G. Gasper and M. Rahman: Basic Hypergeometric Series, Cambridge UP. Cambridge, England, 1990.
- [H] E. Horikawa: Contiguity relations for *q*-hypergeometric function and related quantum group, Proc. Japan Acad., **68** (1992), 157–160.
- [Ja] F.H. Jackson: The application of the basic numbers to Bessel's and Legendre's function, Proc. London Math. Soc., (2) **2** (1905), 192–220.
- [Ji] M. Jimbo: A *q*-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation, Lett. Math. Phys., **11** (1986), 247–252.
- [K] H.T. Koelink: On quantum groups and *q*-special functions, Ph. D. Thesis, Univ. of Leiden, 1991.
- [KHT1] H. Kimura, Y. Haraoka and K. Takano: The generalized confluent hypergeometric functions, Proc. Japan Acad. Ser. A Math. Sci., **68-9** (1992), 289–295.

- [KHT2] H. Kimura, Y. Haraoka and K. Takano: On confluences of the general hypergeometric systems, Proc. Japan Acad. Ser. A Math. Sci., **69**-5 (1993), 99–104.
- [KHT3] H. Kimura, Y. Haraoka and K. Takano: On contiguity relations of the confluent hypergeometric systems, Proc. Japan Acad. Ser. A Math. Sci., **70**-2 (1994), 47–49.
- [Nm] M. Noumi: Quantum Grassmanians and q -hypergeometric series, Centrum voor Wiskunde en Informatica Quarterly, **5**-4 (1992), 293–307.
- [Nk] M. Nakatani: A quantum analogue of the hypergeometric systems $E_{3,6}$, Kyushu J. Math. vol. 49, No.1 (1995), 67–91.
- [NN] M. Noumi and M. Nakatani: q -hypergeometric systems arising from the quantum Grassmannians, in preparation.
- [NUW] M. Noumi, T. Umeda and M. Wakayama: A quantum analogue of Capelli identity and an elementary differential calculus on $GL_q(n)$, to appear in Duke Math. J.
- [NYM] M. Noumi, H. Yamada and K. Mimachi: Finite dimensional representations of the quantum group $GL_q(n; \mathbb{C})$ and the zonal spherical functions on $U_q(n-1) \setminus U_q(n)$, Japan. J. Math., **19** (1993), 31–80.
- [S] T. Sasaki: Contiguity relations of Aomoto-Gel'fand hypergeometric functions and applications to Appel's system F_3 and Goursat's system ${}_3F_2$, SIAM J. Math. Anal., **22** (1991), 821–846.

Department of Mathematics
 Kobe University
 Nada-ku, Kobe 657
 Japan