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Choi, Changsun

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A WEAK-TYPE SUBMARTINGALE INEQUALITY

Changsun CHOI

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Abstract

For a nonnegative submartingale f and an adapted process g, the condition of g being strongly differentially subordinate to f, is generalized. Under this new condition we prove a sharp weak-type inequality which extends an inequality by Burkholder.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$. Consider two adapted processes $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$. Write $f_n = d_0 + \cdots + d_n$ and $g_n = e_0 + \cdots + e_n$ for $n \geq 0$. Set $\|f\| = \sup_{n\geq 0} \mathbb{E}|f_n|$ and $g^* = \sup_{n\geq 0} |g_n|$. This setting is fixed through Sections 1, 2 and 3.

DEFINITION 1.1. We say that *g is differentially subordinate to f* if $|e_n| \leq |d_n|$ for all $n \geq 0$. Also, *g is strongly differentially subordinate to f* if g is differentially subordinate to f and $|\mathbb{E}(e_n \mid \mathcal{F}_{n-1})| \leq |\mathbb{E}(d_n \mid \mathcal{F}_{n-1})|$ for all $n \geq 1$.

Comparing the sizes of f and g under the assumption of differential subordinations has been studied by Burkholder in [1], [2], [3], [4] and [5]. The following weak-type inequalities are from [2] and [5].

THEOREM 1.2. *If f and g are Hilbert space valued martingales and g is differentially subordinate to f, then*

$$\lambda P(g^* \geq \lambda) \leq 2\|f\| \quad \textit{for all} \ \ \lambda > 0$$

and the inequality is sharp.

THEOREM 1.3. *If f is a nonnegative submartingale, g is $\mathbb{R}^\nu$-valued, where ν is a positive integer, and g is strongly differentially subordinate to f, then*

$$\lambda P(g^* \geq \lambda) \leq 3\|f\| \quad \textit{for all} \ \ \lambda > 0$$

and the inequality is sharp.

By the sharpness of the inequality, for example, in Theorem 1.2 we mean that if $0 < \beta < 2$, then the opposite inequality $\lambda P(g^* \geq \lambda) > \beta\|f\|$ holds for some $\lambda > 0$ and some processes f and g, in some probability space, satisfying all the assumptions of Theorem 1.2.

It is interesting to ask what happens to the sharp constant 3 in Theorem 1.3 if we replace the assumption that g is strongly differentially subordinate to f by a more general one:

$$|e_n| \leq \alpha_1 |d_n| \quad \text{for all} \quad n \geq 0; \text{ and} \tag{1.1}$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \leq \alpha_2 |\mathbb{E}(d_n|\mathcal{F}_{n-1})| \quad \text{for all} \quad n \geq 1 \tag{1.2}$$

where $\alpha_1, \alpha_2 \geq 0$ are constants. Of course, we may assume that $\alpha_1 > 0$ and further, considering $\alpha_1 f$ in place of f and α_2/α_1 in place of α_2, that $\alpha_1 = 1$.

2. A weak-type inequality

DEFINITION 2.1. For $\alpha \geq 0$ we say that g is *α-subordinate* to f if

$$|e_n| \leq |d_n| \quad \text{for all} \quad n \geq 0; \text{ and} \tag{2.1}$$

$$|\mathbb{E}(e_n|\mathcal{F}_{n-1})| \leq \alpha |\mathbb{E}(d_n|\mathcal{F}_{n-1})| \quad \text{for all} \quad n \geq 1. \tag{2.2}$$

Let $\mathbb{H}$ be a Hilbert space over $\mathbb{R}$. For $x, y \in \mathbb{H}$ we denote by $x \cdot y$ the inner product of x and y and put $|x|^2 = x \cdot x$.

THEOREM 2.2. *Let $\alpha \geq 0$. If f is a nonnegative submartingale, g is $\mathbb{H}$-valued and g is α-subordinate to f, then*

$$\lambda P(g^* \geq \lambda) \leq (\alpha + 2)\|f\| \quad \textit{for all} \quad \lambda > 0 \tag{2.3}$$

and the inequality is sharp provided $0 \leq \alpha \leq 1$.

PROOF OF THE INEQUALITY (2.3). In order to make the main points of the proof clear we defer some technical details to the next section. Thus, we assume and use Claim 2.3, Claim 2.4, Lemma 2.5 and Lemma 2.6 which are proved later.

We may assume $\|f\| < \infty$ which guarantees integrability of f_n and g_n in view of the assumption (2.1) for all $n \geq 0$. It is enough to consider only $\lambda = 1$ because the inequality (2.3) follows from

$$P(g^* \geq 1) \leq (\alpha + 2)\|f\| \tag{2.4}$$

when we substitute f/λ and g/λ for f and g, respectively.

CLAIM 2.3. *We may further assume that*

$$f_{n-1} > 0 \quad \text{and} \quad |g_{n-1} + te_n| > 0$$

for all $n \geq 1$ *and all* $t \in \mathbb{R}$.

CLAIM 2.4. *It suffices to show*

$$P(|g_n| \geq 1) \leq (\alpha + 2)\mathbb{E}f_n \quad \text{for all} \quad n \geq 0. \tag{2.5}$$

As a matter of fact we will prove the stronger inequality

$$P(f_n + |g_n| \geq 1) \leq (\alpha + 2)\mathbb{E}f_n \quad \text{for all} \quad n \geq 0. \tag{2.6}$$

The inequality (2.6) is equivalent to the inequality

$$\mathbb{E}V(f_n, g_n) \leq 0 \quad \text{for all} \quad n \geq 0 \tag{2.7}$$

where

$$V(x, y) = \begin{cases} -(\alpha + 2)x & \text{if} \quad x + |y| < 1, \\ 1 - (\alpha + 2)x & \text{if} \quad x + |y| \geq 1. \end{cases} \tag{2.8}$$

Put $S = \{(x, y) : x > 0,\ y \in \mathbb{H} \text{ and } |y| > 0\}$. Define a new function U on S by

$$U(x, y) = \begin{cases} (|y| - (\alpha + 1)x)(x + |y|)^{1/(\alpha+1)} & \text{if} \quad x + |y| < 1, \\ 1 - (\alpha + 2)x & \text{if} \quad x + |y| \geq 1. \end{cases} \tag{2.9}$$

Notice that U is continuous and $|U(x, y)| \leq 1 + (\alpha + 2)x$. Hence $U(f_n, g_n)$, which is well-defined by Claim 2.3, is integrable for all $n \geq 0$.

LEMMA 2.5. (a) $V(x, y) \leq U(x, y)$. (b) $U(x, y) \leq 0$ *if* $x \geq |y|$.

From the assumption (2.1) with $n = 0$ we have $f_0 \geq |g_0|$, which, along with (b) of Lemma 2.5, gives $U(f_0, g_0) \leq 0$, hence $\mathbb{E}U(f_0, g_0) \leq 0$. Also, (a) of Lemma 2.5 implies $\mathbb{E}V(f_n, g_n) \leq \mathbb{E}U(f_n, g_n)$ for all $n \geq 0$. Hence, the inequality (2.7) follows from

$$\mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_{n-1}, g_{n-1}) \quad \text{for all} \quad n \geq 1. \tag{2.10}$$

To prove the inequality (2.10) we need

LEMMA 2.6. *There are Borel functions* $\varphi : S \to \mathbb{R}$ *and* $\psi : S \to \mathbb{H}$ *such that*

(a) $-(\alpha+2) \le \varphi(x,y) \le 0$ *and* $0 \le |\psi(x,y)| \le 2$
(b) $\varphi(x,y) + \alpha|\psi(x,y)| \le 0$
(c) *if* $(x,y), (x+h, y+k) \le S$, $|h| \ge |k|$ *and* $|y+tk| > 0$ *for all* $t \in \mathbb{R}$, *then*

$$U(x+h, y+k) - U(x,y) \le \varphi(x,y)h + \psi(x,y)\cdot k,$$

Let $n \ge 1$. The assumption (2.1), Claim 2.3 and (c) of Lemma 2.6 imply

$$U(f_n, g_n) - U(f_{n-1}, g_{n-1}) \le \varphi(f_{n-1}, g_{n-1})d_n + \psi(f_{n-1}, g_{n-1})\cdot e_n$$

which, when conditioned on $\mathcal{F}_{n-1}$, yields

$$\begin{aligned}(2.11)\qquad &\mathbb{E}\left[U(f_n, g_n) \mid \mathcal{F}_{n-1}\right] - U(f_{n-1}, g_{n-1}) \\ &\quad \le \varphi(f_{n-1}, g_{n-1})\mathbb{E}(d_n \mid \mathcal{F}_{n-1}) + \psi(f_{n-1}, g_{n-1})\cdot \mathbb{E}(e_n \mid \mathcal{F}_{n-1})\end{aligned}$$

because d_n and e_n are integrable and $\psi(f_{n-1}, g_{n-1})$ and $\psi(f_{n-1}, g_{n-1})$ are $\mathcal{F}_{n-1}$ measurable and bounded as in (a) of Lemma 2.6. The Cauchy-Schwarz inequality and the assumption (2.2) imply

$$\begin{aligned}\psi(f_{n-1}, g_{n-1})\cdot \mathbb{E}(e_n \mid \mathcal{F}_{n-1}) &\le \psi(f_{n-1}, g_{n-1}) \mid |\mathbb{E}(e_n \mid \mathcal{F}_{n-1})| \\ &\le \alpha|\psi(f_{n-1}, g_{n-1})|\mathbb{E}(d_n \mid \mathcal{F}_{n-1})\end{aligned}$$

where we also used the assumption that f is a submartingale. Thus, from (b) of Lemma 2.6 and the assumption that f is a submartingale we get

$$\begin{aligned}(2.12)\qquad &\mathbb{E}\left[U(f_n, g_n) \mid \mathcal{F}_{n-1}\right] - U(f_{n-1}, g_{n-1}) \\ &\quad \le \left(\varphi(f_{n-1}, g_{n-1}) + \alpha|\psi(f_{n-1}, g_{n-1})|\right)\mathbb{E}(d_n \mid \mathcal{F}_{n-1}) \le 0.\end{aligned}$$

Taking expectation of the final inequality in (2.12), we get

$$\mathbb{E}U(f_n, g_n) = \mathbb{E}\mathbb{E}\big[U(f_n, g_n) \mid \mathcal{F}_{n-1}\big] \le \mathbb{E}U(f_{n-1}, g_{n-1}),$$

proving the inequality (2.10).

We have proved the inequality (2.3) in Theorem 2.2 under the assumption of Claim 2.3, Claim 2.4, Lemma 2.5 and Lemma 2.6.

Basic facts about conditional expectations can be found in [7].

Because we proved the inequality (2.6) a stopping time argument as in the proof of Claim 2.4 in Section 3 shows that a stronger inequality than the one (2.3) in Theorem 2.2 holds:

COROLLARY 2.7. *Let $\alpha \geq 0$. If f is a nonnegative submartingale, g is $\mathbb{H}$-valued and g is α-subordinate to f, then*

$$\lambda P\left(\sup_{n\geq 0}(f_n + |g_n|) \geq \lambda\right) \leq (\alpha+2)\|f\| \quad \textit{for all } \lambda > 0.$$

3. Proof of claims and lemmas

PROOF OF CLAIM 2.3. Suppose that f and g satisfy the assumptions of Theorem 2.2. For each $\epsilon > 0$, the new processes $f+\epsilon$ and (g,ϵ), where (g,ϵ) has value in the standard product Hilbert space $\mathbb{H} \times \mathbb{R}$, satisfy the extra assumption in Claim 2.3 as well as the assumptions of Theorem 2.2. Assuming the inequality (2.4) for these new processes, we have

$$P\big((g,\epsilon)^* \geq 1\big) \leq (\alpha+2)\|f+\epsilon\|. \tag{3.1}$$

Notice that $g^* \leq (g,\epsilon)^*$ and $\|f+\epsilon\| = \|f\| + \epsilon$. Thus

$$P(g^* \geq 1) \leq (\alpha+2)\|f\| + \epsilon(\alpha+2). \tag{3.2}$$

The above inequality (3.2), as $\epsilon \to 0$, yields $P(g^* \geq 1) \leq (\alpha+2)\|f\|$, namely the inequality (2.4), proving Claim 2.3.

PROOF OF CLAIM 2.4. Suppose that f and g satisfy the assumptions of Theorem 2.2. Fix a positive integer N for the moment, set $g_N^* = \sup_{0\leq n\leq N}|g_n|$ and define a stopping time T by

$$T = \inf\{0 \leq n \leq N : |g_n| > 1\} \tag{3.3}$$

if $g_N^* > 1$, and $T = N$ otherwise.

Observe that the stopped processes $f^T = (f_{T\wedge n})_{n\geq 0}$ and g^T with differences processes $\widetilde{d}$ and $\widetilde{e}$ respectively, satisfy the assumptions of Theorem 2.2; one only needs to notice that

$$\widetilde{d}_n = d_n 1_{\{T\geq n\}} \quad \text{and} \quad \mathbb{E}(\widetilde{d}_n \mid \mathcal{F}_{n-1}) = \mathbb{E}(d_n \mid \mathcal{F}_{n-1})1_{\{T\geq n\}} \tag{3.4}$$

and similar facts about $\widetilde{e}$.

Assuming the inequality (2.5) in Claim 2.4 for these new processes with $n = N$, we have

$$P(|g_{T\wedge N}| \geq 1) \leq (\alpha+2)\mathbb{E}f_{T\wedge N}, \quad \text{or} \quad P(|g_T| \geq 1) \leq (\alpha+2)\mathbb{E}f_T \tag{3.5}$$

because $T \leq N$. If $g_N^* > 1$, then $|g_T| \geq 1$. Also, Doob's optional sampling theorem implies $\mathbb{E}f_T \leq \mathbb{E}f_N$ because f is a submartingale and T is bounded by

N. Thus, from the second inequality in (3.5), we have

$$P(g_N^* > 1) \le P(|g_T| \ge 1) \le (\alpha+2)\mathbb{E}f_T \le (\alpha+2)\mathbb{E}f_N \le (\alpha+2)\|f\|. \tag{3.6}$$

Since N was arbitrary and $\{g_N^* > 1\}$ increases to $\{g^* > 1\}$ the inequality (3.6), as $N \to \infty$, gives

$$P(g^* > 1) \le (\alpha+2)\|f\|. \tag{3.7}$$

Finally, for $\gamma > 1$, the above inequality (3.7), applied to the pair γf and γg, gives

$$P\left(g^* > \frac{1}{\gamma}\right) \le (\alpha+2)\gamma\|f\|,$$

which, as $\gamma \to 1$, yields $P(g^* \ge 1) \le (\alpha+2)\|f\|$, namely the inequality (2.4). This proves Claim 2.4.

PROOF OF LEMMA 2.5. Proof of (a). We may assume $x + |y| < 1$. Write $x + |y| = r^{\alpha+1}$. Since $0 < r < 1$, we have

$$V(x,y) - U(x,y) = -r^{\alpha+2} - (1-r)(\alpha+2)x < 0.$$

Proof of (b). Assume $|y| \le x$. Then $|y| - (\alpha+1)x \le |y| - x \le 0$. Thus $U(x,y) \le 0$ if $x + |y| < 1$. If $x + |y| \ge 1$, then $U(x,y) = 1 - (\alpha+2)x \le x + |y| - (\alpha+2)x = |y| - (\alpha+1)x \le 0$.

PROOF OF LEMMA 2.6. Define φ and ψ on S by

$$\varphi(x,y) = \begin{cases} -\dfrac{(\alpha+1)(\alpha+2)x + \alpha(\alpha+2)|y|}{(\alpha+1)(x+|y|)^{\alpha/(\alpha+1)}} & \text{if } x + |y| < 1, \\ -(\alpha+2) & \text{if } x + |y| \ge 1 \end{cases} \tag{3.8}$$

and

$$\psi(x,y) = \begin{cases} \dfrac{(\alpha+2)y}{(\alpha+1)(x+|y|)^{\alpha/(\alpha+1)}} & \text{if } x + |y| < 1, \\ 0 & \text{if } x + |y| \ge 1. \end{cases} \tag{3.9}$$

Proof of (a) and (b). All are clear from the definitions (3.8) and (3.9).
Proof of (c). We fix $(x,y), (x+h, y+k) \in S$ so that $|h| \ge |k|$ and $|y + tk| > 0$ for all $t \in \mathbb{R}$.

To prove the inequality

$$U(x+h, y+k) - U(x,y) \le \varphi(x,y)h + \psi(x,y) \cdot k \tag{3.10}$$

we may assume that $x + |y| \neq 1$. Indeed, with $x + |y| = 1$, we put $x_i = x + 1/i$ and notice $x_i + |y| > 1$. Then the inequality (3.10) follows from the continuity of U, φ and ψ on the region $\{(a, b) \in S : a + |b| \geq 1\}$ and the inequality

$$U(x_i + h, y + k) - U(x_i, y) \leq \varphi(x_i, y)h + \psi(x_i, y) \cdot k$$

as $i \to \infty$. We may further assume that $|h| > |k|$. To see this assume $x + |y| \neq 1$. Put $h_i = h + h/i$ for $i > |h|/(x + h)$ and notice $|h_i| > |k|$. The inequality (3.10) follows from the continuity of U and the inequality

$$U(x + h_i, y + k) - U(x, y) \leq \varphi(x, y)h_i + \psi(x, y) \cdot k$$

as $i \to \infty$.

Put $I = \{t \in \mathbb{R} : x + th > 0\}$. Observe that I is an open ray containing the interval [0,1]. Define a function G on I by

$$G(t) = U(x + th, y + tk). \tag{3.11}$$

Since $x + |y| \neq 1$, we have, from (2.9), (3.8) and (3.9), that $\varphi(x, y) = U_x(x, y)$ and $\psi(x, y) = U_y(x, y)$, the partial derivatives. Hence G is differentiable at 0 and the chain rule gives

$$G'(0) = \varphi(x, y)h + \psi(x, y) \cdot k. \tag{3.12}$$

Thus the desired inequality becomes

$$G(1) - G(0) \leq G'(0) \tag{3.13}$$

which holds because, as we shall show, G is concave.

One may refer to [6] for the basic facts of concave functions which we shall make use of.

In order to show the concavity of G we need a few propositions. On I we consider following functions:

$$A(t) = x + th, \; B(t) = y + tk, \; R(t) = A(t) + |B(t)|,$$

and

$$\begin{cases} G_1(t) = \big(|B(t)| - (\alpha + 1)A(t)\big)R(t)^{1/(\alpha+1)} \\ G_2(t) = 1 - (\alpha + 2)A(t). \end{cases}$$

If no confusion arises we will omit the argument t. Observe that G is continuous, and that $G = G_1$ if $R < 1$, and $G = G_2$ if $R \geq 1$.

PROPOSITION 3.1. *If $R(\tau)=1$, then $G=G_1\wedge G_2$ on a neighborhood of τ.*

PROOF. Let $R(\tau)=1$. From the definitions

$$\begin{aligned} G_1-G_2 &= \big(|B|-(\alpha+1)A\big)R^{1/(\alpha+1)}-1+(\alpha+2)A \\ &= R^{(\alpha+2)/(\alpha+1)}-(\alpha+2)AR^{1/(\alpha+1)}-1+(\alpha+2)A \\ &= (R^{1/(\alpha+1)}-1)C, \end{aligned}$$

where

$$C=\frac{R^{(\alpha+2)/(\alpha+1)}-1}{R^{1/(\alpha+1)}-1}-(\alpha+2)A.$$

As $t\to\tau$ we have $R\to 1$, hence, using l'Hôpital's rule, we get

$$\lim_{t\to\tau} C=(\alpha+2)\big(1-A(\tau)\big)=(\alpha+2)|B(\tau)|>0.$$

On a small neighborhood of τ, we have $C>0$; if $R<1$, then $G_1<G_2$, hence $G=G_1=G_1\wedge G_2$. The case $R\geq 1$ can be treated similarily.

PROPOSITION 3.2. *Both G_1 and G_2 are concave.*

PROOF. G_2 is linear, hence it is concave. Observe that G_1 is smooth, thus it suffices to check $G_1''\leq 0$. Writing

$$G_1=R^{(\alpha+2)/(\alpha+1)}-(\alpha+2)AR^{1/(\alpha+1)},$$

we differentiate G_1 to get

$$G_1'=\frac{\alpha+2}{\alpha+1}R'R^{1/(\alpha+1)}-(\alpha+2)hR^{1/(\alpha+1)}-\frac{\alpha+2}{\alpha+1}AR'R^{-\alpha/(\alpha+1)}$$

and

$$\eta G_1''=R''R+\frac{1}{\alpha+1}(R')^2R-2hR'R-AR''R+\frac{\alpha}{\alpha+1}A(R')^2$$

where

$$\eta=\frac{\alpha+1}{\alpha+2}R^{(2\alpha+1)/(\alpha+1)}.$$

Rearranging terms and inserting $(R')^2R-(R')^2R$, we have

$$\begin{aligned} \eta G_1'' &= \left(R''R-2hR'-AR+(R')^2\right)R+\left(-R+\frac{1}{\alpha+1}R+\frac{\alpha}{\alpha+1}A\right)(R')^2 \\ &= (|k|^2-|h|^2)R-\frac{\alpha}{\alpha+1}|B|(R')^2\leq 0. \end{aligned}$$

Here we used the observation that $A' = h$, $|B|' = R' - h$, $|B||B|' = k \cdot B$ and $|B|R'' = |B||B|'' = |k|^2 - (|B|')^2$. This proves Proposition 3.2.

We return to the proof that G is concave. Since $|h| > |k|$ and

$$R' = h + k \cdot \frac{B}{|B|}$$

we have $|R'| \geq |h| - |k| > 0$, thus R is strictly monotone. Put $I_1 = \{t \in I : R(t) < 1\}$ and $I_2 = \{t \in I : R(t) > 1\}$. Notice, for $k = 1, 2$, that I_k is open and on I_k, $G = G_k$ is concave by Proposition 3.2. Now let $R(\tau) = 1$. By Proposition 3.1 there is a neighborhood of τ where G, being the minimum of concave functions, is itself concave. Therefore, G is locally concave on I, hence it is concave. This completes the proof of Lemma 2.6.

4. Sharpness of the inequality

Assume $0 \leq \alpha \leq 1$ and let $0 < \beta < \alpha + 2$. In the Lebesgue probability space $(\Omega, \mathcal{F}, P)$ on the interval $[0, 1)$ we will construct a filtration $(\mathcal{F}_n)_{n \geq 0}$ and two adapted processes f and g satisfying all the assumptions of Theorem 2.2, but

$$P(g^* \geq 1) > \beta \|f\|.$$

For this we introduce some conventions. First, notice that a partition of $[0, 1)$ generates a σ-field on $[0, 1)$; hence we identify the σ-field with its partition. In the following we write $\mathcal{F}_n$ for the partition which generates the σ-field $\mathcal{F}_n$. Second, for the interval $J = [a, b)$, $0 \leq s \leq 1$ and $u, v \in \mathbb{R}$, we define the left half subinterval J_l, the right half subinterval J_r of J and the step function $J[s; u, v]$ on $[0, 1)$ by

$$J_l = \left[a, \frac{a+b}{2}\right), \quad J_r = \left[\frac{a+b}{2}, b\right)$$

and

$$J[s; u, v](t) = \begin{cases} u & \text{if} \quad a \leq t < a + s(b - a), \\ v & \text{if} \quad a + s(b - a) \leq t < b, \\ 0 & \text{otherwise.} \end{cases}$$

Also, $\operatorname{sgn} x = 1$ if $x > 0$ and $\operatorname{sgn} x = -1$ if $x < 0$. Write $(x, y)c$ for the scalar multiplication $c(x, y)$.

In order to define processes f and g we need to define processes F and G first. Write $F_n = D_0 + \cdots + D_n$ and $G_n = E_0 + \cdots + E_n$ for $n \geq 0$. Put $\mathcal{F}_0 = \{[0, 1)\}$,

$\mathcal{F}_1 = \mathcal{F}_2 = \{[0,1/2),[1/2,1)\}$, $(F_0,G_0) = (F_1,G_1) = (0,0)$ and

$$(D_2,E_2) = (1,1)\left[0,\frac{1}{2}\right)[1;1,0] + (1,-1)\left[\frac{1}{2},1\right)[1;1,0].$$

Also, put

$$(D_3,E_3) = \sum_{J\in\mathcal{F}_2}(1,-\operatorname{sgn}G_2)J\left[\frac{1}{2};1,-1\right]$$

and let $\mathcal{F}_3$ be the smallest refinement of $\mathcal{F}_2$ such that D_3 and E_3 are measurable with respect to $\mathcal{F}_3$. For $n > 1$, we put

$$\begin{aligned}
&(D_{3n-2},E_{3n-2})\\
&\quad= \sum_{\substack{J\in\mathcal{F}_{3n-3}\\ G_{3n-3}=0\text{ on }J}} (1,1)J_l\left[\frac{1}{2n-1};2-2n,1\right] + (1,-1)J_r\left[\frac{1}{2n-1};2-2n,1\right]\\
&(D_{3n-1},E_{3n-1})\\
&\quad= \sum_{\substack{J\in\mathcal{F}_{3n-2}\\ F_{3n-2}=0\text{ on }J}} (1,\alpha\operatorname{sgn}G_{3n-2})J\left[1;\frac{2}{\alpha+1},0\right]\\
&(D_{3n},E_{3n})\\
&\quad= \sum_{\substack{J\in\mathcal{F}_{3n-1}\\ F_{3n-1}=2/(\alpha+1)\text{ on }J}} (1,-\operatorname{sgn}G_{3n-1})J\left[\frac{1}{(\alpha+1)n};2n-\frac{2}{\alpha+1},\frac{-2}{\alpha+1}\right]\\
&\qquad+ \sum_{\substack{J\in\mathcal{F}_{3n-1}\\ |G_{3n-1}|=1\text{ on }J}} (1,-\operatorname{sgn}G_{3n-1})J\left[\frac{1}{2n};-2n+1,1\right]
\end{aligned}$$

where, for $k = -2, -1, 0$, the partition $\mathcal{F}_{3n+k}$ is the refinement of $\mathcal{F}_{3n+k-1}$ by (D_{3n+k},E_{3n+k}).

The motivation of the above construction comes from Burkholder in [4]. The construction can be best understood by plotting the random points $(x,y) = (F_n,G_n)$ in the plane with appropriate weights and then observing (D_{n+1},E_{n+1}), the movement of the points. Thus, for $n \geq 1$, inductively one checks that the points (F_{3n},G_{3n}) are either at $(2n,0)$, $(0,2n)$, or at $(0,-2n)$ with probabilities, say, a_n, b_n and c_n, respectively.

Now let $n > 1$. The move (D_{3n-2},E_{3n-2}) is a martingale one, that is, $\mathbb{E}((D_{3n-2},E_{3n-2}) \mid \mathcal{F}_{3n-3}) = 0$; this way points of (F_{3n-3},G_{3n-3}) at $(0,2n-2)$ and $(-2n+2)$ are fixed, and the points of (F_{3n-3},G_{3n-3}) at $(2n-2,0)$ are spread in four directions, along the lines $|y| = |x-2n+2|$, so that the west bound points stop on the y-axis and the east bound points stop on the line $x = 2n-1$. Observe $\mathcal{F}_{3n-2} = \mathcal{F}_{3n-1}$. Hence, the move (D_{3n-1}, E_{3n-1}), which is a submartingale move for F, carries all points of (F_{3n-2}, G_{3n-2}) at $(0, 2n-2)$ to the position $\big(2/(\alpha+1), 2n-2+2\alpha/(\alpha+1)\big)$, and points at $(0, -2n+2)$ to $\big(2/(\alpha+1), -2n+2-2\alpha/(\alpha+1)\big)$; it leaves points of (F_{3n-2}, G_{3n-2}) at other places where they are. Here by a submartingale move we mean that $\mathbb{E}(D_{3n-1} \mid \mathcal{F}_{3n-2}) \geq 0$. Finally, (D_{3n}, E_{3n}) is a martingale move which splits all the points of (F_{3n-1}, G_{3n-1}), along the lines $|y| = x - 2n$, so that F_{3n} is either 0 or $2n$.

Since $0 \leq \alpha \leq 1$ we have $|E_n| \leq |D_n|$ for all $n \geq 0$. Also, $\mathbb{E}(D_m \mid \mathcal{F}_{m-1}) = \mathbb{E}(E_m \mid \mathcal{F}_{m-1}) = 0$ if $m \neq 3n-1$, and from $\mathcal{F}_{3n-1} = \mathcal{F}_{3n-2}$, we have $|\mathbb{E}(E_{3n-1} \mid \mathcal{F}_{3n-2})| = |E_{3n-1}| = \alpha D_{3n-1} = \alpha \mathbb{E}(D_{3n-1} \mid \mathcal{F}_{3n-2})$; it follows that $\mathbb{E}(D_n \mid \mathcal{F}_{n-1}) \geq 0$ and $|\mathbb{E}(E_n \mid \mathcal{F}_{n-1})| \leq \alpha |\mathbb{E}(D_n \mid \mathcal{F}_{n-1})|$ for all $n \geq 1$. Summing up, F is a nonnegative submartingale, G is adapted, and G is α-subordinate to F.

About a_n, b_n and c_n one has $a_1 = 1/2$, $b_1 = c_1 = 1/4$, and inductively one can check, for $n > 1$, that $a_n + b_n + c_n = 1$, $b_n = c_n$, and

$$a_n = \frac{n-1}{n} a_{n-1} + \frac{a_{n-1}}{(\alpha+1)n(2n-1)} + \frac{b_{n-1} + c_{n-1}}{(\alpha+1)n}.$$

Noting $b_{n-1} + c_{n-1} = 1 - a_{n-1}$, we compute

$$a_n = a_{n-1}\left(1 - \frac{(\alpha+1)(2n-1) + 2(n-1)}{(\alpha+1)n(2n-1)}\right) + \frac{1}{(\alpha+1)n}.$$

With $t_n = a_n - 1/(\alpha+2)$ one has

$$\begin{aligned} t_n &= t_{n-1}\left(1 - \frac{(\alpha+1)(2n-1) + 2(n-1)}{(\alpha+1)n(2n-1)}\right) + \frac{1}{(\alpha+1)(\alpha+2)n(2n-1)} \\ &< t_{n-1}\left(1 - \frac{1}{n}\right) + \frac{1}{n^2} \end{aligned}$$

for $n > 1$. Also, $0 \leq t_1 = \alpha/\big(2(\alpha+2)\big) < 1$ and $t_n > 0$ for all $n > 1$. Thus, if $1 < K < N$, then iteration gives

$$\begin{aligned} 0 < t_N &< t_1 \prod_{n=2}^{N}\left(1 - \frac{1}{n}\right) + \frac{1}{N^2} + \sum_{n=2}^{N-1} \frac{1}{n^2} \prod_{k=n+1}^{N}\left(1 - \frac{1}{k}\right) \\ &< \prod_{n=2}^{N}\left(1 - \frac{1}{n}\right) + \sum_{n=K}^{N} \frac{1}{n^2} + \left(\sum_{n=2}^{K-1} \frac{1}{n^2}\right)\left(\prod_{n=K}^{N}\left(1 - \frac{1}{n}\right)\right) \\ &< \exp\left(-\sum_{n=2}^{N} \frac{1}{n}\right) + \sum_{n=K}^{N} \frac{1}{n^2} + \left(\sum_{n=1}^{K} \frac{1}{n^2}\right) \exp\left(-\sum_{n=K}^{N} \frac{1}{n}\right). \end{aligned}$$

Now, choosing a large K first and letting $N \to \infty$ next, we see, as N tends to ∞, that $t_N \to 0$, hence that $a_N \to 1/(\alpha+2)$. Since $1/\beta > 1/(\alpha+2)$ we may choose N so that

$$\frac{a}{\beta} > a_N, \quad \text{or} \quad 1 > \beta a_N.$$

Finally, we define f and g by

$$(f_n, g_n) = \frac{1}{2N}(F_n, G_n) \quad \text{for} \quad 0 \le n \le 3N$$

and for $n > 3N$

$$(f_n, g_n) = (f_{3N}, g_{3N}) + \sum_{\substack{J \in \mathcal{F}_{3N} \\ g_{3N}=0 \text{ on } J}} (1,1)J_l\left[\frac{1}{2}; 1, -1\right] + (1,-1)J_r\left[\frac{1}{2}; 1, -1\right].$$

Observe that the final movement of (f,g) is a martingale one which spreads the points at $(1,0)$, along the lines $|y| = |x-1|$, so that $|g_n| = 1$ for all $n > 3N$. It is clear that, with the same filtration $(\mathcal{F}_n)_{n \ge 0}$, f is a nonnegative submartingale, g is adapted, and that g is α-subordinate to f. Also,

$$P(g^* \ge 1) = P(|g_{3N+1}| = 1) = 1 \quad \text{and} \quad \|f\| = \mathbb{E}f_{3N+1} = 2 \cdot \frac{a_N}{2} = a_N.$$

Thus,

$$P(g^* \ge 1) > \beta\|f\|.$$

This completes the proof that the inequality (2.3) in Theorem 2.2 is sharp provided $0 \le \alpha \le 1$.

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Department of Mathematics
Korea Advanced Institute of
Science and Technology
Taejon, 305-701, Korea