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MIXED PROBLEM WITH INTEGRAL CONDITIONS FOR A THIRD ORDER PARABOLIC EQUATION

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Abstract

This paper is concerned with the existence and uniqueness of a strong solution to a mixed problem which combine Dirichlet and integral conditions for a third order linear parabolic equation. The proof uses a functional analysis method presented, for instance, in [3], [4] and [8].

1. Formulation of the problem

In the rectangle $\mathcal{Q} = (0, 1) \times (0, T)$, with $T < \infty$, we consider the third order linear parabolic equation:

$$(1.1) \quad \mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2} \left(a(x, t) \frac{\partial u}{\partial x} \right) = f(x, t),$$

which can be considered as a generalization on the linearized Kortweg-de-Vries equation, see, for instance [23].

Condition 1. The coefficient $a(x, t)$ is a real-valued function belonging to $C^2(\overline{\mathcal{Q}})$ such that

$$c_0 \leq a(x, t) \leq c_1, \quad \frac{\partial a(x, t)}{\partial t} \leq c_2.$$

In *Condition 1* and in the rest of the paper, c_i , $i = 1, \dots, 4$, denote strictly positive constants.

We adjoin to (1.1) the initial condition

$$(1.2) \quad \ell u = u(x, 0) = \varphi(x), \quad x \in (0, 1),$$

the Dirichlet condition

$$(1.3) \quad u(0, t) = 0, \quad t \in (0, T),$$

and integral conditions

$$(1.4) \quad \int_0^1 u(x, t) dx = 0, \quad t \in (0, T),$$

$$(1.5) \quad \int_0^1 xu(x, t) dx = 0, \quad t \in (0, T),$$

where φ is a known function.

Condition 2. We shall assume that the function φ satisfies a compatibility conditions with (1.3)–(1.5), i.e.,

$$\varphi(0) = 0, \quad \int_0^1 \varphi(x) dx = 0 \quad \text{and} \quad \int_0^1 x\varphi(x) dx = 0.$$

Problem (1.1)–(1.5) arises, for instance, from the heat transfer theory. In this case, u is a temperature of a slab $0 \leq x \leq 1$, and the integrals in the conditions (1.4) and (1.5) are considered as the average and the weighted average temperature. Mixed problems which combine local and integral conditions for second order parabolic equations were considered by Cannon [10], Batten [1], Ionkin [15], Cannon-Esteva-van der Hoek [11], Canon-van der Hoek [12], [13], Benouar-Yurchuk [2], Yurchuk [22], Cahlon-Kulkarni-Shi [9], and Shi [21].

Recently, problems of this type that have non-linearity in the boundary conditions have been investigated in Jones et al. [16] and Jumahron-McKee [17], [18]. Mixed problems with only integral conditions for a second order parabolic equation have been studied by Bouziani-Benouar [7], and for a $2m$ -parabolic equation in Bouziani [5].

In this paper, we prove existence and uniqueness of a strong solution of problem (1.1)–(1.5). For this, we consider the problem (1.1)–(1.5) as a solution of the operator equation

$$(1.6) \quad Lu = \mathcal{F}$$

where $L = (\mathcal{L}, \ell)$, with domain of definition $D(L)$ consisting of functions $u \in L^2(\mathcal{Q})$ such that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial t \partial x^2} \in L^2(\mathcal{Q})$ and u satisfies conditions (1.3)–(1.5); the operator L is considered from B to F , where B is the Banach space consisting of all functions $u(x, t)$ having a finite norm

$$\|u\|_B = \left(\int_{\mathcal{Q}} \left(\mathfrak{I}_x \frac{\partial u}{\partial t} \right)^2 dx dt + \sup_{0 \leq \tau \leq T} \int_0^1 (1-x) \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \right)^{1/2}$$

and satisfying the conditions (1.3)–(1.5), where $\Im_x g = \int_0^x g(\xi, t) d\xi$, and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, \varphi)$ for which the norm

$$\|\mathcal{F}\|_F = \left(\int_{\mathcal{Q}} (1-x)^2 f^2 dx dt + \int_0^1 (1-x) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \right)^{1/2}$$

is finite. Then, we establish an energetic inequality:

$$(1.7) \quad \|u\|_B \leq c \|Lu\|_F$$

and we show that the operator L has a closure $\bar{L}$.

DEFINITION. A solution of the operator equation

$$\bar{L}u = \mathcal{F}$$

is called a strong solution of the problem (1.1)–(1.5).

Since points of the graph $\bar{L}$ are limits of sequences of points of the graph of L , we can extend (1.7) to apply to strong solution by taking limits, i.e.,

$$\|u\|_B \leq c \|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}).$$

From this inequality we obtain the uniqueness of a strong solution if it exists, and the equality of sets $R(\bar{L})$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of the problem (1.1)–(1.5) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in F .

2. An energetic inequality and its consequences

THEOREM 1. Let Condition 1 be fulfilled. Then for any function $u \in D(L)$ we have the inequality

$$(2.1) \quad \|u\|_B \leq c \|Lu\|_F,$$

where c is a positive constant independent of u .

PROOF. Multiplying the equation (1.1) by $Mu = -\Im_x \left((1-x) \frac{\partial u}{\partial t} + \Im_x \frac{\partial u}{\partial t} \right)$ and integrating over $\mathcal{Q}^\tau$, where $\mathcal{Q}^\tau = (0, 1) \times (0, \tau)$, we get

$$(2.2) \quad \begin{aligned} \int_{\mathcal{Q}^\tau} \mathcal{L}u \cdot Mu dx dt &= - \int_{\mathcal{Q}^\tau} \frac{\partial u}{\partial t} \cdot \Im_x \left((1-x) \frac{\partial u}{\partial t} + \Im_x \frac{\partial u}{\partial t} \right) dx dt \\ &+ \int_{\mathcal{Q}^\tau} \frac{\partial^2}{\partial x^2} \left(a(x, t) \frac{\partial u}{\partial x} \right) \cdot \Im_x \left((1-x) \frac{\partial u}{\partial t} + \Im_x \frac{\partial u}{\partial t} \right) dx dt. \end{aligned}$$

Integrating by parts each term of (2.2) with the use of conditions (1.2)–(1.5), we obtain

$$\begin{aligned}
 (2.3) \quad & - \int_{\mathcal{Q}^\tau} \frac{\partial u}{\partial t} \mathfrak{S}_x \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) dx dt \\
 & = - \int_0^\tau \mathfrak{S}_x \frac{\partial u}{\partial t} \mathfrak{S}_x \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) \Big|_{x=0}^{x=1} dt \\
 & \quad + \int_{\mathcal{Q}^\tau} \mathfrak{S}_x \frac{\partial u}{\partial t} \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) dx dt \\
 & = \frac{3}{2} \int_{\mathcal{Q}^\tau} \left(\mathfrak{S}_x \frac{\partial u}{\partial t} \right)^2 dx dt + \frac{1}{2} \int_0^\tau (1-x) \left(\mathfrak{S}_x \frac{\partial u}{\partial t} \right)^2 \Big|_{x=0}^{x=1} dt \\
 & = \frac{3}{2} \int_{\mathcal{Q}^\tau} \left(\mathfrak{S}_x \frac{\partial u}{\partial t} \right)^2 dx dt,
 \end{aligned}$$

$$\begin{aligned}
 (2.4) \quad & \int_{\mathcal{Q}^\tau} \frac{\partial^2}{\partial x^2} \left(a(x, t) \frac{\partial u}{\partial x} \right) \cdot \mathfrak{S}_x \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) dx dt \\
 & = \int_0^\tau \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial u}{\partial x} \right) \cdot \mathfrak{S}_x \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) \Big|_{x=0}^{x=1} dt \\
 & \quad - \int_{\mathcal{Q}^\tau} \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial u}{\partial x} \right) \cdot \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) dx dt \\
 & = \int_0^\tau a(x, t) \frac{\partial u}{\partial x} \cdot \left((1-x) \frac{\partial u}{\partial t} + \mathfrak{S}_x \frac{\partial u}{\partial t} \right) \Big|_{x=0}^{x=1} dt \\
 & \quad + \int_{\mathcal{Q}^\tau} a(x, t) \frac{\partial u}{\partial x} \cdot (1-x) \frac{\partial^2 u}{\partial x \partial t} dx dt \\
 & = \frac{1}{2} \int_0^1 (1-x) a(x, t) \left(\frac{\partial u}{\partial x} \right)^2 \Big|_{t=0}^{t=\tau} dx \\
 & \quad - \frac{1}{2} \int_{\mathcal{Q}^\tau} (1-x) \frac{\partial a(x, t)}{\partial t} \left(\frac{\partial u}{\partial x} \right)^2 dx dt \\
 & = \frac{1}{2} \int_0^1 (1-x) a(x, \tau) \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \\
 & \quad - \frac{1}{2} \int_0^1 (1-x) a(x, 0) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
 & \quad - \frac{1}{2} \int_{\mathcal{Q}^\tau} (1-x) \frac{\partial a(x, t)}{\partial t} \left(\frac{\partial u}{\partial x} \right)^2 dx dt.
 \end{aligned}$$

Putting (2.3) and (2.4) into (2.2), we have

$$\begin{aligned}
 (2.5) \quad & \frac{3}{2} \int_{\mathcal{Q}_\tau} \left(\Im_x \frac{\partial u}{\partial t} \right)^2 dx dt + \frac{1}{2} \int_0^1 (1-x) a(x, \tau) \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \\
 & = - \int_{\mathcal{Q}_\tau} f \cdot \Im_x \left((1-x) \frac{\partial u}{\partial t} + \Im_x \frac{\partial u}{\partial t} \right) dx dt \\
 & \quad + \frac{1}{2} \int_0^1 (1-x) a(x, 0) \left(\frac{\partial \varphi}{\partial x} \right)^2 dx \\
 & \quad + \frac{1}{2} \int_{\mathcal{Q}_\tau} (1-x) \frac{\partial a(x, t)}{\partial t} \left(\frac{\partial u}{\partial x} \right)^2 dx dt.
 \end{aligned}$$

The first integral on the right-hand side of (2.5) can be estimated as

$$\begin{aligned}
 (2.6) \quad & - \int_{\mathcal{Q}_\tau} f \cdot \Im_x \left((1-x) \frac{\partial u}{\partial t} + \Im_x \frac{\partial u}{\partial t} \right) dx dt \\
 & = - \int_{\mathcal{Q}_\tau} f \cdot \left((1-x) \Im_x \frac{\partial u}{\partial t} + 2 \Im_x^2 \frac{\partial u}{\partial t} \right) dx dt \\
 & = - \int_{\mathcal{Q}_\tau} f \cdot (1-x) \Im_x \frac{\partial u}{\partial t} dx dt - 2 \int_0^\tau \Im_x f \cdot \Im_x^2 \frac{\partial u}{\partial t} \Big|_{x=0}^{x=1} dt \\
 & \quad + 2 \int_{\mathcal{Q}_\tau} \Im_x f \cdot \Im_x \frac{\partial u}{\partial t} dx dt \\
 & \leq \frac{1}{2} \int_{\mathcal{Q}_\tau} (1-x)^2 f^2 dx dt + 2 \int_{\mathcal{Q}_\tau} (\Im_x f)^2 dx dt + \int_{\mathcal{Q}_\tau} \left(\Im_x \frac{\partial u}{\partial t} \right)^2 dx dt.
 \end{aligned}$$

In light of the inequality $\int_{\mathcal{Q}_\tau} (\Im_x f)^2 dx dt \leq 4 \int_{\mathcal{Q}_\tau} (1-x)^2 f^2 dx dt$, we get

$$\begin{aligned}
 & - \int_{\mathcal{Q}_\tau} f \cdot \Im_x \left((1-x) \frac{\partial u}{\partial t} + \Im_x \frac{\partial u}{\partial t} \right) dx dt \\
 & \leq \frac{17}{2} \int_{\mathcal{Q}_\tau} (1-x)^2 f^2 dx dt + \int_{\mathcal{Q}_\tau} \left(\Im_x \frac{\partial u}{\partial t} \right)^2 dx dt,
 \end{aligned}$$

where the second integral will be absorbed in the left-hand side of (2.5).

According to *Condition 1*, we then get

$$\begin{aligned}
 (2.7) \quad & \int_{\mathcal{Q}_\tau} \left(\Im_x \frac{\partial u}{\partial t} \right)^2 dx dt + c_0 \int_0^1 (1-x) \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \\
 & \leq 17 \int_{\mathcal{Q}_\tau} (1-x)^2 f^2 dx dt + c_1 \int_0^1 (1-x) (\varphi')^2 dx \\
 & \quad + c_2 \int_{\mathcal{Q}_\tau} (1-x) \left(\frac{\partial u}{\partial x} \right)^2 dx dt.
 \end{aligned}$$

Using Lemma 1 in [9], we have

$$(2.8) \quad \int_{\mathcal{Q}^\tau} \left(\Im_x \frac{\partial u}{\partial t} \right)^2 dx dt + \int_0^1 (1-x) \left(\frac{\partial u(x, \tau)}{\partial x} \right)^2 dx \\ \leq c_3 \left(\int_{\mathcal{Q}} (1-x)^2 f^2 dx dt + \int_0^1 (1-x) (\varphi')^2 dx \right),$$

where

$$c_3 = \frac{\max(17, c_1)}{\min(1, c_0)} \exp(c_2 T).$$

The right-hand side of (2.8) is independent of τ , hence replacing the left-hand side by its upper bound with respect to τ from 0 to T , we obtain the desired inequality, where $c = c_3^{1/2}$. $\square$

PROPOSITION 1. *The operator L from B to F admits a closure.*

PROOF. Suppose that $\{u_n\} \in D(L)$ is a sequence such that

$$(2.9) \quad u_n \rightarrow 0 \quad \text{in } B$$

and

$$(2.10) \quad Lu_n \rightarrow \mathcal{F} = (f, \varphi) \quad \text{in } F;$$

we must show that $f \equiv 0$ and $\varphi \equiv 0$.

According to (2.9) we get

$$u_n \rightarrow 0 \quad \text{in } \mathcal{D}'(\mathcal{Q}).$$

By virtue of the continuity of derivation of $\mathcal{D}'(\mathcal{Q})$ in $\mathcal{D}'(\mathcal{Q})$, we deduce that

$$(2.11) \quad (1-x)\mathcal{L}u_n \rightarrow 0 \quad \text{in } \mathcal{D}'(\mathcal{Q}).$$

Further, according to (2.10), we have

$$(1-x)\mathcal{L}u_n \rightarrow (1-x)f \quad \text{in } L^2(\mathcal{Q}),$$

thus we have

$$(2.12) \quad (1-x)\mathcal{L}u_n \rightarrow (1-x)f \quad \text{in } \mathcal{D}'(\mathcal{Q}).$$

By virtue of the uniqueness of the limit in $\mathcal{D}'(\mathcal{Q})$ we see that $f \equiv 0$.

On the other hand, (2.10) implies that

$$(2.13) \quad \sqrt{1-x} \frac{d\ell u_n}{dx} \rightarrow \sqrt{1-x} \varphi' \quad \text{in } L^2(0,1).$$

Moreover, since by virtue of (2.9) and the fact that

$$\int_0^1 (1-x) \left(\frac{d\ell u_n}{dx} \right)^2 dx \leq \|u_n\|_B^2, \quad \forall n,$$

we have

$$(2.14) \quad \sqrt{1-x} \frac{d\ell u_n}{dx} \rightarrow 0 \quad \text{in } L^2(0,1).$$

Now the uniqueness of the limit in $L^2(0,1)$ implies that $\varphi \equiv 0$. $\square$

Theorem 1 is valid for strong solutions, i.e., we have the inequality

$$(2.15) \quad \|u\|_B \leq c \|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}).$$

Hence we obtain

COROLLARY 1. *A strong solution of the problem (1.1)–(1.5) is unique if it exists, and depends continuously on $\mathcal{F} = (f, \varphi) \in F$.*

COROLLARY 2. *The range $R(\bar{L})$ of the operator $\bar{L}$ is closed in F , and $R(\bar{L}) = \overline{R(L)}$.*

3. Existence of solutions

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in D(L)$ and for arbitrary $\mathcal{F} = (f, \varphi) \in F$.

THEOREM 2. *Suppose the conditions of Theorem 1 are satisfied. Then the problem (1.1)–(1.5) admits a unique strong solution $u = \bar{L}^{-1}\mathcal{F} = \overline{L^{-1}\mathcal{F}}$.*

PROOF. First we prove that $R(L)$ is dense in F for the special case where $D(L)$ is reduced to $D_0(L)$, where $D_0(L) = \{u/u \in D(L) : \ell u = 0\}$. $\square$

PROPOSITION 2. *Let the conditions of Theorem 2 be satisfied. If, for $(1-x)\omega \in L^2(Q)$ and for all $u \in D_0(L)$, we have*

$$(3.1) \quad \int_Q (1-x)^2 \mathcal{L}u\omega dx dt = 0,$$

then ω vanishes almost everywhere in Q .

PROOF. The equality (3.1) can be written as follows:

$$(3.2) \quad \int_{\mathcal{Q}} (1-x)^2 \frac{\partial u}{\partial t} \cdot \omega dx dt = \int_{\mathcal{Q}} (1-x)^2 \frac{\partial^2}{\partial x^2} \left(a(x, t) \frac{\partial u}{\partial x} \right) \cdot \omega dx dt.$$

If we put

$$u = \mathfrak{S}_t(e^{c_4 \tau} z) = \int_0^t e^{c_4 \tau} z(x, \tau) d\tau,$$

where c_4 is a constant such that $c_4 c_0 - c_2 \geq 0$, and $z, \frac{\partial z}{\partial x}, \frac{\partial}{\partial x} \left(a \frac{\partial \mathfrak{S}_t(e^{c_4 \tau} z)}{\partial x} \right), (1-x) \frac{\partial^2}{\partial x^2} \left(a \frac{\partial \mathfrak{S}_t(e^{c_4 \tau} z)}{\partial x} \right) \in L^2(\mathcal{Q})$, then, u satisfies the conditions (1.3)–(1.5). As a result of (3.2), we obtain the equality

$$(3.3) \quad \int_{\mathcal{Q}} e^{c_4 t} (1-x)^2 z \omega dx dt = \int_{\mathcal{Q}} (1-x)^2 \frac{\partial^2}{\partial x^2} \left(a \frac{\partial \mathfrak{S}_t(e^{c_4 \tau} z)}{\partial x} \right) \omega dx dt$$

The left-hand side of (3.3) shows that the mapping

$$L^2(\mathcal{Q}) \ni z \rightarrow \int_{\mathcal{Q}} (1-x)^2 \frac{\partial^2}{\partial x^2} \left(a \frac{\partial \mathfrak{S}_t(e^{c_4 \tau} z)}{\partial x} \right) \omega dx dt$$

is a continuous linear functional of z . From the right-hand side of (3.3) there follows that this is true if the function ω has the following properties:

$$(1-x) \frac{\partial \omega}{\partial x}, \mathfrak{S}_t^*(a\omega), (1-x) \mathfrak{S}_t^* \left(a \frac{\partial \omega}{\partial x} \right), \frac{\partial}{\partial x} \mathfrak{S}_t^*(a\omega), \frac{\partial}{\partial x} \left((1-x) \mathfrak{S}_t^* \left(a \frac{\partial \omega}{\partial x} \right) \right), \\ (1-x)^2 \mathfrak{S}_t^* \left(a \frac{\partial^2 \omega}{\partial x^2} \right), \frac{\partial}{\partial x} \left((1-x)^2 \mathfrak{S}_t^* \left(a \frac{\partial^2 \omega}{\partial x^2} \right) \right) \in L^2(\mathcal{Q}),$$

and

$$\omega_{x=0} = 0, \quad \frac{\partial \omega}{\partial x} \Big|_{x=0} = 0, \quad \mathfrak{S}_t^*(a\omega)|_{x=1} = 0.$$

In terms of the given function ω , from the equality (3.3) we determine the function

$$z = -4(1-x) \frac{\partial \omega}{\partial x} + 12\omega - 8 \int_0^x \frac{\omega(\xi, t)}{1-\xi} d\xi.$$

Integrating by parts with respects to ξ , we have

$$\mathfrak{S}_x z = -4 \int_0^x (1-\xi) \frac{\partial \omega(\xi, t)}{\partial \xi} d\xi + 12 \int_0^x \omega(\xi, t) d\xi - 8 \int_0^x d\xi \int_0^\xi \frac{\omega(\eta, t)}{1-\eta} d\eta$$

$$\begin{aligned}
&= 8(1-x) \int_0^x \frac{\omega(\xi, t)}{1-\xi} d\xi - 4(1-x)\omega(x, t) \\
&= -4(1-x) \int_0^x \frac{1}{(1-\xi)^2} \frac{\partial((1-\xi)^2 \omega(\xi, t))}{\partial \xi} d\xi.
\end{aligned}$$

Integrating once more by parts with respect to ξ , we obtain

$$\begin{aligned}
\Im_x^2 z &= -4 \int_0^x (1-\xi) \int_0^\xi \frac{1}{(1-\eta)^2} \frac{\partial((1-\eta)^2 \omega(\eta, t))}{\partial \eta} d\eta d\xi \\
&= 2 \int_0^x d(1-\xi)^2 \int_0^\xi \frac{1}{(1-\eta)^2} \frac{\partial((1-\eta)^2 \omega(\eta, t))}{\partial \eta} d\eta \\
&= 2(1-x)^2 \int_0^x \frac{1}{(1-\xi)^2} \frac{\partial((1-\xi)^2 \omega(\xi, t))}{\partial \xi} d\xi - 2(1-x)^2 \omega(x, t).
\end{aligned}$$

Hence it follows that

$$(3.4) \quad (1-x)^2 \omega = -\frac{1}{4}((1-x)\Im_x z + 2\Im_x^2 z),$$

$$(3.5) \quad z(0, t) = 0,$$

$$(3.6) \quad \Im_1 z = 0,$$

$$(3.7) \quad \Im_1^2 z = 0.$$

Replacing $(1-x)^2 \omega$ in (3.3) by its representation (3.4), we obtain

$$\begin{aligned}
(3.8) \quad & - \int_{\mathcal{Q}} e^{c_4 t} z ((1-x)\Im_x z + 2\Im_x^2 z) dx dt \\
&= - \int_{\mathcal{Q}} \frac{\partial^2}{\partial x^2} \left(a(x, t) a \frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} \right) ((1-x)\Im_x z + 2\Im_x^2 z) dx dt.
\end{aligned}$$

Integrating by parts in the right-hand side of (3.8) with respect to x and t by taking into account (3.5)–(3.7) yields

$$\begin{aligned}
& - \int_{\mathcal{Q}} e^{c_4 t} z ((1-x)\Im_x z + 2\Im_x^2 z) dx dt \\
&= - \int_0^T \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} \right) ((1-x)\Im_x z + 2\Im_x^2 z) \Big|_{x=0}^{x=1} dt
\end{aligned}$$

$$\begin{aligned}
& + \int_{\mathcal{Q}} \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} \right) ((1-x)z + \Im_x z) dx dt \\
& = \int_0^T a(x, t) \frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} ((1-x)z + \Im_x z) \Big|_{x=0}^{x=1} dt \\
& \quad - \int_{\mathcal{Q}} a(x, t) \frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} (1-x) \frac{\partial z}{\partial x} dx dt \\
& = -\frac{1}{2} \int_0^1 e^{-c_4 t} (1-x) a(x, t) \left(a \frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} \right)^2 \Big|_{t=0}^{t=T} dx \\
& \quad - \frac{1}{2} \int_{\mathcal{Q}} e^{-c_4 t} (1-x) \left(c_4 a(x, t) - \frac{\partial a(x, t)}{\partial t} \right) \left(\frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} \right)^2 dx dt.
\end{aligned}$$

By *Condition 1*, we obtain

$$\begin{aligned}
(3.9) \quad & - \int_{\mathcal{Q}} e^{c_4 t} z ((1-x)\Im_x z + 2\Im_x^2 z) dx dt \\
& \leq -\frac{1}{2} (c_4 c_0 - c_2) \int_{\mathcal{Q}} e^{-c_4 t} (1-x) \left(\frac{\partial \Im_t(e^{c_4 \tau} z)}{\partial x} \right)^2 dx dt \leq 0.
\end{aligned}$$

Integrating by parts in the left-hand side of (3.9) with respect to x , we obtain

$$\frac{3}{2} \int_{\mathcal{Q}} e^{c_4 t} (\Im_x z)^2 dx dt \leq 0$$

and thus $z = 0$; hence $\omega = 0$. This proves Proposition 2. $\square$

We return to the proof of Theorem 2. We have already noted that it is sufficient to prove that the set $R(L)$ is dense in F .

Suppose that, for some $W = (\omega, \omega_0) \in F$ and for all $u \in D(L)$, it holds

$$(3.10) \quad (Lu, W)_F = \int_{\mathcal{Q}} (1-x)^2 \mathcal{L}u \omega dx dt + \int_0^1 (1-x) \frac{\partial \ell u}{\partial x} \cdot \frac{\partial \omega_0}{\partial x} dx dt = 0.$$

Then we must prove that $W = 0$. Putting $u \in D_0(L)$ in (3.10), we obtain

$$\int_{\mathcal{Q}} (1-x)^2 \mathcal{L}u \omega dx dt = 0, \quad u \in D_0(L).$$

Hence Proposition 2 implies that $\omega = 0$. Thus, (3.10) takes the form

$$(3.11) \quad \int_0^1 (1-x) \frac{\partial \ell u}{\partial x} \cdot \frac{\partial \omega_0}{\partial x} dx dt = 0, \quad u \in D(L).$$

Since the range of the trace operator ℓ is dense in the Hilbert space with the norm $\left(\int_0^1 (1-x) \left(\frac{\partial \ell u}{\partial x} \right)^2 dx dt \right)^{1/2}$, the equality (3.11) implies that $\omega_0 = 0$ (we

recall that ω_0 satisfies *Condition 2*). Hence $W = 0$. This completes the proof of Theorem 2.

REMARK. We can prove that our results still hold for a mixed problem for a certain $(2m + 1)$ -parabolic equation which combine local and integral conditions, i.e.,

$$(3.12) \quad \mathcal{L}u = \frac{\partial u}{\partial t} + (-1)^m \frac{\partial^{2m}}{\partial x^{2m}} \left(a(x, t) \frac{\partial u}{\partial x} \right) = f(x, t),$$

$$(3.13) \quad \ell u = u(x, 0) = \varphi(x),$$

$$(3.14) \quad \frac{\partial^k u}{\partial x^k} \Big|_{x=0} = \frac{\partial^{k-1} u}{\partial x^{k-1}} \Big|_{x=1} = 0, \quad 0 \leq k \leq n \leq m,$$

$$(3.15) \quad \int_0^1 x^l u(x, t) dx = 0, \quad 0 \leq l \leq m - n.$$

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