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UNKNOTTING TUNNELS OF MONTESINOS LINKS

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Abstract

Let L be a link in S^3. The *tunnel number* $t(L)$ of L is the minimal number of mutually disjoint arcs $\{\tau_i\}$ "properly embedded" in the pair (S^3, L) such that the complement of an open regular neighborhood of $L \cup (\cup\tau_i)$ is a handlebody. In the above, if the arc system consists of only one arc, it is called an *unknotting tunnel* of L. In [8], Morimoto-Sakuma-Yokota have determined the knot types of Montesinos knots with tunnel number one. In this paper, by using a method similar to those in [1] and [8], we determine the link types of 2-component Montesinos links with tunnel number one (Theorem 2.2), and determine the tunnel numbers of 2-component prime links up to 9 crossings (Theorem 2.6).

1. Spatial graphs associated with unknotting tunnels

A 2-component link $L = K_1 \cup K_2$ in S^3 is said to be *strongly invertible* if there is an involution h of the pair (S^3, L) such that $Fix(h)$ is a circle which

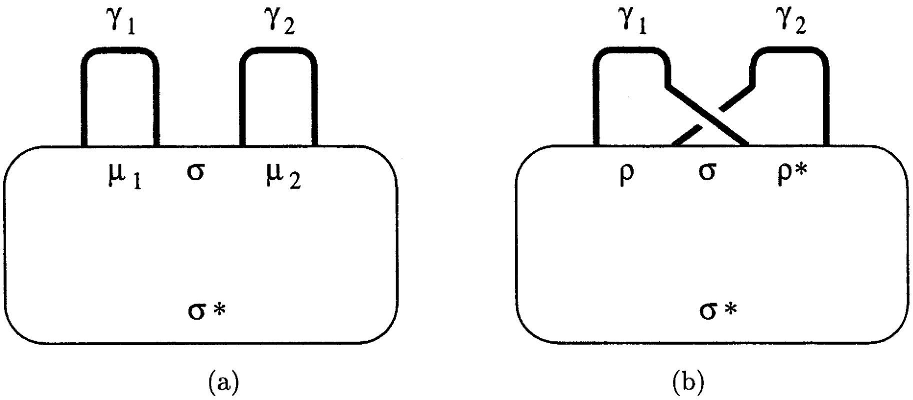


Fig. 1.1.

intersects K_i $(i = 1, 2)$ in two points. We call h a *strong inversion of* L. Let p be the projection $S^3 \to S^3/h$, and put $O = p(Fix(h))$, $\gamma_i = p(K_i)$ $(i = 1, 2)$, $\gamma = \gamma_1 \cup \gamma_2$ and $G(L, h) = O \cup \gamma$. Then, by [12], S^3/h is again the 3-sphere, O is the trivial knot, γ_i $(i = 1, 2)$ is an arc such that $\gamma_1 \cap \gamma_2 = \emptyset$ and $\gamma_i \cap O = \partial\gamma_i$. We call $G(L, h)$ the *spatial graph associated with h.* If $lk(K_1, K_2)$ is even, then σ, σ^*, μ_1 and μ_2 denote the edges of $G(L, h)$ as illustrated in Figure 1.1 (a). If $lk(K_1, K_2)$ is odd, then σ, σ^*, ρ and ρ^* denote the edges of $G(L, h)$ as illustrated in Figure 1.1 (b).

Then, by arguments similar to [1, Proof of Theorem 2.1] and [8, Theorem 1.2], we obtain the following theorem (*cf.* Figure 1.2 and Figure 1.3).

THEOREM 1.1. *A 2-component link L in S^3 admits an unknotting tunnel if and only if L admits a strong inversion h such that $G(L, h)$ has a 3-bridge decomposition. In this case, the unknotting tunnel τ is a subarc of $Fix(h)$.*

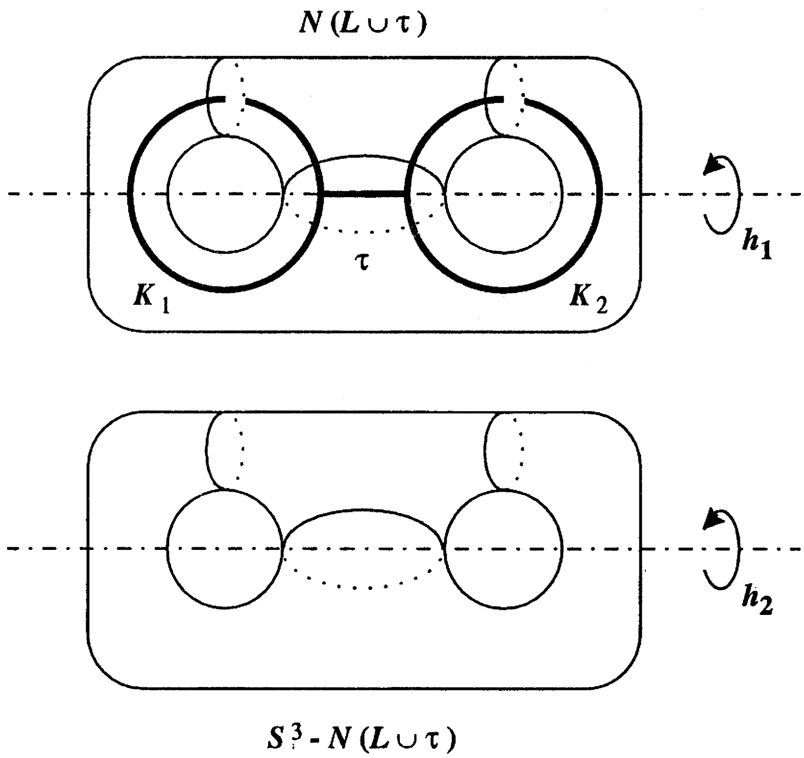


Fig. 1.2.

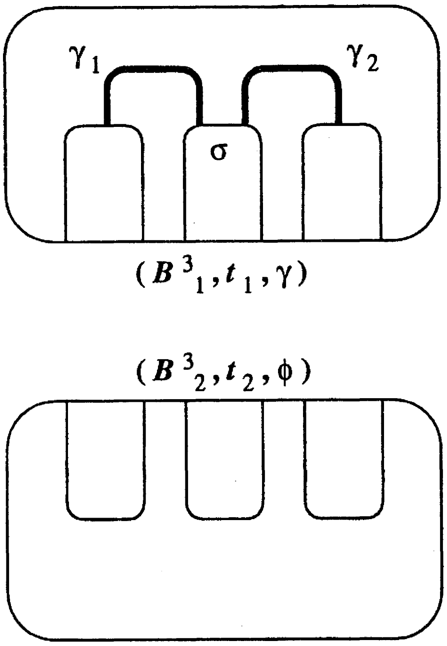


Fig. 1.3.

We say that $G(L,h)$ has a 3-*bridge decomposition* if (S^3, O, γ) is decomposed into (B_1, t_1, γ) and $(B_2, t_2, \emptyset)$, where (B_i, t_i) is a 3-strand trivial tangle for $i = 1, 2$ and γ_1 and γ_2 are two "trivial" arcs in (B_1, t_1) as illustrated in Figure 1.3 (*cf.* [8, Definition 1.1]).

COROLLARY 1.2. *Suppose a* 2-*component link* L *in* S^3 *admits an unknotting tunnel* τ, *and let* h *be the strong inversion given by Theorem* 1.1. *Then the following holds.*

(1) *Put* $\sigma = p(\tau)$, *where* p *is the projection* $S^3 \to S^3/h$. *Then the knot* $\gamma_1 \cup \sigma \cup \gamma_2 \cup \sigma^*$ *in* S^3/h *is trivial.*

(2) *Let* $(B, t) = (\overline{S^3/h - N(\sigma \cup \gamma)}, \overline{G(L,h) - N(\sigma \cup \gamma)})$. *Then* (B, t) *is a* 3-*strand trivial tangle.*

2. Unknotting tunnels of 2-component Montesinos links

A Montesinos link $L = M(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r))$ with r branches is the link in S^3 as illustrated in Figure 2.1 (*a*).

Here r, b, α_i and β_i are integers such that $r \geq 0$, $\alpha_i \geq 2$ and *g.c.d.*$(\alpha_i, \beta_i) = 1$. The symbol $\boxed{\frac{\beta}{\alpha}}$ stands for a rational tangle of slope β/α (*cf.* Figure 2.1 (*b*)). If

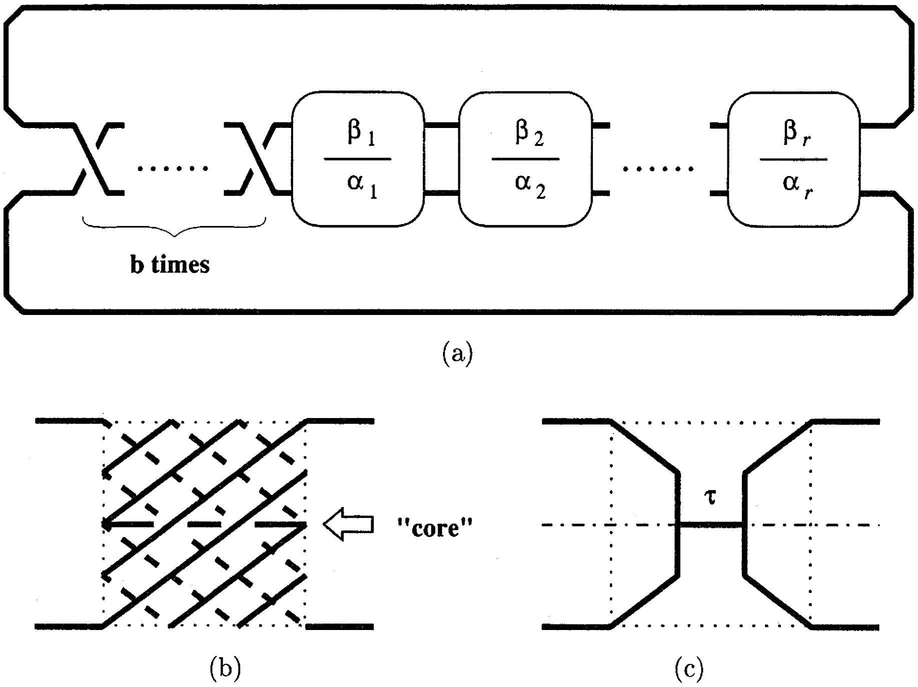


Fig. 2.1.

we forget the chart on the boundary, a rational tangle is merely a 2-strand trivial tangle as illustrated in Figure 2.1 (*c*); we call the image of the arc τ in 2.1 (*b*) in a rational tangle the *core* of the rational tangle. The following proposition is well-known (*cf.* [3, Chapter 12]):

PROPOSITION 2.1.

(1) *Suppose* $r = 2$. *Then* L *is a 2-bridge link* $S(p,q)$ *of type* (p,q), *where* $p = |b\alpha_1\alpha_2 - \alpha_1\beta_2 - \alpha_2\beta_1|$ *and* q *is an integer relatively prime to* p. *In particular,* L *is the trivial knot, if and only if* $b\alpha_1\alpha_2 - \alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1$.

(2) *Suppose* $r \geq 3$. *Then* L *is not a 2-bridge link, and it is classified by the Euler number*

$$e(L) = b - \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i}$$

and the vector

$$v(L) = \left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \dots, \frac{\beta_r}{\alpha_r}\right) \in (\boldsymbol{Q}/\boldsymbol{Z})^r$$

up to cyclic permutation and reversal of the order.

In this section, we prove the following theorem:

THEOREM 2.2. *Let $L = M(b;(\alpha_1,\beta_1),(\alpha_2,\beta_2),\dots,(\alpha_r,\beta_r))$ be a 2-component Montesinos link with r branches. Then L has tunnel number one if and only if one of the following conditions holds up to cyclic permutation of the indice*:

(1) $r = 2$.

(2) $r = 3$ *and* $\alpha_1 = 2$.

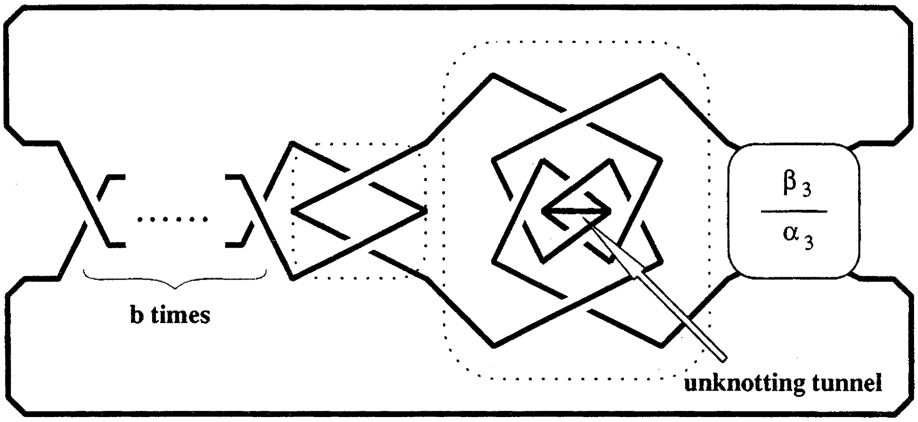


Fig. 2.2.

PROOF. First, we prove the "*if*" part of Theorem 2.2. Suppose the condition (1) is satisfied, then L is a 2-bridge link, and therefore it has tunnel number one (*cf.* [1]). Suppose the condition (2) is satisfied. Then, since L has two components, we may assume α_2 is even and α_3 is odd. Then by [8, Figure 2.2], we can see that the core of the rational tangle of slope β_2/α_2 is an unknotting tunnel.

Next, we prove the "*only if*" part of Theorem 2.2. The following result has been proved by Klimenko-Sakuma [6, Theorem B].

PROPOSITION 2.3. *The extended triangle group*

$$\langle x_1,\dots,x_r \mid x_1^2 = \cdots = x_r^2 = (x_1x_2)^{\alpha_1} = \cdots (x_rx_1)^{\alpha_r} = 1\rangle$$

is genarated by two elements if and only if one of the following conditions is satisfied up to permutation of the indices:

(a) $r = 3$, $\alpha_1 = 2$, *and* $\alpha_2 \not\equiv 0 \pmod 2$.

(b) $r = 3$, $\alpha_1 = \alpha_2 = 3$, *and* $\alpha_3 \not\equiv 0 \pmod 3$.

Suppose L has tunnel number one. Then the knot group $G(L) = \pi_1(S^3 - L)$ is generated by two elements. And, it is well-known (*cf.* [3, Chapter 12]) that the knot group of a Montesinos link $M(b;\ (\alpha_1,\beta_1),\ (\alpha_2,\beta_2),\dots,(\alpha_r,\beta_r))$ has an epimorphism to the reflection group

$$\langle x_1,\dots,x_r \mid x_1^2=\cdots=x_r^2=(x_1x_2)^{\alpha_1}=\cdots(x_rx_1)^{\alpha_r}=1\rangle .$$

Hence, the condition (a) or (b) in Proposition 2.3 must be satisfied.

Suppose the condition (a) is satisfied, then the condition (2) in Theorem 2.2 is satisfied. Suppose the condition (b) is satisfied, that is, $L = M(b;(3,1),(3,\pm1),(\alpha,\beta))$ $(\alpha\not\equiv 0 \pmod 3)$. Since L has two components, we see $\alpha\equiv\pm1 \pmod 6$. Note that $\alpha\geq 3$ and hence L is non-elliptic in the sense of [2]. Let $Sym(S^3,L)$ be the symmetry group of L, i.e., the group of all pairwise isotopy classes of diffeomorphisms of the pair (S^3,L). Let $Sym_+(S^3,L)$ be the subgroup of $Sym(S^3,L)$ generated by diffeomorphisms which preserve the orientation of S^3. The following result has been proved by Boileau-Zimmermann [2].

PROPOSITION 2.4. *There is an exact sequence*

$$1\longrightarrow \boldsymbol{Z}_2\longrightarrow Sym_+(S^3,L)\xrightarrow{\Psi} D_+(\boldsymbol{v}(L))\longrightarrow 1,$$

where $D_+(\boldsymbol{v}(L))$ is the group of those dihedral permutations of the components of the vector $\boldsymbol{v}(L)=(\beta_1/\alpha_1,\beta_2/\alpha_2,\dots,\beta_r/\alpha_r)\in(\boldsymbol{Q}/\boldsymbol{Z})^r$ which preserve $\boldsymbol{v}(L)$.

Suppose $L=M(b;(3,1),(3,-1),(\alpha,\beta))$, then $Sym_+(S^3,L)$ is isomorphic to $\boldsymbol{Z}_2$ by the above proposition and its generator is realized by an involution of (S^3,L) interchanging the components of L (*cf.* [2, Section 3]. Hence L is not invertible and has no unknotting tunnels. Thus, in the following, we assume $L=M(b;(3,1),(3,1),\ (\alpha,\beta))$ when $\alpha\equiv\pm1 \pmod 6$. Then by Proposition 2.4

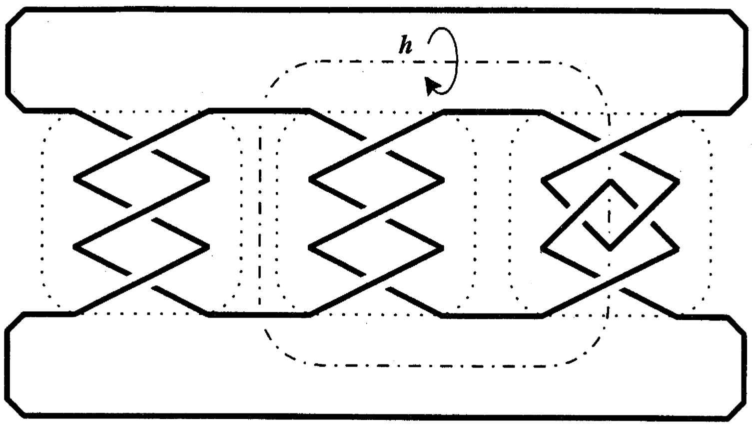
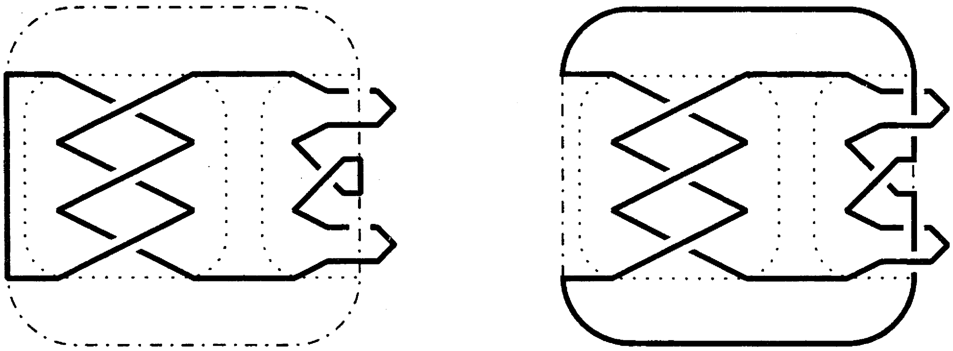


Fig. 2.3.



(a) (b)

Fig. 2.4.

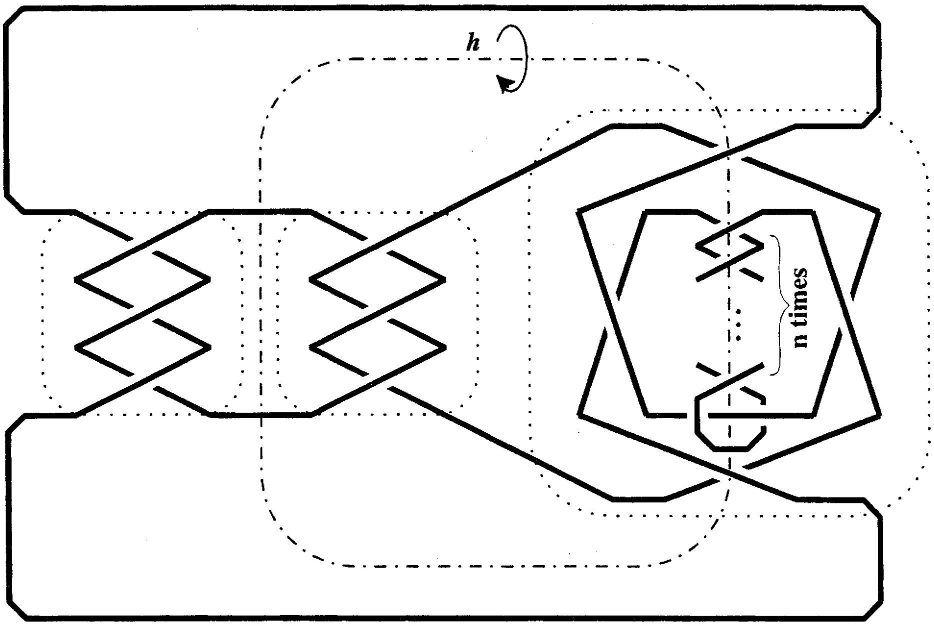


Fig. 2.5.

and the result of [11], L has precisely one strong inversion h, which is illustrated in Figure 2.3 (*cf.* also Figure 2.5).

Now, suppose that h is associated with an unknotting tunnel τ. By Corollary 1.2, the constituent knot $C = \gamma_1 \cup \sigma \cup \gamma_2 \cup \sigma^*$ of $G(L, h)$ must be trivial, where $\sigma = p(\tau)$.

Suppose $lk(K_1, K_2)$ is even. The constituent knot $C = \gamma_1 \cup \sigma \cup \gamma_2 \cup \sigma^*$ of

$G(L,h)$ is the knot as illustrated in Figure 2.4 (a) ($cf.$ [8, Lemma 2.4 (2)]). Then, we can see that it is a trefoil knot by using the fact that α is even. This is a contradiction.

Suppose $lk(K_1,K_2)$ is odd. Put $C_\sigma = \gamma_1 \cup \sigma \cup \gamma_2 \cup \sigma^*$ and $C_\rho = \gamma_1 \cup \rho \cup \gamma_2 \cup \rho^*$. Then the constituent knot in Corollary 1.2 (1) is equal to C_σ or C_ρ. By the previous argument, we see C_σ is nontrivial. Hence C_ρ must be the trivial knot. On the other hand we can see, by using [8, Lemma 2.4], that C_ρ is $M(0;(3,1),(\alpha,(b\alpha+\beta)/2))$ ($cf.$ Figure 2.4 (b)). Since L has two components, we see $(b\alpha+\beta)/2 \equiv 0 \pmod 2$. Since C_ρ is trivial, we see $3\times(b\alpha+\beta)/2-\alpha\times 1=\pm 1$, by Proposition 2.1 (1). Hence $(\alpha,\beta)=(-6n\pm 1, 4n)$ for some n. Hence L is equivalent to the Montesinos link $M(0;(3,1),(3,1),(-6n\pm 1,4n))$ ($cf.$ Figure 2.5).

Let $(B,t) = (\overline{S^3/h - N(\sigma\cup\gamma)}, \overline{G(L,h)-N(\sigma\cup\gamma)})$ be the tangle illustrated by the solid lines in Figure 2.6. We can see that the tangle (B,t) in

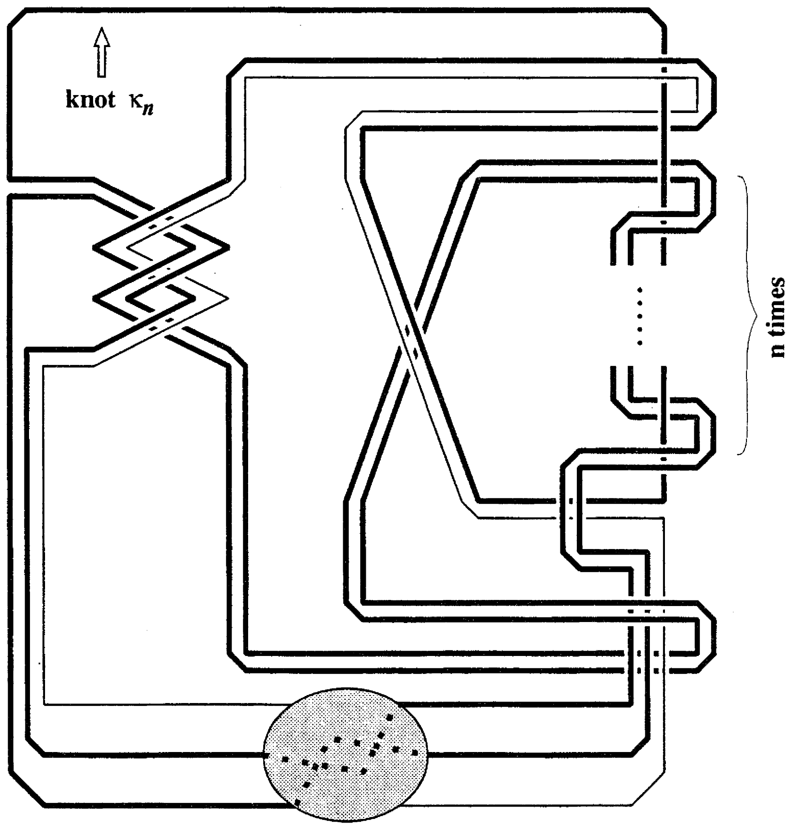


Fig. 2.6.

Corollary 1.2 is equivalent to the tangle ilustrated in Figure 2.6. Let s be the component of t illustrated by the thin solid line in Figure 2.6. Then $(B, t-s)$ must be a 2-strand trivial tangle since (B, t) is a trivial tangle by Corollary 1.2 (2). Hence any knot obtained from $(B, t-s)$ by attaching a 2-strand trivial tangle must be a 2-bridge knot. Let κ_n be the knot obtained from $(B, t-s)$ by attaching the rational tangle of slope $2n$. The knot in Figure 2.6 which is illustrated by the thick solid lines and the thick broken lines is κ_n. Then we can see that κ_n is isotopic to the knot obtained from c_1 by $1/n$ surgery along c_2 indicated in Figure 2.7.

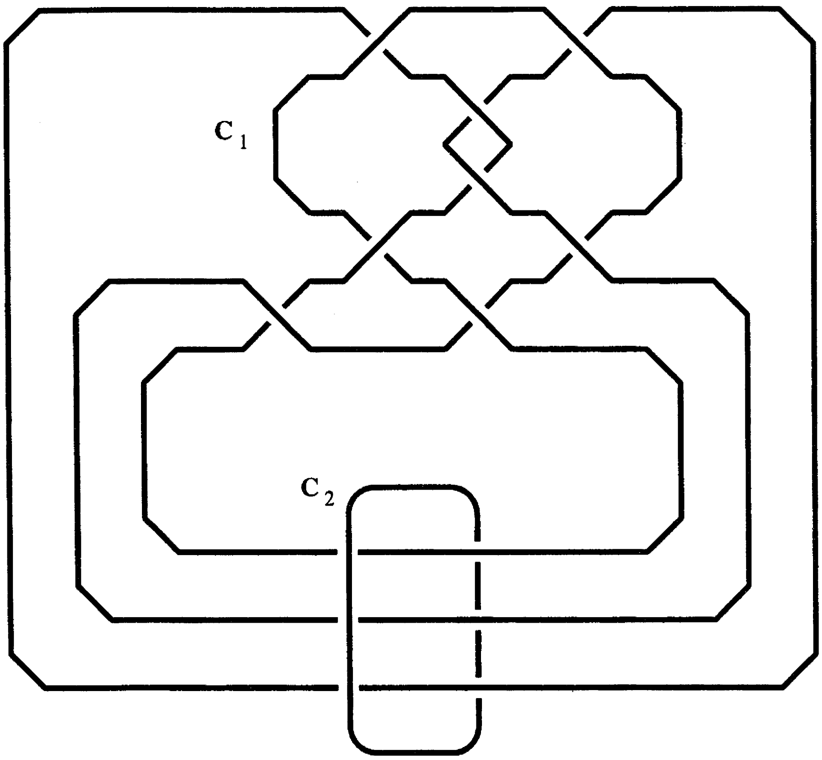


Fig. 2.7.

The two-variable Alexander polynomial $\Delta(c_1 \cup c_2; x, y)$ of the link $c_1 \cup c_2$ is given by

$$\begin{aligned}\Delta(c_1 \cup c_2; x, y) = {} & 2x^2 - 3x^3 + x^4 - y + 3xy - 3x^2y \\ & + 3x^3y - x^4y + y^2 - 3xy^2 + 2x^2y^2.\end{aligned}$$

Therefore by [5, Theorem 3.1 and Corollary 3.2], the Alexander polynomial $\Delta(\kappa_n;t)$ of κ_n is given by

$$\begin{aligned}\Delta(\kappa_n;t) &= \Delta(c_1\cup c_2;t,t^{2n}) \\ &= 2t^2-3t^3+t^4-t^{2n}+3t\,(t^{2n})-3t^2\,(t^{2n}) \\ &\quad +3t^3\,(t^{2n})-t^4\,(t^{2n})+(t^{2n})^2-3t\,(t^{2n})^2+2t^2\,(t^{2n})^2 \\ &= 2t^2-3t^3+t^4-t^{2n}+3t^{2n+1}-3t^{2n+2} \\ &\quad +3t^{2n+3}-t^{2n+4}+t^{4n}-3t^{4n+1}+2t^{4n+2}.\end{aligned}$$

By using the following result of Murasugi [7], we see that κ_n is not a 2-bridge knot. This completes the proof of Theorem 2.2.

LEMMA 2.5. *If K is a knot in S^3 with two bridges, then the Alexander polynomial of K, $\Delta(K;t)=t^n(a_0+a_1t+\cdots+a_mt^m)$, satisfies $\Delta(K;t)\equiv\frac{1-t^\lambda}{1-t}$ (mod 2) for some odd integer λ.*

At the end of this paper, we apply our method to the prime links up to 9 crossings. In the following, we use the numbering in the table of Rolfsen's book. The bridge indices of these links are equal to 2 or 3. So, their tunnel numbers are equal to 1 or 2. Moreover, the symmetry groups of hyperbolic links up to 9-crossings are determined by [4], except 9^2_{24} and 9^2_{40}. For these exceptional links, we use the computer program `Snappea (for Power Macintosh)` by Weeks [13], and determine their symmetry groups.

So, by Theorem 1.1, the problem to determine the tunnel numbers of these links is reduced to a problem concerning spatial graphs. However, for most of the spatial graphs arising from these links, the problem is settled by the method described in Corollary 1.2. And for the non-simple links 9^2_{42} and 9^2_{61}, the problem is settled by the result of Muños-Uchida [9].

THEOREM 2.6. *Let L be a prime link with at most 9 crossings. Then $t(L)=2$ if and only if L is equivalent to one of the following links; otherwise, $t(L)=1$:*

7^2_6

8^2_n *with* $n=13,14$

9^2_n *with* $n=23,24,29\sim 42,53\sim 61$

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