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Przytycki, Józef H.

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FUNDAMENTALS OF KAUFFMAN BRACKET SKEIN MODULES

Józef H. PRZYTYCKI

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Abstract

Skein modules are the main objects of an algebraic topology based on knots (or position). In the same spirit as Leibniz we would call our approach *algebra situs*. When looking at the panorama of skein modules[1], we see, past the rolling hills of homologies and homotopies, distant mountains – the Kauffman bracket skein module, and farther off in the distance skein modules based on other quantum invariants. We concentrate here on the basic properties of the Kauffman bracket skein module; fundamental properties in further development of the theory. In particular we consider the relative Kauffman bracket skein module, and we analyze skein modules of I-bundles over surfaces.

History of skein modules from my personal perspective

I would like to use this opportunity, of informal presentation, to give my personal history of algebraic topology based on knots (a more formal account was given in [Pr-7]).

In July 1986 I left Poland invited by Dale Rolfsen for a visiting position at UBC. In January of 1987, Jim Hoste gave a talk at the first Cascade Mountains Conference (in Vancouver) and described his work on multivariable generalization of the Jones-Conway ([HOMFLY] [PT]) polynomial of links in S^3. He was convinced that his construction works for 2 colors when the first color is represented only by a trivial component. He had already succeeded in the case of 2-component 2-bridge links. His method, following Nakanishi [Na], was to analyze link diagrams in an annulus (the trivial component being z axis). We immediately noticed (with Jim) that the analogous construction for the

[1]This is an extended version of a part of the talk "Panorama of skein modules", given at *Low Dimensional Topology Conference*; Madeira, Portugal, January, 1998. Chapters 1–4 are based on the notes prepared for the course: "Topics in Topology" taught at GWU in Fall semester 1996.

Kauffman bracket polynomial has an easy solution [H-P-1]. In March of 1987 I got a manuscript on "Invariants of colored links" [H-K]. I read it carefully and I was trying to generalize the work of Hoste and Kidwell to $n+1$ colors (n of which are used to color a trivial link of n components). This led may attention to the possible torsion[2] related to the Dirac trick (more precisely with 2-torsion in the mapping class group of the 2-sphere). I discussed it with Darryl McCullough, whom I visited in the first part of April. On the last weekend of my stay in Oklahoma, I was struck by the idea that I am not really analyzing colored links, but a (monochromatic) knot theory in the solid torus or the connected sum of solid tori. Knots are formally added and taken modulo the skein relation coming from the Jones-Conway polynomial, in the case of [H-K], or from the Kauffman bracket polynomial in the case of [H-P-1]. For me, this warm April day in Oklahoma was the birth of *skein modules*[3].

Skein relations have their origin in an observation by Alexander ([A1], 1928) that his polynomials of three links L_+, L_- and L_0 in S^3 are linearly related (here L_+, L_- and L_0 denote three links which are identical except in a small ball as shown in Fig. 0.1). Conway rediscovered the Alexander observation and normalized the Alexander polynomial so that it satisfies the skein relation

$$\Delta_{L_+}(z) - \Delta_{L_-}(z) = z\Delta_{L_0}(z)$$

([Co-1], 1969). For my invention/discovery of skein modules, it was probably crucial that I had read Conway's famous paper [Co-1] and the following it work by Giller [Gi], Kauffman [Ka-1, Ka-2, Ka-3], and Lickorish and Millett [L-M-1][4].

Still in April of 1987 (visiting C. Gordon and J. Hempel in Texas), I interpreted several known facts in the language of skein modules. First was the observation that the existence of Jones type polynomials could be interpreted as saying that the related skein modules in S^3 were free and generated by the unknot. Then the results of Hoste-Kidwell and Hoste-Przytycki interpret (compute) skein modules of the solid torus. The Kauffman method allowed the computation of the Kauffman bracket skein module of the product of a surface and the interval (this

[2]In November of 1984, I gave a talk at Warsaw seminar about the recently discovered Jones polynomial, and when describing the Jones skein relation and the Alexander skein relation I was asked by Pawel Traczyk whether we really need any restrictions on coefficients. I realized then that even if restrictions are needed we should not assume them from the beginning but instead we should analyze their character.

[3]I was hesitant about what to call these new objects. I thought that Conway's "linear skein" was to parochial for the "big word" I envisioned for the concept. Still I wanted to keep "skein" acknowledging Conway's vision [Co-1]. The name "skein group" would be natural (like homology group) but misleading. Finally, I decided for the *skein module.*

[4]Only later I learned that in the late seventies Conway advocated the idea of considering the free $Z[z]$-module over oriented links in an oriented 3-manifold and dividing it by the submodule generated by his skein relation [Co-2] [Co-3]; Conway called the resulting module "linear skein".

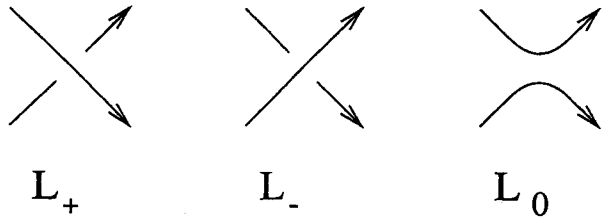


Fig. 0.1.

is a very important fact which I will discuss in details later in the talk). Relative skein modules of the disc (Temperley-Lieb algebra, Hecke algebra and Birman-Murakami-Wenzl algebra as they are known now) was computed/interpreted and shown to be free of $\frac{1}{n+1}\binom{2n}{n}$ (resp. $n!$ and $(2n-1)(2n-3)\cdots 1$) generators. I was told by Pawel Traczyk, in summer of 1986, about easy proofs of these facts.

I wrote the introductory paper on skein modules in May of 1987 [Pr-1]. My initial definition of the Kauffman bracket skein module was rather clumsy: it used unframed links up to regular isotopy. It worked well for $M = F \times I$, in particular for a handlebody, but then I was forced to consider a Heegaard decomposition of a manifold.

In May of 1988 (at the conference in Annapolis) I got a paper by Vladimir Turaev, in which he introduced (independently) the concept of skein modules [Tu-1]. He pointed there importance of framing in definitions of some skein modules.

1. Skein modules of 3-manifolds

Our goal is to build an algebraic topology based on knots[5]. We call the main object used in the theory *a skein module* and we associate it to any 3-dimensional manifold. Skein modules are quotients of free modules over ambient isotopy classes of links in a 3-manifold by properly chosen local (skein) relations. The choice of relations is a delicate task; we should take into account several factors:

(i) Is the module we obtain accessible (computable)?

(ii) How precise are our modules in distinguishing 3-manifolds and links in them?

(iii) Does the module we obtain admit some additional structure (e.g. filtration, gradation, multiplication, Hopf algebra structure)?

From a practical point of view there is yet a fourth important factor

[5]One would like to say, in the spirit of Leibniz: *algebra situs*.

(iv) The "historical factor" in the choice of (skein) relations: the relations of links which were already studied (possibly for totally different reasons) will be compared with the new structures, just to see how they work in the new setup. For example, if we consider the Jones skein relation we can be sure that even for S^3 we get a nontrivial result.

The idea of the skein module should become more apparent after we consider the main example of the talk, the Kauffman bracket skein module.

2. The Kauffman bracket skein module

The skein module based on the Kauffman bracket skein relation is, so far, the most extensively studied object of the *algebraic topology based on knots.* We describe in this section the basic properties of the Kauffman Bracket Skein Module (KBSM) and list manifolds for which the structure of the module is known. In the third section, we give the detailed proof of the structure of KBSM of a 3-manifold being an interval bundle over a surface. We extend the analysis to the case of the Relative Kauffman Bracket Skein Module (RKBSM). In the fourth section we discuss the torsion in KBSM. In particular, we investigate in details the case of the RKBSM of a product of a surface and the interval $(F \times I)$.

DEFINITION 2.1 ([Pr-1, H-P-3]). Let M be an oriented 3-manifold, $\mathcal{L}_{fr}$ the set of unoriented framed links in M (including the empty link, $\emptyset$), R any commutative ring with identity and A an invertible element in R. Let $S_{2,\infty}$ be the submodule of $R\mathcal{L}_{fr}$ generated by skein expressions $L_+ - AL_0 - A^{-1}L_\infty$, where the triple L_+, L_0, L_∞ is presented in Fig. 2.1, and $L \sqcup T_1 + (A^2 + A^{-2})L$, where T_1 denotes the trivial framed knot. We define the Kauffman bracket skein module, $\mathcal{S}_{2,\infty}(M; R, A)$, as the quotient $\mathcal{S}_{2,\infty}(M; R, A) = R\mathcal{L}_{fr}/S_{2,\infty}$.

Notice that $L^{(1)} = -A^3 L$ in $\mathcal{S}_{2,\infty}(M; R, A)$, where $L^{(1)}$ denotes a link obtained from L by twisting the framing of L by a full twist in a positive direction.

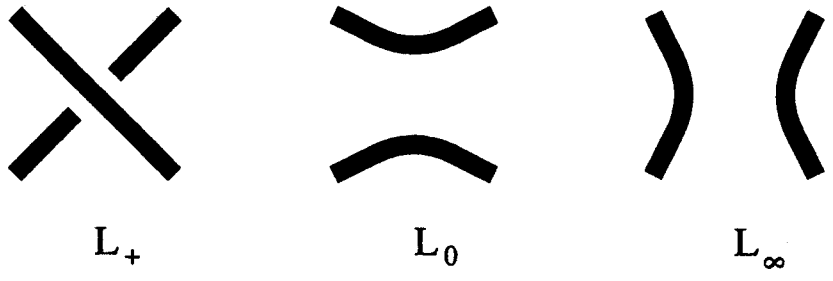


Fig. 2.1.

We call this the framing relation. We use the simplified notation $\mathcal{S}_{2,\infty}(M)$ for $\mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], A)$.

We list below several elementary properties of KBSM including description of the KBSM of any compact 3-manifold using generators and relators.

PROPOSITION 2.2. (1) *An orientation preserving embedding of 3-manifolds $i : M \to N$ yields the homomorphism of skein modules $i_* : \mathcal{S}_{2,\infty}(M; R, A) \to \mathcal{S}_{2,\infty}(N; R, A)$. The above correspondence leads to a functor from the category of 3-manifolds and orientation preserving embeddings (up to ambient isotopy) to the category of R-modules (with a specified invertible element $A \in R$).*

(2) (i) *If N is obtained from M by adding a 3-handle to it (i.e. capping off a hole), and $i : M \to N$ is the associated embedding, then $i_* : \mathcal{S}_{2,\infty}(M; R, A) \to \mathcal{S}_{2,\infty}(N; R, A)$ is an isomorphism.*

(ii) *If N is obtained from M by adding a 2-handle to it, and $i : M \to N$ is the associated embedding, then $i_* : \mathcal{S}_{2,\infty}(M; R, A) \to \mathcal{S}_{2,\infty}(N; R, A)$ is an epimorphism.*

(3) *If $M_1 \sqcup M_2$ is the disjoint sum of 3-manifolds M_1 and M_2 then*

$$\mathcal{S}_{2,\infty}(M_1 \sqcup M_2; R, A) = \mathcal{S}_{2,\infty}(M_1; R, A) \otimes \mathcal{S}_{2,\infty}(M_2; R, A).$$

(4) *(Universal Coefficient Property) Let $r : R \to R'$ be a homomorphism of rings (commutative with 1). We can think of R' as an R module. Then the identity map on $\mathcal{L}_{fr}$ induces the isomorphism of R' (and R) modules:*

$$\bar{r} : \mathcal{S}_{2,\infty}(M; R, A) \otimes_R R' \to \mathcal{S}_{2,\infty}(M; R', r(A)).$$

(5) *Let $(M, \partial M)$ be a 3-manifold with the boundary ∂M, and let γ be a simple closed curve on the boundary. Let $N = M_\gamma$ be the 3-manifold obtained from M by adding a 2-handle along γ. Furthermore let $\mathcal{L}_{fr}^{gen}$ be a set of framed links in M generating $\mathcal{S}_{2,\infty}(M; R, A)$.*
Then $\mathcal{S}_{2,\infty}(N; R, A) = \mathcal{S}_{2,\infty}(M; R, A)/J$, where J is the submodule of $\mathcal{S}_{2,\infty}(M; R, A)$ generated by expressions $L - sl_\gamma(L)$, where $L \in \mathcal{L}_{fr}^{gen}$ and $sl_\gamma(L)$ is obtained from L by sliding it along γ (i.e. handle sliding).

(6) *Let M be an oriented compact manifold and consider its Heegaard decomposition (that is M is obtained from the handlebody H_n by adding 2 and 3-handles to it), then M has a presentation as follows: generators of $\mathcal{S}_{2,\infty}(M; R, A)$ are generators of $\mathcal{S}_{2,\infty}(H_n; R, A)$ and relators are yielded by 2-handle slidings.*

PROOF. (1) i_* is well defined because if framed links L_1 and L_2 are ambient isotopic in M then $i(L_1)$ and $i(L_2)$ are ambient isotopic in N. Furthermore any skein triple L_+, L_0, L_∞ in M, is sent by i to a skein triple in N. Finally $i(T_1)$ is a trivial framed knot in N. Notice that if $i_* : M \to N$ is an orientation reversing embedding then i_* is a Z-homomorphism and $i(Aw) = A^{-1}i(w)$.

(2) (i) It holds because the cocore of a 3-handle is 0-dimensional[6].

(ii) It holds because the cocore of a 2-handle is 1-dimensional.

(3) This is a consequence of the well known property of short exact sequences, [B1]:
If $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ are short exact sequences of R-modules then $0 \to A' \otimes B + A \otimes B' \to A \otimes B \to A'' \otimes B'' \to 0$ is a short exact sequence.

(4) The exact sequence of R modules

$$\mathcal{S}_{2,\infty}(R, A) \to R\mathcal{L}_{fr} \to \mathcal{S}_{2,\infty}(M; R, A) \to 0$$

leads to the exact sequence of R' modules ([C-E], Proposition II.4.5):

$$\mathcal{S}_{2,\infty}(R, A) \otimes_R R' \to R\mathcal{L}_{fr} \otimes_R R' \to \mathcal{S}_{2,\infty}(M; R, A) \otimes_R R' \to 0.$$

Now, applying the "five lemma" to the commutative diagram with exact rows (see for example [C-E] Proposition I.1.1)

$$\begin{array}{ccccccc} \mathcal{S}_{2,\infty}(R, A) \otimes_R R' & \to & R\mathcal{L}_{fr} \otimes_R R' & \to & \mathcal{S}_{2,\infty}(M; R, A) \otimes_R R' & \to & 0 \\ \downarrow epi & & \downarrow iso & & \downarrow \bar{r} & & \\ \mathcal{S}_{2,\infty}(R', \bar{r}(A)) & \to & R'\mathcal{L}_{fr} & \to & \mathcal{S}_{2,\infty}(M; R', \bar{r}(A)) & \to & 0 \end{array}$$

we conclude that $\bar{r}$ is an isomorphism of R' (and R) modules.

(5) It follows from (2) (ii) because any skein relation can be performed in M, and the only difference between KBSM of M and N lies in the fact that some nonequivalent links in M can be equivalent in N; the difference lies exactly in the possibility of sliding a link in M along the added 2-handle (that is L is moving from one side of the cocore of the 2-handle to another).

(6) It follows from (5) and (2) (i).

□

In the next theorem we list manifolds for which the exact structure of the Kauffman bracket skein module has been computed.

[6] A manifold N is obtained from an n-dimensional manifold M by attaching a p-handle, $D^p \times D^{n-p}$, to M, if $N = M \cup_f D^p \times D^{n-p}$ where $f : \partial D^p \times D^{n-p}$ is an embedding. $D^p \times \{0\}$ is a core of the handle and $\{0\} \times D^{n-p}$ is a cocore of the handle [R-S].

THEOREM 2.3 ([Ka-4, Pr-1, H-P-4, H-P-5, H-P-6, Bu-2]).

(a) $\mathcal{S}_{2,\infty}(S^3) = Z[A^{\pm 1}]$, *more precisely:* $\emptyset$ *is the generator of the module and* $L = \langle L \rangle T_1 = (-A^2 - A^{-2})\langle L \rangle \emptyset$ *where* $\langle L \rangle$ *is the Kauffman bracket polynomial of a framed link* L.

(b) $\mathcal{S}_{2,\infty}(F \times [0,1])$ *is a free module generated by links (simple closed curves) on* F *with no trivial component (but including the empty link). Here* F *denotes an oriented surface (see also Theorems* 3.1 *and* 3.9). *This applies in particular to a handlebody, because* $H_n = P_n \times I$, *where* H_n *is a handlebody of genus* n *and* P_n *is a disc with* n *holes.*

(c) $\mathcal{S}_{2,\infty}(L(p,q))$ *is a free* $Z[A^{\pm 1}]$ *module and it has* $[p/2]+1$ *generators, where* $[x]$ *denotes the integer part of* x.

(d) $\mathcal{S}_{2,\infty}(S^1 \times S^2) = Z[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} Z[A^{\pm 1}]/(1 - A^{2i+4})$

(e) *The skein module of the complement of the torus knot ot type* $(k,2)$ *is a free* $Z[A^{\pm 1}]$*-module generated by links* $U_{i,j}$ $(i \geq 0,\ \frac{k-1}{2} \geq j \geq 0)$, *where* $U_{i,j}$ *is composed of* i *meridians and* j *curves* λ, *as illustrated in Fig.* 2.2.

(f) *Let* W *be the classical Whitehead manifold, then* $\mathcal{S}_{2,\infty}(W)$ *is infinitely generated torsion free but not free.*

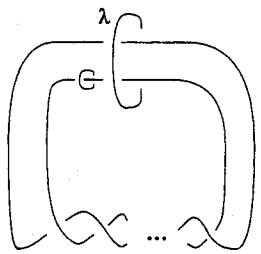


Fig. 2.2.

In [H-P-5], (f) is proved for a large class of genus one Whitehead type manifolds. For the classical Whitehead manifold it seems feasible to find the exact structure of $\mathcal{S}_{2,\infty}(W)$, and we plan to address that in a future paper.

We prove (b), with its generalizations, in the next section.

3. KBSM and relative KBSM of $F \times I$ and $F \hat{\times} I$

The understanding of the Kauffman bracket skein module of the product of a surface and the interval is the first step to understand KBSM of a general 3-manifold. Furthermore the case of $F \times I$ is relatively easy to understand because

we can project links onto the surface and work with diagrams of links. This can be generalized to twisted I-bundles over F and one can have reasonable hope that the method can work for other 3-manifolds by projecting links to spines of 3-manifolds. The relative case is described in Theorem 3.9.

THEOREM 3.1. *Let M be an oriented 3-manifold which is either equal to $F \times I$, where F is an oriented surface, or equal to a twisted I bundle over F ($F\hat{\times}I$), where F is an unoriented surface. Then the KBSM, $\mathcal{S}_{2,\infty}(M; R, A)$, is a free R-module with a basis $B(F)$ consisting of links in F without contractible components (but including the empty link).*

PROOF. We will give here the proof of Theorem 3.1 which is based on the original proof of Kauffman on the existence of his bracket polynomial. Let M be an oriented 3-manifold which is an I-bundle over a surface F[7]. Let $B(F)$ consist of all links in F which have no trivial components (including $\emptyset$). Furthermore each link is equipped with an arbitrary, but specific framing (to be concrete we can assume that if a knot in F preserves the orientation of F then we choose as its framing the regular neighborhood of K in F ("blackboard" framing), if K is changing the orientation on F than its regular neighborhood is a Möbius band so to get a framing we perform a positive half twist on it). Now one can quickly see that $B(F)$ is a generating set of $\mathcal{S}_{2,\infty}(M; R, A)$. Namely every link in M has a regular projection on F and any link can be reduced by skein relations so that a projection has no crossings. Then another relation allows us to eliminate trivial components and finally the framing relations allow us to adjust framing. We will prove that $B(F)$ is a basis for $\mathcal{S}_{2,\infty}(M; R, A)$. First we need however to consider the space of link diagrams (for a nonorientable surface F the proof is still straightforward but requires great care). □

DEFINITION 3.2. (a) A link diagram (or marked diagram) on F is a 4-valent graph in F (allowing loops without vertices) such that one corner of a neighborhood of each vertex is marked. F does not need to be oriented for this definition.

(b) Let $\mathcal{D}$ be a set of link diagrams on F (up to isotopy of F), and $R\mathcal{D}$ the free module over $\mathcal{D}$. The skein space of diagrams, $\mathcal{SD}$ is defined as a quotient:

$$\mathcal{SD}(F; R, A) = R\mathcal{D}/(\times - A\asymp - A^{-1})(,\ D \sqcup T_1 + (A^2 + A^{-2})D).$$

LEMMA 3.3. *Let $B^d(F)$ denote the set of link diagrams in F without vertices and without trivial components (but allowing $\emptyset$). We can identify $B^d(F)$*

[7]Because M is oriented therefore for γ in F changing orientation of F, the restriction of the I-bundle to γ is a nontrivial bundle (Möbius band). For γ preserving orientation of F, the bundle is trivial (an annulus).

with the set $B(F)$ with framing ignored. $B^d(F)$ is a subset of the set of link diagrams, so we have a homomorphism $\phi : RB^d(F) \to \mathcal{SD}(F; R, A)$ defined by associating to a link diagram in F, $\gamma \in B^d(F)$, its class in $\mathcal{SD}(F; R, A)$.

Then ϕ is an isomorphism.

PROOF. For any $D \in \mathcal{D}$ we can use the first relation to eliminate all crossings, and the second to eliminate trivial components of $\mathcal{D}$. Thus ϕ is an epimorphism.

To show that it is a monomorphism we will construct the inverse map, ψ. First we define a map $\hat{\psi} : R\mathcal{D} \to RB^d(F)$. Let $D \in \mathcal{D}$. We define $\hat{\psi}(D)$ as follows: Choose any ordering $p_1, \ldots, p_n$ of crossings of D, and use the formula $D^{p_i}_{\times} = AD^{p_i}_{\smile\frown} + A^{-1}D^{p_i}_{)(}$, for each crossing, until all crossings are eliminated. The upper index denotes the crossing at which we perform a smoothing (crossing elimination). The result does not depend on the order of the crossings since we can make any transposition of adjacent (with respect to ordering) pairs and get the same result:

$$\begin{aligned}(D^{p}_{\times})^{q}_{\times} &= A(D^{p}_{\smile\frown})^{q}_{\times} + A^{-1}(D^{p}_{)(})^{q}_{\times} \\ &= A^2(D^{p}_{\smile\frown})^{q}_{\smile\frown} + (D^{p}_{\smile\frown})^{q}_{)(} + (D^{p}_{)(})^{q}_{\smile\frown} + A^{-2}(D^{p}_{)(})^{q}_{)(}\end{aligned}$$

and

$$\begin{aligned}(D^{q}_{\times})^{p}_{\times} &= A(D^{q}_{\smile\frown})^{p}_{\times} + A^{-1}(D^{q}_{)(})^{p}_{\times} \\ &= A^2(D^{q}_{\smile\frown})^{p}_{\smile\frown} + (D^{q}_{\smile\frown})^{p}_{)(} + (D^{q}_{)(})^{p}_{\smile\frown} + A^{-2}(D^{q}_{)(})^{p}_{)(}\end{aligned}$$

After smoothing all crossings we eliminate trivial components by the relation $D \sqcup T_1 = (-A^2 - A^{-2})D$ (there is no ambiguity in the reduction). Thus D is uniquely expressed as a linear combination of elements of $B^d(F)$, and we define $\hat{\psi}(D)$ as this linear combination (which lies in $RB^d(F)$). Therefore $\hat{\psi}$ is well defined. Now $\hat{\psi}$ descends to $\psi : \mathcal{SD}(F; R, A) \to RB^d(F)$ because $\hat{\psi}(\times - A\asymp - A^{-1})( ) = 0$ and $\hat{\psi}(D \sqcup T_1 + (A^2 + A^{-2})D) = 0$. Now, obviously, $\psi\phi = Id$, thus ϕ is a monomorphism. □

Our goal is to prove that $B(F)$ is a basis of the Kauffman bracket skein module $\mathcal{S}_{2,\infty}(M; R, A)$ where M is an oriented 3-manifold being an I bundle over a surface F. Because we would like to consider the case of orientable and unorientable surface simultaneously, it is convenient to consider half-integer framings of links, that is, to allow embedded Möbius bands. This suggests the following definition.

DEFINITION 3.4. Let M be any oriented 3-manifold, $\mathcal{L}_{fr}$ the set of embeddings of annuli and Möbius bands in M (up to an ambient isotopy of M) and $\bar{R}$ a commutative ring with identity with a chosen invertible element $\bar{A}$ (we define $A = -\bar{A}^2$ and we will often write $\sqrt{-A}$ for $\bar{A}$. Let $\bar{R}\bar{\mathcal{L}}_{fr}$ denote a free $\bar{R}$ module over $\bar{\mathcal{L}}_{fr}$ and let $\bar{S}_{2,\infty}$ denote the submodule of $\bar{R}\bar{\mathcal{L}}_{fr}$ generated by expressions $L_+ - AL_0 - A^{-1}L_\infty$, and $L^{1/2} - (\sqrt{-A})^3 L$, where $L^{1/2}$ denotes L with its framing twisted by a half twist in a positive direction. As before, for convenience, we allow the empty link, $\emptyset$, and add the relation $T_1 = (-A^2 - A^{-2})\emptyset$. Then we define $\bar{S}_{2,\infty}(M, \bar{R}, \bar{A}) = \bar{R}\bar{\mathcal{L}}_{fr}/S_{2,\infty}$.

Consider the $\bar{R}$-homomorphism $g : \mathcal{SD}(F; \bar{R}, A) \to \bar{S}_{2,\infty}(M, \bar{R}, \bar{A})$ defined on the basic elements $\gamma \in B^d(F)$ by $g(\gamma) = \gamma^{f_r}$ where γ^{f_r} is a framed link obtained from γ by giving it the blackboard framing (it may be an annulus or a Möbius band). Using our skein relations, in a similar manner as before, we see that g is an epimorphism. If D is any marked diagram we can describe the framed link $g(D)$ as follows: we resolve every crossing of D according to the rule given in Fig. 3.1 and giving the link $g(D)$ the blackboard framing (the orientation of M in a neighborhood of the crossing should agree with that of R^3 from Fig. 3.1).

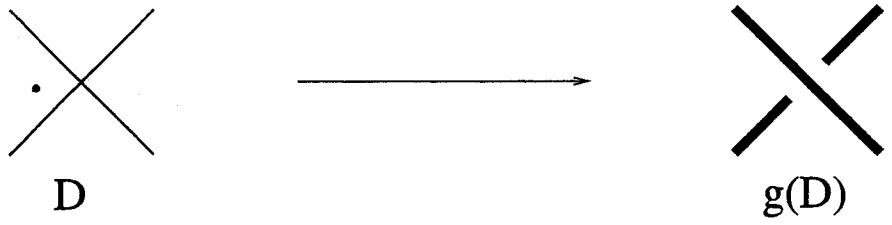


Fig. 3.1.

Consider now the following lemma concerning Reidemeister moves on diagrams.

LEMMA 3.5. *Consider the moves $\bar{R}_1, \bar{R}_2, \bar{R}_3$ on marked diagrams (described below). In $\mathcal{SD}(F; \bar{R}, A)$ they satisfy:*

($\bar{R}_1$) $\times\!\bigcirc = -A^3\,)$ *and* $\times\!\bigcirc = -A^{-3}\,)$, *where* $\bar{R}_1(\,) = \times\!\bigcirc$ *or* $\times\!\bigcirc$.

($\bar{R}_2$) $\bar{R}_2(D) = D$, *where* $\bar{R}_2(\,)(\,) = \asymp$.

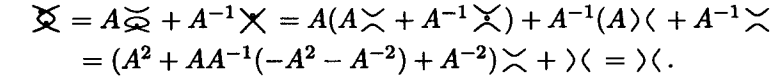
($\bar{R}_3$) $\bar{R}_3(D) = D$, *where* $\bar{R}_3(\times) = \times$.

PROOF. ($\bar{R}_1$)

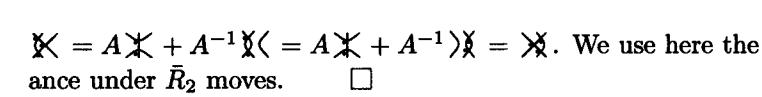
$$\times\!\bigcirc = A\,) \sqcup O + A^{-1}\,) = (A(-A^2 - A^{-2}) + A^{-1})\,) = -A^3\,).$$

$$\times\!\bigcirc = A^{-1}\,) \sqcup O + A\,) = (A + A^{-1}(-A^2 - A^{-2}))\,) = -A^{-3}\,).$$

($\bar{R}_2$)



($\bar{R}_3$)



To use the lemma in the proof that g is a monomorphism, we need a variant of Reidemeister's theorem for marked diagrams:

PROPOSITION 3.6. *Let $\hat{g} : \mathcal{D} \to \bar{\mathcal{L}}_{fr}$ be a map given by Fig.* 3.1. *Then two marked diagrams, D_1 and D_2, represent the same framed link, $\hat{g}(D_1) = \hat{g}(D_2)$, if and only if one can go from D_1 to D_2 using Reidemeister moves $\bar{R}_i^{\pm 1}$ and an isotopy of F, and additionally, for corresponding link components of D_1 and D_2, their Tait numbers are the same. One should notice here that for a knot diagram the Tait number is independent on orientation of the knot. Precisely for a knot diagram D we defne Tait $(D) = \Sigma_p \mathrm{sgn}(p)$, where* $\mathrm{sgn}(\times) = 1$ *and* $\mathrm{sgn}(\times) = -1$.

PROOF. The proposition can be deduced from the classical Reidemeister theorem and the result from the PL topology; Theorem 6.2 in [Hud][8] . □

Our goal is to show that the epimorphism $g : \mathcal{SD}(F; \bar{R}, A) \to \bar{\mathcal{S}}_{2,\infty}(M, \bar{R}, \bar{A})$ is an isomorphism. We use Lemma 3.5 and Proposition 3.6 to construct the map inverse to g. Let $\hat{h} : \bar{R}\bar{\mathcal{L}}_{fr} \to \mathcal{SD}(F; \bar{R}, A)$ be a homomorphism defined as follows: choose a representative of a link $L \in \bar{\mathcal{L}}_{fr}$ which has a regular projection on F. Let D_L be a marked diagram on F constructed as in Fig. 3.1, and let $t(L)$ be the number (possibly half-integer) of positive twists which should be performed on the blackboard framing of D_L to get the framing of L. Then we define $\hat{h}(L) = (-A^3)^{t(L)} D_L$. $\hat{h}(L)$ is well defined by Lemma 3.5 and Proposition 3.6. Furthermore $\hat{h}(L_+ - AL_- - A^{-1}L_\infty) = 0$, $\hat{h}(L \sqcup T_1 + (A^2 + A^{-2})L) = 0$ and $\hat{h}(L^{1/2} - \sqrt{-A^3}L) = 0$ so $\hat{h}$ descends to $h : \mathcal{S}_{2,\infty}(M; R, A) \to \mathcal{SD}(F; \bar{R}, A)$. Of course $hg = Id$ so g is a monomorphism, as required. □

We can now complete the proof of Theorem 3.1. Because $g : \mathcal{SD}(F; \bar{R}, A) \to \bar{\mathcal{S}}_{2,\infty}(M, \bar{R}, \bar{A})$ is an isomorphism, therefore $g(B^d(F))$ is a basis of $\bar{\mathcal{S}}_{2,\infty}(M, \bar{R}, \bar{A})$. On the other hand $B(F)$, whose elements may differ from elements of $g(B^d(F))$ only by framing, also forms a basis of $\bar{\mathcal{S}}_{2,\infty}(M; \bar{R}, \bar{A})$. Thus they are linearly independent in $\mathcal{S}_{2,\infty}(M; R, A)$. Because $B(F)$ generates $\mathcal{S}_{2,\infty}(M; R, A)$ it is a basis of this module. The proof of Theorem 3.1 is completed. □

[8]It follows from the theorem that if C is a compact subset of a manifold M and $F : M \times I \to M$ is the isotopy of M then there is another isotopy $\hat{F} : M \times I \to M$ such that $F_0 = \hat{F}_0$, $F_1/C = \hat{F}_1/C$ and there exists a number N such that the set $\{x \in M \mid \hat{F}/\{x\} \times (k/N, (k+1)/N)$ is not constant$\}$ sits in a ball embedded in M; [Hud], Corollary 6.3.

As an immediate corollary of Theorem 3.1 we obtain the structure of KBSM of the projective space RP^3 [9].

COROLLARY 3.7. *$\mathcal{S}_{2,\infty}(RP^3; R, A) = R \oplus R$. As a basis of KBSM we can take $\emptyset$ and a generator of the fundamental group of RP^3.*

PROOF. By Proposition 2.2 (i) $\mathcal{S}_{2,\infty}(RP^3; R, A) = \mathcal{S}_{2,\infty}(RP^3 - \mathrm{int}(D^3); R, A)$ and $RP^3 - \mathrm{int}(D^3)$ is equal to the twisted I-bundle over a projective plane $(RP^2 \tilde{\times} I)$. By Theorem 3.1, $\mathcal{S}_{2,\infty}(RP^2 \tilde{\times} I; R, A)$ is a free R-module with basis $B(RP^2)$, which has two elements: the empty link and the noncontractible curve on RP^2. □

One can generalize Theorem 3.1 to relative skein modules, as long as a surface F has a boundary.

DEFINITION 3.8 (Relative Kauffman Bracket Skein Module). Let $x_1, x_2, \ldots, x_{2n}$ be a set of $2n$ (framed[10]) points in ∂M, where M is an oriented 3-manifold. Let $\mathcal{L}_{fr}(n)$ be a family of relative framed links in $(M, \partial M)$ such that $L \cap \partial M = \partial L = \{x_i\}$, considered up to an ambient isotopy fixing ∂M. Let R be a commutative ring with identity and A an invertible element in R. Let $S_{2,\infty}(n)$ be the submodule of $R\mathcal{L}_{fr}(n)$ generated by the Kauffman bracket skein relations. We define the Relative Kauffman Bracket Skein Module (RKBSM) as the quotient:

$$\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A) = R\mathcal{L}_{fr}(n)/S_{2,\infty}(n).$$

We list below a few useful properties of relative skein modules:

PROPOSITION 3.9. (a) *There is a functor from the category of oriented 3-manifolds with $2n$ framed points on the boundary and orientation preserving embeddings (up to ambient isotopy fixed on the boundary) to the category of R-modules (with a specified invertible element $A \in R$). The functor sends an embedding $i : (M, \{x_i\}_{i=1}^{2n}) \to (N, \{y_i\}_{i=1}^{2n})$ into R-modules morphism $\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A) \to \mathcal{S}_{2,\infty}(N, \{y_i\}_1^{2n}; R, A)$.*

(b) *Adding a 3-handle to M (outside x_i) does not change the RKBSM, and adding a 2-handle is adding only relations to RKBSM (handle slidings yield relations); compare Proposition 2.2 (2).*

[9] Corollary 3.7, obtained in the summer of 1988, is a special case of a computation for lens spaces (Theorem 2.3 (c)) obtained in the fall of 1989. In fact the corollary helped the authors of [H-P-4] to predict the general structure of the KBSM of lens spaces. Corollary 3.7 was also independently proven by J. Drobotukhina [Dr].

[10] A framed point in ∂M is an interval in ∂M. Thus a relative framed link intersects ∂M in framed points.

(c) *The relative KBSM depends only on the distribution of boundary points* $\{x_i\}$ *among boundary components of* M*, but not on the exact position of* $\{x_i\}$*. In particular if* ∂M *is connected, we can write shortly* $\mathcal{S}_{2,\infty}(M,n;R,A)$ *instead of* $\mathcal{S}_{2,\infty}(M,\{x_i\}_1^{2n};R,A)$.

(d) *The relative KBSM satisfies Universal Coefficient Property, compare Proposition* 2.2 (4).

(e) *For a disjoint sum of 3-manifolds we have:*

$$\mathcal{S}_{2,\infty}(M_1 \sqcup M_2, \{x_i, y_i\}_1^{2n}; R, A) = \mathcal{S}_{2,\infty}(M_1, \{x_i\}_i^{2n}, R, A) \otimes \mathcal{S}_{2,\infty}(M_2, \{y_i\}_1^{2n}; R, A).$$

THEOREM 3.10. *Let* $M = F\bar{\times}I$ *that is* $M = F \times I$ *or* $M = F\hat{\times}I$*. Then*

(a) *Let* $\partial F \neq \emptyset$ *then* $\mathcal{S}_{2,\infty}(M,\{x_i\}_1^{2n};R,A)$ *is a free* R*-module. Consider all* x_i *to lie on* $\partial F \times \{\frac{1}{2}\}$*. Then the basis of the module* $\mathcal{S}_{2,\infty}(M,\{x_1\}_1^{2n};R,A)$ *is composed of relative links on* F *without trivial components.*

(b) *In the case of* $F_{g,0}$*, closed surface of genus* g $(F \neq S^2)$*, the situation is more delicate so we stop on the following observation:* $\mathcal{S}_{2,\infty}(F_{g,0}\bar{\times}I;\{x_i\}_1^{2n};R,A) = \mathcal{S}_{2,\infty}(F_{g,1}\bar{\times}I;\{x_i\}_1^{2n};R,A)/(I)$*, where* $F_{g,1} = F_{g,0} - \mathrm{int}(D^2)$ *and assuming* $x_i \in \partial D^2$*, ideal* (I) *is generated by moves in which arcs go above* D^2*.*

PROOF. The proof of (a) is the same as that of Theorem 3.1; as before relative link diagrams representing the same link are related by Reidemeister moves. In the case (b) it is no longer true as we need also handle sliding. $F_{g,0}\bar{\times}I$ is obtained from $F_{g,1}\bar{\times}I$ by adding the 2-handle along ∂D^2. Now (b) follows from Proposition 3.9 (b). □

In the case F is a closed surface the question whether $\mathcal{S}_{2,\infty}(F\bar{\times}I,\{x_i\}_1^{2n};R,A)$ is free is open in general. If not all x_i lie on the same boundary component of $F \times I$ then the skein mdule has a torsion in the case of F being a sphere or a torus. We prove it in the last section. We propose the following conjecture, which easily holds for $F = S^2$ and otherwise which we are able to confirm only for F being a torus and $n = 1$.

CONJECTURE 3.11. *Let* F *be a closed surface and* $x_i \in F \times \{0\}$ *for any* i*. Then the skein module* $\mathcal{S}_{2,\infty}(F \times I,\{x_i\}_1^{2n};R,A)$ *is free.*

Theorem 3.10 can be nicely illustrated by the following corollary.

COROLLARY 3.12. (a) $\mathcal{S}_{2,\infty}(D^2 \times I,\{x_i\}_1^{2n};R,A)$ *is a free* R *module of* $\frac{1}{n+1}\binom{2n}{n}$ *free generators.*

(b) $\mathcal{S}_{2,\infty}(\circledcirc \times I,\{x_i\}_1^{2n};R,A)$ *is a free* $R[\alpha]$ *module with* $\binom{2n}{n}$ *free generators, where* α *is represented by a longitude of the annulus.*

PROOF. Corollary 3.12 (a) describes, well known module structure of the Temperley-Lieb algebra with basis consisting of the nth Catalan number of elements. (b) follows from the work of Jones and Tom Dieck [Jo,To]. We provide here a short, self-contained proof. In lieu of Theorem 3.9, it suffices to count the crossless connections (by arcs) of $2n$ points in the disc and annulus. We offer an amazingly simple calculation for both cases simultaneously. Let C_n be the number of connections in the disc and D_n be the number of connections in the annulus (all points x_i are on the "outside" circle of the annulus). Connection arcs in the disc cut the disk into $n+1$ pieces. To get a connection in the annulus we have to put a "table" in D^2 (remove a disk from D^2). Thus $D_n = (n+1)C_n$. On the other hand, any arc of a connection in the annulus has a first point (with respect to some fixed orientation of the annulus) and any choice of n points leads to a unique connection system, for which given points are first[11]. Therefore $D_n = \binom{2n}{n}$ and thus $C_n = \frac{1}{n+1}\binom{2n}{n}$. □

For an oriented surface F, let $x_1, x_2, \ldots, x_n \in F \times \{1\}$ and $x_{n+1}, x_{n+2}, \ldots, x_{2n} \in F \times \{0\}$, where x_i and x_{n+i} project to the same point of F. Then $\mathcal{S}_{2,\infty}(F \times I, \{x_i\}_{i=1}^{2n}; R, A)$ has an algebra structure with the product of relative links $L_1 \cdot L_2$ defined by placing L_1 in $F \times [\frac{1}{2}, 1]$ and L_2 in $F \times [0, \frac{1}{2}]$. For $F = D^2$ the algebra is the Temperley-Lieb algebra, TL_n. For $F = ◎$ we call the algebra, annular Temperley-Lieb algebra and denote by ATL_n. The algebra related to ATL_n was first computed by Jones in the context of affine Hecke algebras [Jo] and independently by Tom Dieck (who denoted it by TB_n), [To]. Corollary 3.12 supports a relatively short proof of the structure of ATL_n.

THEOREM 3.13 ([Jo,To]). (a) $ATL_n = \mathcal{S}_{2,\infty}(◎ \times I, \{x_i\}_{i=1}^{2n}; R, A) = R[\alpha]\langle t, e_1, \ldots, e_{n-1}\rangle / \mathcal{I}(n)$ *where* $\mathcal{I}(n)$ *is the ideal generated by the expressions*:

$$e_i e_j = e_j e_i \text{ for } |j-i| \geq 2,\ e_i e_j e_i = e_i \text{ for } |j-i| = 1,\ te_i = e_i t$$
$$\text{for } i > 1,\ e_1 t e_1 = \alpha e_1,\ e_i^2 = (-A^2 - A^{-2})e_i,\ t^2 = -A^{-2}\alpha t - A^{-4}.$$

(b) ATL_n, *as an* $R[\alpha]$ *module, is freely generated by* $\binom{2n}{n}$ *words of the form*

[11]The easy way to see this unique system of arcs which comes from a choice of n points is to imagine the following children's game: n girls and n boys stand by the wall of a room. The result of the game is that each girl will give a hand to a boy to her right (no crossings), but not necessarily to the boy on her immediate right. In the first round each girl looks to her right and gives her hand, if she has a boy to her immediate right. Connected pairs are out of the game which restarts with the remaining children. At the end of the game all of the children are paired up (exactly as needed in a connection system). No table was needed in this game (algorithm). The table was necessary for the inverse function: given connection arcs, decide on which end is a girl and on which end is a boy.

(for convenience we write e_0 for t) :

$$e_{i_1}e_{i_1-1}\cdots e_1e_0e_{i_2}e_{i_2-1}\cdots e_1e_0\cdots e_{i_s}e_{i_s-1}\cdots e_1e_0$$
$$\cdot e_{a_1}e_{a_1-1}\cdots e_{b_1+1}e_{b_1}\cdots e_{a_k}e_{a_k-1}\cdots e_{b_k+1}e_{b_k}$$

where $s,k \geq 0$, $0 \leq i_1 < i_2 \cdots < i_s < a_1 < a_2 < \cdots < a_k < n$, $0 < b_i \leq a_i < n$, *and* $0 < b_1 < b_2 < \cdots < b_k < n$.

We finish this section by offering the following very useful observation (compare [P-S-2]).

PROPOSITION 3.14. *Consider a 3-manifold* $(M, \{x_i\}_{i=1}^{2n})$ *and let* x_{2n+1} *and* x_{2n+2} *lie on the some boundary component of* M. *Consider the R-homomorphism of RKBSM*

$$i_{\#} : \mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A) \to \mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n+2}, R, A)$$

generated by the identity map and with convention that $i_{\#}(L)$ *has the point* x_{2n+1} *connected to the point* x_{2n+2} *by a framed arc close to boundary (we push out of the boundary framed arc joining* x_{2n+1} *and* x_{2n+2} *in* ∂M). *Then* $i_{\#}$ *is a monomorphism if one assumes that* $A^2 + A^{-2}$ *is not an anihilator of any non-zero element of* $\mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A)$ *(i.e.* $(A^2 + A^{-2})x = 0 \Rightarrow x = 0$).

PROOF. Consider the R-homomorphism

$$i'_{\#} : \mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n+2}, R, A) \to \mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A)$$

given by connecting x_{2n+1} and x_{2n+2} in ∂M and pushing it inside M. Now clearly $i'_{\#}i_{\#}(L) = (-A^2 - A^{-2})(L)$, thus $i'_{\#}i_{\#}(u) = (-A^2 - A^{-2})(u)$ for any $u \in \mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A)$. Therefore $i'_{\#}i_{\#}$ is a monomorphism iff $A^2 + A^{-2}$ is not an anihilator of any non-zero element of $\mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, R, A)$. A monomorphism of $i_{\#}$ follows from a monomorphism of $i'_{\#}i_{\#}$. □

4. Torsion in KBSM

In all of the examples above the module is torsion free except in the case of $S^1 \times S^2$. In fact a non-separating S^2 in M always yields a torsion in $\mathcal{S}_{2,\infty}(M)$. It is enough to use the framing relation to see a torsion: Let L be a framed link cutting a non-separating S^2 exactly in one point. We can twist S^2 twice, twisting also the framing of L twice and then undo this by an isotopy of M. Thus $(A^6 - 1)L = 0$ in $\mathcal{S}_{2,\infty}(M)$. It is less obvious that a separating S^2 can often yield a torsion.

CONJECTURE 4.1 ([Kir]). *If $M = M_1 \# M_2$, where M_i is not equal to S^3, possibly with holes, then $\mathcal{S}_{2,\infty}(M)$ has a torsion element.*

We have proven the Conjecture 4.1 only partially. In an example in [Pr-7] we use the first homology groups of summands of the connected sum. One can extend it employing $SL(2,C)$ representations of summands (particularly if summands are hyperbolic).

THEOREM 4.2. (a) ([Pr-7]). *If M_1 and M_2 have first homology groups that are not 2-torsion groups, then the conjecture holds.*

(b) *If there are representations $\rho_i : \pi_1(M_i) \to SL(2,C)$, such that the image $\rho_i(\pi_1(M_i))$ is not in the center of $SL(2,C)$ (which is composed of $\pm Id$) then the conjecture holds.*

In the case of M containing an incompressible torus we are able to show the following theorem (the first homology group is used in the proof of (a) and $SL(2,C)$ representations in the proof of (b)).

THEOREM 4.3. (a) *Let M be a 3-manifold allowing embedded non-separating torus. Then $\mathcal{S}_{2,\infty}(M)$ has a torsion element.*

(b) *Let M be a manifold and ∂M contains a torus, $\partial_1 M$. Assume that there is a representations $\rho : \pi_1(M) \to SL(2,C)$, such that the image $\rho(\pi_1(M))$ is not conjugated to upper triangular subgroup of matrices (Borel subgroup) in $SL(2,C)$ and $\rho(\pi_1(\partial_1 M))$ is not in the center of $SL(2,C)$, then the double of M along $\partial_1 M$ has a torsion element in its KBSM. In particular the double of the complement of a hyperbolic knot has a torsion element in its KBSM; see* [Ve].

The relative case similar to Theorem 4.3 is considered in Lemma 4.4 (b).

We go back now to our main task of analyzing the relative skein module of the product of a surface and the interval.

LEMMA 4.4. (a) *If $M = S^2 \times I$ and not all x_i are on the same boundary component of M, then the relative Kauffman bracket skein module of $\mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, Z[A^{\pm 1}], A)$ has a torsion.*

(b) *The relative Kauffman bracket skein module of $M = T^2 \times I$, $\mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^{2n}, Z[A^{\pm 1}], A)$ has a torsion if not all x_i are on the same boundary component of M.*

PROOF. (a-1) For $n = 1$ and $M = S^2 \times I$ one uses the standard "Dirac trick" that is if we twist framing twice on the arc joining x_1 with x_2 in any relative link, we get the relative link ambient isotopic to L this $(A^6 - 1)L = 0$. L is not zero as the following easy argument shows: L represents a nontrivial element in the relative homology group with Z_2 coefficients,

$H_1(M, \{x_1, x_2\}; Z_2)$. On the other hand we have a Z-homomorphism $\mathcal{S}_{2,\infty}(M, \{x_1, x_2\}) \to CH_1(M, \{x_1, x_2\}; Z_2)$ given by sending an unoriented framed link L to an element in $H_1(M, Z_2)$ it represent and A to ω_A in C, where ω_A is the primitive sixth root of unity (that is it satisfies: $\omega_A + \omega_A^{-1} = 1$). Thus $L \neq 0$ in $\mathcal{S}_{2,\infty}(M, \{x_1, x_2\})$; [Pr-1].

(b-1) We will describe the construction in details as it has several uses and generalizations. Let $T_b^2 = T^2 \times 1/2$ be the "middle" torus and L a relative link which cuts T_b^2 in exactly one point. Further let λ be a noncontractible curve on T_b^2. We can represent the link $L \sqcup \lambda$ putting λ "on the top of" L or "below" L. We use the fact that on the torus λ can be isotoped on the "other" side of $L \cap T_b^2$. Thus in the RKBSM, $0 = (A - A^{-1})(L\lambda^{-1} - L\lambda)$ as illustrated in Fig. 4.1.

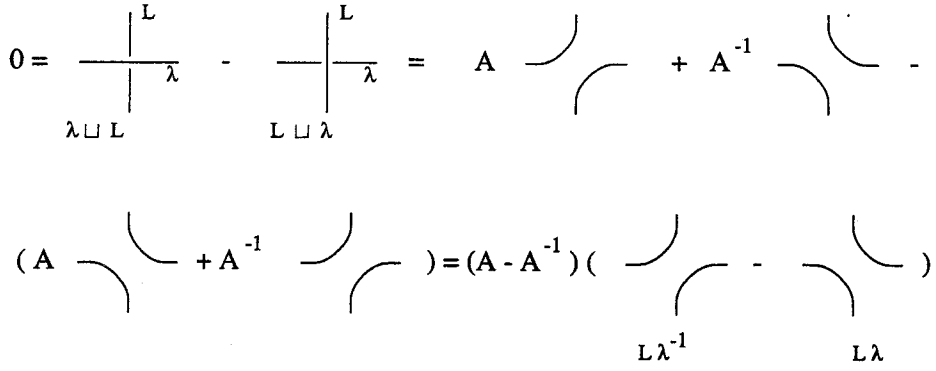


Fig. 4.1.

It remains to see that $L\lambda \neq L\lambda^{-1}$ in $\mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^2, Z[A^{\pm 1}], A)$. We show that it is not zero even for $A = -1$. In fact $\mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^2; Z, -1)$ is isomorphic to $Z[Z \oplus Z]$ (it follows from [P-S-1, P-S-2]. The isomorphism $\phi : \mathcal{S}_{2,\infty}(M, \{x_i\}_{i=1}^2; Z, -1) \to Z[Z \oplus Z]$ send an arc, γ, joining x_1 with x_2 (with an orientation from x_1 to x_2) to the (minus) element in $H_1(M, Z)$ represented by the arc γ with endpoints connected by the straight vertical line joining x_1 with x_2. In fact we only need much weaker fact that ϕ is well defined and $\phi(L\lambda) \neq \phi(L\lambda^{-1})$ in $H_1(M, Z)$.

(b-2) Let $x_1, x_2 \in T^2 \times \{1\}$ and $x_3, x_4 \in T^2 \times \{0\}$. Consider a link K composed of two arcs (joining x_1 with x_3, and x_2 and x_4, each cutting $T_b^2 = T^2 \times 1/2$ in exactly one point (c_1 and c_2 respectively). We choose λ like in (b-1) and again use "two sides" of c_1 and c_2 (assumed to be closed

together) on the torus. We get after some computation (see Fig. 4.2) $(A^2-A^{-2})((K_2^{-1}\lambda K_1' \sqcup K_2' K_1^{-1}) - (K_2^{-1}K_1' \sqcup K_2'\lambda^{-1}K_1^{-1})) = 0$. We have to show that $(K_2^{-1}\lambda K_1' \sqcup K_2' K_1^{-1}) - (K_2^{-1}K_1' \sqcup K_2'\lambda^{-1}K_1^{-1})$ is not 0. Let us denote our two links by L_1 and L_2. To distinguish L_1 from L_2 in the RKBSM we use Theorem 4.3 (b) (result which depends on $SL(2,C)$ representations) and some topological reasoning (embedding M in bigger manifold and extending relative links to links without boundary).
Let M_2 be a manifold obtained from $M = T^2 \times [0,1]$ by adding to it two 1-handles, first with a core joining x_1 with x_2 and the second with a core joining x_3 with x_4. Any relative link in M can be extended to a link in M_2 by adding to it the cores of 1-handles. This gives us a homomorphism:

$$\psi : \mathcal{S}_{2,\infty}(M, \{x_1, x_2, x_3, x_4\}; R, A) \to \mathcal{S}_{2,\infty}(M_2; R, A).$$

We want to show that $\psi(L_1) \neq \psi(L_2)$ in $\mathcal{S}_{2,\infty}(M_2; Z[A^{\pm 1}], A)$. By Universal Coefficient Property, it suffices to show this in $\mathcal{S}_{2,\infty}(M_2; C, -1)$. Clearly M_2 is the double of the manifold M_1 obtained from $T^2 \times [\frac{1}{2}, 1]$ by adding a 1-handle with a core joining x_1 with x_2. The fundamental group $\pi_1(M_1) = (Z \oplus Z) * Z$. We can use Theorem 4.3 (b), because there exists a required representation $\rho : \pi_1(M) \to SL(2,C)$. We can build the representation concretely, to distinguish $\psi(L_1)$ from $\psi(L_2)$. For example

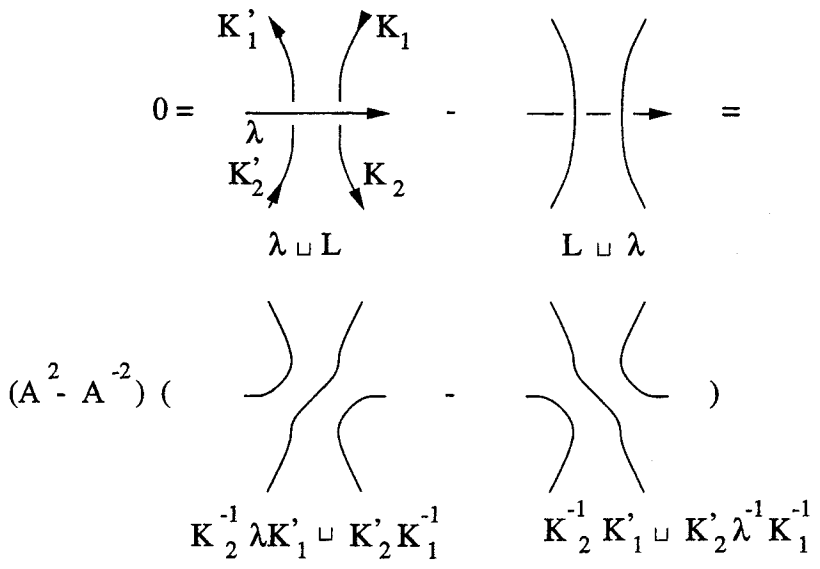


Fig. 4.2.

we can send generators of $(Z \oplus Z)$ to the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and the generator of the fundamental group composed of $K_1\lambda$, K_1' and the core of the added handle to

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(compare [Ve]).

(a-n), (b-n) For any other number of points $(2n)$, Lemma 4.4 follows immediately from (a-1) or (b-1) for an odd n, and from (a-2) or (b-2) for an even n. (a-2) is proven using Theorem 4.2(a) in a manner applied in (b-2), that is adding two 1-handles. □

PROBLEM 4.5. *Let F be an oriented closed surface of genus greater than* 1, *and not all points x_i lie on the same boundary component of $F \times [0,1]$. Does the RKBSM, $\mathcal{S}_{2,\infty}(F \times [0,1], \{x_i\}_1^{2n})$ have a torsion element?*

The answer to the above question can shed a light into the role of incompressible surfaces in the structure of Kauffman bracket skein modules of 3-manifolds.

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Department of Mathematics
George Washington University
Washington, DC 20052, U.S.A.
E-mail: przytyck@gwu.edu