Kobe University Repository : Kernel

PDF issue: 2026-11-22

Torsion in the KBSM of a 3-manifold caused by an incompressible torus and detected by a complete hyperbolic structure

Veve, Michael A.

(Citation)
Kobe Journal of Mathematics,16(2):109-118

(Issue Date)
1999-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0003681

(URL)
https://hdl.handle.net/20.500.14094/E0003681

KOBE UNIVERSITY

VEVE, M.A.
KOBE J. MATH.,
16 (1999) 109—118

TORSION IN THE KBSM OF A 3-MANIFOLD CAUSED BY AN INCOMPRESSIBLE TORUS AND DETECTED BY A COMPLETE HYPERBOLIC STRUCTURE†

Michael A. VEVE

(Received July 3, 1998)
(Revised June 14, 1999)

Abstract

We describe an example of a 3-manifold whose Kauffman Bracket Skein Module (KBSM) has torsion. The 3-manifold is the double of a hyperbolic knot complement. The torsion is shown to exist using representations of hyperbolic knots and an idea due to D. Bullock about algebra homomorphisms induced by representations into $SL_2(C)$ [2]. This example is due to J.H. Przytycki. A general conjecture based on this example is formulated.

1. Introduction

The Kauffman Bracket Stein Module (KBSM) is an algebraic object which we can associate with any 3-manifold. It belongs to a larger family of algebraic objects called skein modules. The building blocks of the skein module of a 3-manifold are links (loops up to ambient isotopy). More precisely, skein modules are the quotient of the free R-module (R is a commutative ring with unit) of links by some submodule generated by carefully chosen relations. For the KBSM, denoted by $\mathcal{S}_{2,\infty}(M;R,A)$ where A is a unit in R, the relations are the same local relations that arise in the definition of the Kauffman Bracket Polynomial of a link.

Contrast the definition of a skein module with the definition of the first homology group $H_1(M)$. $H_1(M)$ is built starting from formal linear combinations of 1-simplices, taking the subgroup of 1-cycles and dividing by the subgroup of 1-boundaries. The main difference between skein modules and $H_1(M)$ (as well as the fundamental group $\pi_1(M)$) is that skein modules (most of them, anyway) are sensitive to "knottedness" while neither $H_1(M)$ nor $\pi_1(M)$ can discriminate

†Notes from a colloquium talk given by the author at GWU on November 14, 1997.

between homotopic knots in a 3-manifold.

The earliest incarnation of a skein module comes from [1] where Alexander noted that his polynomial of three links L_+, L_-, and L_0 are linearly related. Here L_+, L_-, and L_0 represent links that are identical except in a small ball in which they differ as shown in Figure 1.

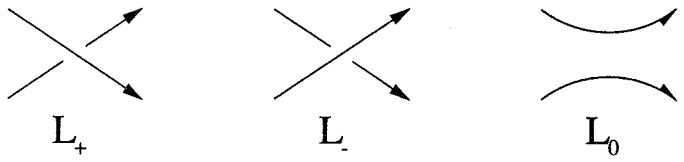


Fig. 1. Alexander's Linear Skein Relation

In [3], Conway rediscovered this observation by Alexander and normalized the Alexander polynomial to yield the skein relation

$$\Delta_{L_+}(z) - \Delta_{L_-}(z) = z\Delta_{L_0}(z).$$

Conway's idea was further developed by Giller [4], Kauffman [7, 8, 9], and Lickorish and Millett [10] (but only for subspaces of S^3). A paper by Hoste and Kidwell [5] about invariants of colored links motivated Przytycki to formally introduce the idea of a skein module [12]. Skein modules were also introduced independently by Turaev [15].

Some central questions regarding skein modules are

(1) How easy are they to compute?
(2) How good are they at distinguishing 3-manifolds and the links that live in them?
(3) What additional structure does the skein module have?

Torsion can arise in $\mathcal{S}_{2,\infty}(M; R, A)$ for 3-manifolds that contain an incompressible torus. A complete hyperbolic structure on the 3-manifold gives us a way to detect this torsion. This example was worked out by J.E. Przytycki. Much of the work relies on an observation of D. Bullock [2].

2. Definitions

DEFINITION 2.1. Let M be an oriented 3-manifold, R a commutative ring with identity, and $A \in R$ a unit of R. A framed link is an embedded annulus in which the central curve of the annulus determines an unframed link. Let $\mathcal{L}^{fr}$ be the set of unoriented framed links in M, including the empty link which we denote by $\emptyset$. $R\mathcal{L}^{fr}$ is the set of formal linear combinations of framed links in M

with coefficients in R (i.e. - $R\mathcal{L}^{fr}$ is a free R-module over $\mathcal{L}^{fr}$). Let $S\mathcal{L}^{fr}$ denote the submodule of $R\mathcal{L}^{fr}$ generated by local relations shown in Figure 2.

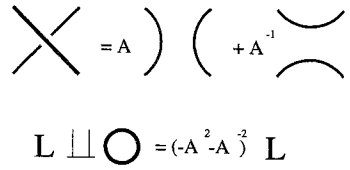


Fig. 2. Kauffman Bracket Skein Relations

Then the *Kauffman Bracket Skein Module (KBSM)* of M is defined to be

$$\mathcal{S}_{2,\infty}(M;R,A) = R\mathcal{L}^{fr} \Big/ \left\{ \begin{array}{l} L_+ - AL_0 - A^{-1}L_\infty \\ L \cup T_1 + (A^2 + A^{-2})L \end{array} \right\}.$$

where T_1 is the trivial framed link of one component (the trivial framed unknot).

The local relations are the ones that arise in the definition of the Kauffmann bracket polynomial of a link. By local relations we mean that the three links L_+, L_0, and L_∞ are three framed links that are identical outside some small neighborhood.

It can quickly be seen that $L^{(1)} = -A^3 L$ (where $L^{(1)}$ is L with a positive twist) in $\mathcal{S}_{2,\infty}(M;R,A)$ which we call the *framing relation*.

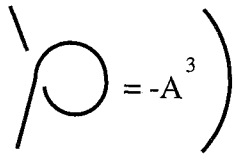


Fig. 3. Framing relations

The torsion that we will observe in $\mathcal{S}_{2,\infty}(M;R,A)$ will be of the form

$$(A^2 - A^{-2})[k_1 - k_2] = 0.$$

Here k_1 and k_2 will be distinct elements of $\mathcal{S}_{2,\infty}(M;R,A)$ such that $k_1 - k_2 \neq 0$ in $\mathcal{S}_{2,\infty}(M;R,A)$ and $A^2 - A^{-2} \neq 0$ in R.

DEFINITION 2.2. Let M be a 3-manifold and let $F \subset M$ be a surface such that $\partial F \subset \partial M$. Furthermore, assume that F is not homeomorphic to S^2, P^2,

or a disk that can be pushed into ∂M. We say that F is *incompressible* in M if every loop (simple closed curve) on F which bounds an open disk in $M \setminus F$ also bounds a disk in F.

REMARK 2.3. If $\pi_1(F) \hookrightarrow \pi_1(M)$ embeds injectively, then F is incompressible. If F is 2-sided then the converse is also true (this follows the Loop Theorem of Papakyriakopoulus).

DEFINITION 2.4. Let F be a surface and let $\gamma \in M$ be a loop. We define the *geometric index*, denoted by $i(F, \gamma)$, between F and γ to be the minimum number of intersections of γ with F where we allow γ to move up to homotopy. If $i(F, \gamma)$ is non-zero then we say that γ geometrically cuts F non-trivially.

A general conjecture is that if a 3-manifold M which contains an incompressible torus T^2 such that there is a loop in M which geometrically cuts T^2 non-trivially then $\mathcal{S}_{2,\infty}(M; Z[A^{\pm}1], A)$ has torsion.

3. Example of torsion

Let $M = M_1 \bigcup_{T^2} M_2$ where M_1 and M_2 are 3-manifolds such that $\partial M_1 = \partial M_2 = T^2$, T^2 is incompressible in M, and M_1 and M_2 are glued together by some surface homeomorphism of T^2.

First, pick a basepoint x in T^2. Then take an element $k_1 \in \pi_1(M_1)$ which is not in the image of $\pi_1(T^2) \to \pi_1(M_1)$ in such a way that a small arc of it lies on T^2.

Orient k_1 as shown in Figure 4. Take a similar type of element $k_2 \in \pi_1(M_2)$ and give it an orientation so that we can take the connected sum $k = k_1 \sharp k_2$. It is easy to see that the resulting loop k will be a non-trivial element of $\pi_1(M)$ and that $i(k, T^2) = 2$.

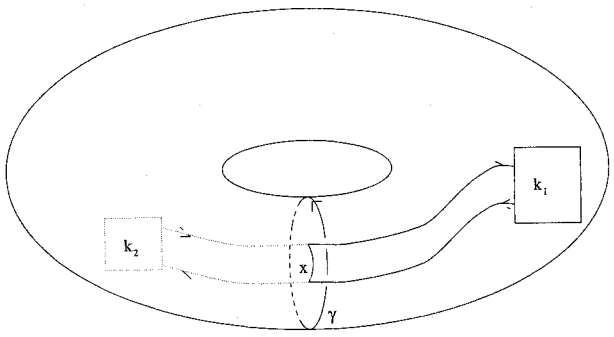


Fig. 4. $k_1 \in \pi_1(M_1)$, $k_2 \in \pi_1(M_2)$, $\gamma \in \pi_1(T^2)$

Now take the non-contractible curve γ on T^2 as shown in Figure 4.

Observe that we can slide γ all the way around T^2 to position it on the other side of the two intersection points of $k \cap T^2$ by an obvious ambient isotopy. Look closely at a local patch of T^2 that contains the two intersection points as well as γ. Project k_1 and k_2 onto this patch to get the local link diagram shown in Figure 5.

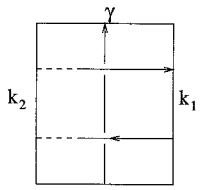


Fig. 5. Local patch before ambient isotopy

The action of the ambient isotopy changes both crossings to give us the local link diagram shown in Figure 6.

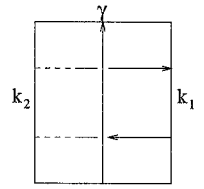


Fig. 6. Local patch after ambient isotopy

Since these two link diagrams are related by an ambient isotopy they represent the same link in $\mathcal{S}_{2,\infty}(M; R, A)$. This gives us the equation

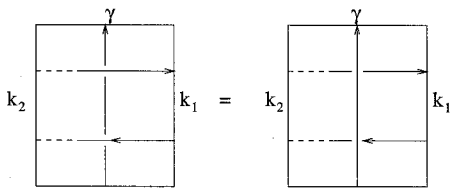


Fig. 7. Link diagram equation in $\mathcal{S}_{2,\infty}(M; R, A)$

Using the skein relations, we can break these two crossings to get

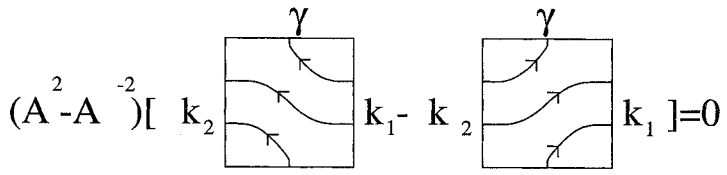


Fig. 8.

Orienting γ induces orientations on the knots in the above equation so rewriting this equation with respect to the original orientations we gave to k_1 and k_2 gives us

$$(A^2 - A^{-2})[\gamma k_2^{-1} k_1 - \gamma k_1 k_2^{-1}] = 0.$$

Our goal is to find γ, k_1, and k_2 so that the two curves $\gamma k_2^{-1} k_1$ and $\gamma k_1 k_2^{-1}$ are distinct elements of $\mathcal{S}_{2,\infty}(M; R, A)$.

Notice that the two curves $\gamma k_2^{-1} k_1$ and $\gamma k_1 k_2^{-1}$ are the same in $H_1(M)$.

4. Bullock's idea connecting $SL_2(C)$ representations with $\mathcal{S}_{2,\infty}(M; R, A)$

Recall that $SL_2(C)$ is the set of all 2-by-2 matrices with determinant equal to unity and entries drawn from the complex numbers. The characteristic polynomial of a matrix A is the polynomial

$$f_\lambda(A) = \det(A - \lambda I).$$

If $A \in SL_2(C)$, then we get

$$\begin{aligned} f_\lambda(A) &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc \\ &= \lambda^2 - tr(A)\lambda + 1. \end{aligned}$$

The Cayley-Hamilton theorem states that a matrix A is a root of its own characteristic polynomial. Thus,

$$A^2 - tr(A)A + I = O.$$

When $A \in SL_2(C)$, then A is invertible so that we can rewrite this equation (after multiplying through by A^{-1}) as

$$A - tr(A)I + A^{-1} = O,$$

$$A + A^{-1} = tr(A)I.$$

Let $B \in SL_2(C)$. Then we get

$$BA + BA^{-1} = tr(A)B.$$

We can take the trace of this equation to get

$$tr(BA) + tr(BA^{-1}) = tr(A)tr(B). \tag{1}$$

A few other useful properties of the trace function are

$$\begin{aligned} tr(A+B) &= tr(A) + tr(B), \\ tr(AB) &= tr(BA), \\ tr(\alpha A) &= \alpha tr(A)I, \end{aligned}$$

and when $A \in SL_2(C)$

$$tr(A) = tr(A^{-1}).$$

In [2], a representation $\varphi : \pi_1(M) \to SL_2(C)$ yields an algebra homomorphism $\widehat{\varphi} : \mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1) \to C$.

First, a fact which we shall not prove is that $\mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1)$ is an algebra where multiplication of two links is defined as their disjoint sum.

The idea behind the construction of the desired algebra homomorphism is as follows:

(1) Construct an algebra using $\pi_1(M)$ and divide by relations akin to (1).
(2) Show that this algebra is isomorphic to the algebra $\mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1)$.
(3) Show that the desired algebra homomorphism comes from the character of a representation $\varphi : \pi_1(M) \to SL_2(C)$.

This will imply that loops with different characters must be distinct in the algebra $\mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1)$ and, hence, must be distinct in $\mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], A)$. In the final section of this paper we will construct such loops.

We start with $\pi_1(M)$ and construct the symmetric tensor algebra $R[\pi_1(M)]$ consisting of polynomials with coefficients in R and variables $x_g, g \in \pi_1(M)$ (here $e \in \pi_1(M)$ is the identity element). Now define

$$S^+(\pi_1(M), R) = R[\pi_1(M)] \Big/ \left\{ \begin{array}{l} x_e + 2 \\ x_g x_h + x_{gh} + x_{gh^{-1}} \end{array} \right\}.$$

These relations are not chosen by accident. They look very similar to the trace identities (the first one being the $tr(I) = 2$) derived earlier (just a small sign difference).

FACT. $S^+(\pi_1(M), R) \cong \mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1)$ as algebras [11].

PROPOSITION 4.1. *$S^+(\pi_1(M), R)$ has the following properties:*

(1) $x_g = x_g^{-1}$.
(2) $x_{gh} = x_{hg}$.
(3) $x_{ghg^{-1}} = x_h$.

PROOF.

(1) $x_e x_g = -x_{eg} - x_{eg^{-1}} = x_e x_g^{-1}$, so that $x_g = x_g^{-1}$.
(2) $x_g x_h = -x_{gh} - x_{gh^{-1}}$ and

$$\begin{aligned} x_h x_g &= -x_{hg} - x_{hg^{-1}} \\ &= -x_{hg} - x_{(gh^{-1})^{-1}} \\ &= -x_{hg} - x_{gh^{-1}}. \end{aligned}$$

Thus $-x_{gh} - x_{gh^{-1}} = -x_{hg} - x_{gh^{-1}}$, so that $x_{gh} = x_{hg}$.

(3) $x_{gh} x_{g^{-1}} = -x_{ghg^{-1}} - x_{ghg} = -x_h - x_{ghg}$ so that $x_{ghg^{-1}} = x_h$.

□

Topologically, the first and third properties mean that to get distinct elements in the algebra $S^+(\pi_1(M), R)$ we need only consider unoriented loops in different conjugacy classes (change in basepoint does not matter) of $\pi_1(M)$.

FACT. A representation $\varphi : \pi_1(M) \to SL_2(C)$ yields an algebra homomorphism $\widehat{\varphi} : \mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1) \to C$ given by $\widehat{\varphi}(K) = -tr(\varphi([K]))$ where $[K]$ is the representative of a loop $K \in \mathcal{S}_{2,\infty}(M; Z[A^{\pm 1}], -1)$.

5. A 3-manifold in which $\mathcal{S}_{2,\infty}(M; R, A)$ has torsion

DEFINITION 5.1. Let M_1 be an orientable 3-manifold and with $\partial M_1 = T^2$. Let M_2 be homeomorphic to M_1 but with opposite orientation. Now let $M = M_1 \bigcup_{Id} M_2$ where M_1 and M_2 are glued by the identity map. The resulting 3-manifold M is called the *double* of M_1.

Let M be an orientable, complete hyperbolic 3-manifold. Recall that the orientation preserving isometries of hyperbolic space are isomorphic to $PSL_2(C)$. The holonomy map $H_0 : \pi_1(M) \to PSL_2(C)$ can be lifted to $\widetilde{H}_0 : \pi_1(M) \to SL_2(C)$. Let $\gamma \in \pi_1(M)$ be as in Figure 4 and we may assume that $\widetilde{H}_0(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The final step is to find $k_1, k_2 \in \pi_1(M)$ so that $\gamma k_2^{-1} k_1$ and $\gamma k_1 k_2^{-1}$ have different traces. This implies that they are distinct elements of $\mathcal{S}_{2,\infty}(M; C, -1)$. Note that $\widetilde{H}_0(\gamma)$ and $\widetilde{H}_0(k_1)$ do not commute.

In fact, if $k_1 \in \pi_1(M)$ and $\widetilde{H}_0(k_1) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, $x \neq 0$, then the loop $k_2 = \gamma k_1$ is a loop in $\mathcal{S}_{2,\infty}(M; C, -1)$ such that $\gamma k_2^{-1} k_1$ and $\gamma k_1 k_2^{-1}$ are distinct elements of $\mathcal{S}_{2,\infty}(M; C, -1)$. This follows from the following simple computation.

$$\widetilde{H}_0(k_2) = \begin{pmatrix} 1+x & 1 \\ x & 1 \end{pmatrix}.$$

Thus,

$$\gamma k_2^{-1} k_1 = \begin{pmatrix} x^2 - x + 1 & x \\ x^2 & x+1 \end{pmatrix},$$

$$\gamma k_1 k_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$\begin{aligned} tr(\gamma k_2^{-1} k_1) &= x^2 + 2, \\ tr(\gamma k_1 k_2^{-1}) &= 2, \end{aligned} \tag{2}$$

which are not equal whenever $x \neq 0$.

6. Concrete example – The double of 4_1

Riley [13] showed that hyperbolic knots have faithful parabolic representations whose image is discrete. The figure-8 knot 4_1 is one such example.

According to Riley [13] and Thurston [14], there is a representation of $\pi_1(S^3 \setminus 4_1) \to PSL_2(C)$ generated by $H_0(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $H_0(k_1) = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$, $\omega^2 + \omega + 1 = 0$. This representation lifts to a representation of $\pi_1(S^3 \setminus 4_1) \to SL_2(C)$ generated by $\widetilde{H}_0(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\widetilde{H}_0(k_1) = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$, where ω is a third primitive root of unity. So by (2), their traces are different. Hence, the KBSM of the double of 4_1 has torsion.

7. Conjecture

We formulate the following conjecture:

CONJECTURE 7.1. *Let $\widehat{M}$ denote a 3-manifold obtained from M by capping off all S^2 boundary components of M. If $\widehat{M}$ has an incompressible torus T^2 that is not parallel to the boundary, then $\mathcal{S}_{2,\infty}(\widehat{M}; R, A)$ has torsion.*

For irreducible M, P. Scott proved that T^2 has a closed curve cutting T^2 with geometric index equal to 2 (see [6]). This allows for the possible construct of a candidate for torsion as specified in Section 3.

References

[1] J.W. Alexander, Topological invariants of knots and links, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[2] D. Bullock, Estimating A. Skein Module With $SL_2(C)$ Characters, Proc. Amer. Math. Soc. **125** (1997), 1835–1839.

[3] J.H. Conway, An enumeration of knots and links, Computational problems in abstract algebra (ed. J. Leech), Pergamon Press, 1969, 329–358.

[4] C.A. Giller, A family of links and the Conway calculus, Trans. Amer. Math. Soc. **270** (1982), 75–109.

[5] J. Hoste and M. Kidwell, Dichromatic link invariants, Trans. Amer. Math. Soc., **321** (1990), 197–229.

[6] W. Jakobsche and J.H. Przytycki, Topologia 3-Wymiarowych Rozmaitości, Wydawnictwa Uniwersytetu, Warszawskiego, 1987, Theorem 7.3, 122–125.

[7] L.H. Kauffman, Combinatorics and knot theory, Contemporary Math. **20** (1983), 181–200.

[8] L.H. Kauffman, Formal knot theory, Mathematical Notes, **30**, Princeton University Press, 1983.

[9] L.H. Kauffman, On knots, Annals of Math. Studies, **115**, Princeton University Press, 1987.

[10] W.B.R. Lickorish and K.C. Millett, A polynomial invariant of oriented links, Topology **26** (1987), 107–141.

[11] J.H. Przytycki and A.S. Sikora, Skein algebra of a group, Banach Center Pub. **42** (1998), 297–306.

[12] J.H. Przytycki, Skein modules of 3-manifolds, Bull. Ac. Pol. Math. **39** (1991), 91–100.

[13] R. Riley, An elliptical path from parabolic representations to hyperbolic structures, Topology of Low-dimensional manifolds (ed. R. Fenn), Lecture notes in Math. **722** (1979), 281–288.

[14] W.P. Thurston, The geometry and topology of 3-manifolds, Mimeographed notes, Princeton University, 1979.

[15] V.G. Turaev, The Conway and Kauffman modules of a solid torus, Zap. Nauch. Sem. Lomi **167** (1988), 79–89. English translation: J. Soviet math. **52** (1990), 2799–2805.

Department of Mathematics
The George Washington University
Washington, DC 20052, U.S.A.
E-mail: veve@gwu.edu