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PERIODIC COINCIDENCE FOR MAPS OF SPHERES

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Abstract

Suppose that $t\colon S^m \to S^m$ is a free periodic map of period q of an m-sphere. Let $f\colon S^m \to Y$ be map of S^m into a polyhedron Y. We show that if $q(dim\, Y) \leq m$ then there exists a point $x \in S^m$ and a prime factor p of q such that f maps each orbit of the induced $\mathbf{Z}_{\frac{q}{p}}$-action on the orbit of x to a single point. In particular, if q is a prime then there exists an orbit of t mapped to a single point. The inequality $q(dim\, Y) \leq m$ is an improvement of that obtained by Gonçalves and Pergher in [3]. Our result also extends an unpublished result proved in [5] and resolves a question on the existence of antipodal coincidence which was left open in [4].

Let S be a space with a periodic map $t\colon S \to S$ of period q. A map $f\colon S \to Y$ to a space Y is said to have a *q-coincidence* if f maps the orbit of a point of S to a single point, i.e., if there exists a point $x \in S$ such that $fx = ftx = \cdots = ft^{q-1}x$. The classical Borsuk-Ulam theorem says that if $X = S^m$, $Y = \mathbf{R}^k$, $q = 2$ and t is the antipodal map of S^m, then every map $f\colon S^m \to \mathbf{R}^k$ has a 2-coincidence (an antipodal coincidence) if $k \leq m$.

In [4] we constructed an example to show that for each k and $m < 2k$, there exists a map of S^m into a contractible k-dimensional polyhedron Y without an antipodal coincidence. In particular, there exists such a map of S^3 into a contractible 2-dimensional polyhedron, a fact which answers a question posed by Conner and Floyd ([2], p. 89). On the other hand,we showed that if $m > 2k$ then every map $S^m \to Y$ of S^m into a k-dimensional polyhedron Y has an antipodal coincidence. For $m = 2k$ we were able to prove the existence of an antipodal coincidence, but only under an additional assumption that $H_k(Y; \mathbf{Z}_{/2}) = 0$. In a later paper [5], the second author showed that this additional assumption could be dropped. The latter paper remained unpublished because we found a paper by E. V. Ščepin [7] which contained the same result, although proved in a different way. In this paper we will prove an analogous result for free periodic maps of

spheres of any period. The index of the set consisting of coincidence points of a $\mathbf{Z}_{p^\alpha}$-action, where p is prime, was studied by Neža Mramor-Kosta in [6].

If q is not prime, it is useful to generalize the concept of q-coincidnece as follows. Let $t\colon S \to S$ and $f\colon S \to Y$ be as above, let p be a factor of q and let $G(x) = \{x, tx, \dots, t^{q-1}x\}$ be an orbit of a point x of S. Let $t_{(x,p)}\colon G(x) \to G(x)$ be the restriction of $t^{\frac{q}{p}}$ to $G(x)$. Then $t^{\frac{q}{p}}$ generates an acion of $\mathbf{Z}_{/p}$ on $G(x)$ with $\frac{q}{p}$ orbits. We say that f *has a* (q,p)*-coincidence* if there exists a point $x \in S$ such that f maps each of these oribits to a single point.

We will prove the following theorem.

THEOREM. *Let S^m be an m-sphere with a free periodic map t of period q and $f\colon S^m \to Y$ a map of S^m into a k-dimensional polyhedron Y. If $qk < m$ then there exists a prime factor p of q such that f has a (q,p)-coincidence.*

COROLLARY. *Suppose that $t\colon S^m \to S^m$ is a periodic map of a prime period p and $f\colon S^m \to Y$ a map into a k-dimensional polyhedron Y. If p is prime and $pk < m$ then f has a p-coincidence.*

This theorem and the corollary weaken the inequality $q(\dim Y) < m$ obtained by Gonçalves and Pergher in [3] (Theorems B and A) to $q(\dim Y) \leq m$ thus yielding a sharper conclusion in the cases where $\dim Y$ is odd or if $q = 2$. Of course, if $\dim Y$ is even and $q > 2$ then $q(\dim Y) \leq m$ already implies $q(\dim Y) < m$. Some arguments in the proof are similar to those used by Gonçalves and Pergher. However, while they assume that Y is a CW-complex, we restrict ourselves to compact polyhedra.

Deleted products. The q-fold product $Y^q = Y \times \cdots \times Y$ admits an action of $\mathbf{Z}_{/q}$ generated by the cyclic permutation $T\colon (y_0, \dots, y_{q-1}) \mapsto (y_{q-1}, y_0, \dots, y_{q-2})$. For each prime factor p of q. The subgroup G_p of $\mathbf{Z}_{/q}$ generated by $t^{\frac{q}{p}}$ is isomorphic to $\mathbf{Z}_{/p}$. We write $(Y^q)^{G_p}$ for its fixed point set and $F := \bigcup_p (Y^q)^{G_p}$, where p runs over all prime factors of q. The deleted q-fold product of Y is $Y_0^{(q)} := Y^q - F$; it is precisely the part of Y^q where the $\mathbf{Z}_{/q}$-action is free.

Let X be a space with a periodic map $t\colon X \to X$ of period q. A map $f\colon X \to Y$ induces a $\mathbf{Z}_{/q}$-equivalent map from $X \to Y^q$ by $x \mapsto (fx, ftx, \dots, ft^{q-1}x)$. If $f\colon S^m \to Y$ has no (q,p)-coincidence for any prime factor p of q then $f(S^m) \subset Y_0^{(q)}$ so that f induces an equivariant map $g\colon Y \to Y_0^{(q)}$.

LEMMA. *If Y is a k-dimensional polyhedron then the cohomology restriction homomorphism $i_*\colon H^{qk}(Y^q) \longrightarrow H^{qk}Y_0^{(q)}$ is surjective.*

PROOF. The q-fold Cartesian product of a rectilinear embedding of Y in a Euclidian space E embeds Y^q as a subpolyhedron of the q-fold product E^q. The fixed point sets $(E^q)^{G_p}$ are unions of some diagonal subspaces of E^q, and

the fixed point sets $(Y^q)^{G_p}$ become intersections of Y^q with them so that F is a subpolyhedron of Y^q (in some triangulation). Let N be a regular neighborhood of F in Y^q and let M be the closure of its complement. Then Y^q is the union $Y^q = M \cup N$ such that $Y_0^{(q)}$ deformation retracts to M, N deformation retracts to F, and $dim(M \cap N) \leq qk - 1$. Therefore the top portion of the Mayer-Vietoris cohomology sequence of the triad (Y^q, M, N) looks as follows.

$$\cdots \longrightarrow H^{qk-1}(M \cap N) \longrightarrow H^{qk}(Y^q) \xrightarrow{i_*} H^{qk} y_0^{(q)} \longrightarrow 0$$

□

The Smith Sequence and cohomology properties of a free $\mathbf{Z}_{/q}$-action. In the rest of this paper cohomology with coefficients mod q (resp. mod p) will be used. The coefficient group $\mathbf{Z}_{/q}$ (or $\mathbf{Z}_{/p}$) will be suppressed from the notation. The orbit space of a space X with a group action will be denoted by $\overline{X}$.

Suppose that $\mathbf{Z}_{/q}$ acts freely on a simplicial complex X and let $\Lambda := \mathbf{Z}_{/q}[\mathbf{Z}_{/q}]$. Let $\sigma = 1 + t + t^2 + \cdots + t^{q-1}$ and $\tau = 1 - t$ be in Λ, let $C(X)$ be the chain complex of X and let $\Lambda_\sigma := im\,\sigma$ and $\Lambda_\tau = im\,\tau$. Then Λ_σ and Λ_τ fit into two short exact sequences

$$0 \to \Lambda_\sigma \xrightarrow{\iota} \Lambda \xrightarrow{\kappa} \Lambda_\tau \to 0$$
$$0 \to \Lambda_\tau \xrightarrow{\iota'} \Lambda \xrightarrow{\kappa'} \Lambda_\sigma \to 0.$$

In these sequences ι and ι' are the inclusions and κ and κ' are defined by σ and τ respectively. Since $C(X)$ is free over Λ, each of these sequences leads to a short exact of chain complexes. We will be using the sequence arising from the first of the above sequences:

$$0 \to C(X)_\sigma \xrightarrow{\iota} C(X) \xrightarrow{\kappa} C(X)_\tau \to 0.$$

This short exact sequence generates a long cohomology exact sequence; it is a *Smith sequence* associated with the $\mathbf{Z}_{/q}$-action:

$$\cdots \xrightarrow{\kappa^*} H^j X \xrightarrow{\iota^*} H^j_\sigma X \longrightarrow H^{j+1}_\tau X \xrightarrow{\kappa^*} H^{j+1} X \xrightarrow{\iota^*} \cdots$$

Smith homology and cohomology were originally defined by Smith in [8] and in a series of subsequent papers. A systematic exposition of the Smith theory can be found in [1]. The Smith sequence is most useful if q is prime.

If the $\mathbf{Z}_{/q}$-action is free, the group $H^j_\sigma X$ is naturally isomorphic to the ordinally cohomology group $H^j \overline{X}$ of the orbit space.

The standard periodic map of an m-sphere S^m of period 2 is the antipodal map. If q is a odd and m is odd, $m = 2n+1$, then the standatd periodic map $t\colon S^{2n+1} \to S^{2n+1}$ of period q is the scalar multiplication by a complex q-th root of one. That is, if S^{2n+1} is the unit sphere in the complex $(n+1)$-dimensional space $\mathbf{C}^{n+1}$ then $t\colon S^{2n+1} \to S^{2n+1}$ is given by $t(z) = e^{\frac{2\pi\sqrt{-1}}{q}} z$. This map generates the standard complex representation of the cyclic group $\mathbf{Z}_{/q}$ and it acts freely on S^{2n+1}. The orbit space of t, $L_q^{2n+1} = \overline{S}^{2n+1}$, is a lens space $L_q^{2n+1} = L_q^{2n+1}(1,\dots,1)$. In particular, if $q = 2$ then L_2^{2n+1} is just the real projective $(2n+1)$-space.

A universal example of a $\mathbf{Z}_{/q}$-actions is the standard action of $\mathbf{Z}_{/q}$ on the unit sphere S^∞ in the infinite-dimensional complex vector space S^∞. The orbit space is the infinite-dimensional lens space L_q^∞. Since $H^j(S^\infty) = 0$ for $j > 0$, the Smith sequence is:

$$0 \to H_\tau^0(S^\infty) \xrightarrow{\cong} H^0(S^\infty) \xrightarrow{0} H^0(L_q^\infty) \xrightarrow{\cong} H_\tau^1(S^\infty) \to 0$$
$$\to H^1(L_q^\infty) \xrightarrow{\cong} H_\tau^2(S^\infty) \to 0 \to \cdots \to 0 \to H^j(L_q^\infty) \xrightarrow{\cong} H_\tau^{j+1}(S^\infty) \to 0 \cdots$$

and so it becomes a sequence of zeroes and isomorphisms

$$0 \longrightarrow \mathbf{Z}_{/q} \xrightarrow{\cong} \mathbf{Z}_{/q} \xrightarrow{0} \mathbf{Z}_{/q} \xrightarrow{\cong} \mathbf{Z}_{/q} \xrightarrow{\cong} 0 \longrightarrow \mathbf{Z}_{/q} \xrightarrow{\cong} \mathbf{Z}_{/q} \longrightarrow \cdots .$$

Suppose now that p is prime. The classifying space for $\mathbf{Z}_{/p}$ is the infinite-dimensional lens space L_p^∞. If $p = 2$, $L_2^\infty = P^\infty$, the real infinite dimensional projective space. The cohomology algebra $H^*(P^\infty)$ is isomorphic to $\mathbf{Z}_{/2}[u]$, where $[u] \in H^1(P^\infty)$ is the universal characteristic class of the involution t. If p is an odd prime, $H^*(L_p^\infty)$ is isomorphic to $\bigwedge_p[u] \oplus \mathbf{Z}_{/p}[v]$, where $[u] \in H^1(L_p^\infty)$ is the universal characteristic class of $\mathbf{Z}_{/p}$, $v = \beta u \in H^2(L_p^\infty)$ and $\beta\colon H^1 \to H^2$ is the Bockstein homomorphism associated with the sequence

$$0 \longrightarrow \mathbf{Z}_{/p} \longrightarrow \mathbf{Z}_{/p^2} \longrightarrow \mathbf{Z}_{/p} \longrightarrow 0$$

If X is a $\mathbf{Z}_{/p}$-space and $\phi\colon \overline{X} \to L_p^\infty$ is a classifying map for X, $u_X := \phi^* u$ is the characteristic class of the $\mathbf{Z}_{/p}$-action on X. The $\mathbf{Z}_{/p}$-*index*, $\operatorname{Ind}^{\mathbf{Z}_{/p}} X$, of X is the largest integer j such that $H^j(L_p^\infty)$ is not contained in the kernel of ϕ^*. In particular, if $p = 2$ then $\operatorname{Ind}^{\mathbf{Z}_{/2}} X$ is the largest exponent j such that $u_X^j \neq 0$.

PROOF OF THE THEOREM. This proof we will only use the structure of the cohomology ring of the orbit space of the action. Since the cohomology ring of an arbitrary free $\mathbf{Z}_{/q}$-action is isomorphic to that of the standard action, we may assume that the action is standard.

We will assume that $q > 2$; the case $q = 2$ is just a little simpler (compare [5]). Let $m = 2n+1$. We will show that if $qk = m$ then there is no $\mathbf{Z}_{/q}$-equivariant map from S^{qk} to $Y_0^{(q)}$.

Suppose that $g\colon S^{qk} \to Y_0^{(q)}$ is an $\mathbf{Z}_{/q}$-equivariant map. Let p be any prime factor of q. Then g is also $\mathbf{Z}_{/p}$-equivariant. Consider the $\mathbf{Z}_{/p}$-action on S^{2n+1} generated by $\mathbf{Z}_{/\frac{q}{p}}$ and the corresponding $\mathbf{Z}_{/p}$-action on $Y_0^{(q)}$. Since the Smith sequence is natural with respect to equivariant maps, g induces a map of the Smith sequence of $Y_0^{(q)}$ to that of S^{qk}. The top-dimensional portion of the diagram of the map is

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_\tau^{qk}(S^{qk}) & \xrightarrow{\iota^*} & H^{qk}(S^{qk}) & \xrightarrow{\kappa} & H_\sigma^{qk}(S^{qk}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle g_\tau{}^*} & & \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g_\sigma{}^*} & & \\
\cdots & \longrightarrow & H_\tau^{qk}(Y_0^{(q)}) & \xrightarrow{\iota^*} & H^{qk}(Y_0^{(q)}) & \xrightarrow{\kappa^*} & H_\sigma^{qk}(Y_0^{(q)}) & \longrightarrow & 0.
\end{array}$$

The group $H_\sigma^{qk}(S^{qk})$ is isomorphic to the cohomology group of the orbit space, which is the lens space L_p^{qk}. Under this isomorphism $g_\sigma{}^* = \overline{g}^*$, the map induced by the map $L_p^{qk} \to \overline{Y_0^{(q)}} = Y_0^{(q)}/\mathbf{Z}_{/p}$ of the orbit spaces, and the top portion of the diagram is

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_\tau^{qk}(S^{qk}) & \xrightarrow{\iota^*} & H^{qk}(S^{qk}) & \xrightarrow{\kappa^*} & H^{qk}(L_p^{qk}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle g_\tau{}^*} & & \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle \overline{g}^*} & & \\
\cdots & \longrightarrow & H_\tau^{qk}(Y_0^{(q)}) & \xrightarrow{\iota^*} & H^{qk}(Y_0^{(q)}) & \xrightarrow{\kappa^*} & H^{qk}(\overline{Y_0^{(q)}}) & \longrightarrow & 0
\end{array}$$

Since $u_{Y_0^{(q)}}$ and $v_{Y_0^{(q)}}$ are natural with respect to equivariant maps,

$$g_\sigma{}^*((u_{Y_0^{(q)}})(v_{Y_0^{(q)}})^{\frac{qk-1}{2}}) = \overline{g}^*((u_{Y_0^{(q)}})(v_{Y_0^{(q)}})^{\frac{qk-1}{2}}) \neq 0.$$

On the other hand we will show that $\overline{g}^* = 0$. Since $\kappa^*\colon H^{qk}(Y_0^{(q)}) \to H^{qk}(\overline{Y_0^{(q)}})$ is surjective, it suffices to show that the middle vertical arrow g^* is zero. Consider the following commutative triangle.

$$\begin{array}{ccc}
H^{qk}(Y^q) & \xrightarrow{k*} & H^{qk}(S^{qk}) \\
 & i* \searrow & \uparrow g* \\
 & & H^{qk}(Y_0^{(q)})
\end{array}$$

By Lemma, the map $i*$ of the Mayer-Vietoris sequence is surjective, so to show that $g* = 0$ it suffices to show that $H^{qk}(Y^q) \xrightarrow{k*} H^{qk}(S^{qk})$ is zero. But this follows from the fact the generators of $H^{qk}(Y^q)$ are decomposable into cup

products of p-dimensional classes and the generator of $H^{qk}(S^{qk})$ is not so decomposable. □

COMMENTS. We were able to prove the theorem under the assumption $qk \leq m$ which is weaker than $qk < m$ because we only showed that there is no equivariant map from S^{qk} to $Y_0^{(q)}$; we did not attempt to show that $\mathrm{Ind}^{\mathbf{Z}/p} Y_0^{(q)} < 2n$ as it is essentially done in [3]. Of course, if $\mathrm{Ind}^{\mathbf{Z}/p}(Y_0^{(q)}) < 2n$ then an equivariant map $S^{qk} \to Y_0^{(q)}$ cannot exist, but this is not alway so for $qk = m$. In fact, the $\mathbf{Z}/p$-index of the q-fold deleted product $Y_0^{(q)}$ of a k-dimensional polyhedron can very well be equal to qk.

EXAMPLE. Let Y be one of Kuratowski's non-planar graphs (the 1-skeleton of a 4-simplex; or the 1-skeleton of a 3-simplex together with an interval joining the centers of its two opposite edges), then the $\mathbf{Z}/2$-index of the deleted product $Y_0^{(2)}$ is two.

This is related to the work of W.T. Wu ([9], [10]) on embeddings of spaces in Euclidian spaces. In [10], p. 210, Wu shows that he deleted squares of the two Kuratowski's graphs have both index 2.

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