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C_n -MOVES AND POLYNOMIAL INVARIANTS FOR LINKS

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1. Introduction

In this paper we consider oriented links in the 3-sphere S^3 . A C_n -move ($n \geq 1$) is a local move on oriented links (see Figure 1.1) defined by K. Habiro [4].

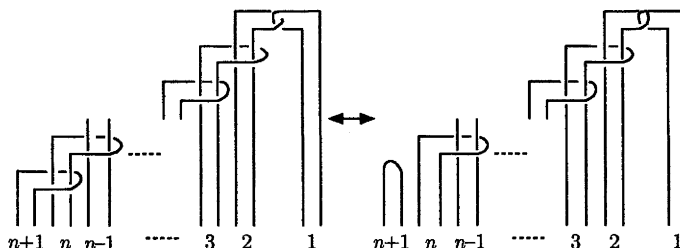


Fig. 1.1.

This local move is closely related to Vassiliev invariants introduced by V.A. Vassiliev [17]. In fact the following theorems are known:

THEOREM 1.1 ([3, 10, 14]). *Let L and L' be oriented links in S^3 and n a positive integer. If L and L' can be transformed into each other by C_n -moves, then $v(L) = v(L')$ for any Vassiliev invariant v of order less than n .*

THEOREM 1.2 ([5, 15]). *Let K and K' be oriented knots in S^3 and n a positive integer. Then K and K' can be transformed into each other by C_n -moves if and only if $v(K) = v(K')$ for any Vassiliev invariant v of order less than n .*

By $a_n(L)$ we denote the coefficient of z^n of the Conway polynomial $\nabla_L(z)$ for an oriented link L . We set $V^{(n)}(L) = \frac{d^n}{dt^n} V_L(t)|_{t=1}$, where $V_L(t)$ is the Jones polynomial for L . It is well-known that $a_n(L)$ and $V^{(n)}(L)$ are Vassiliev invariants of order n ([1]). Let L and L' be oriented links which can be transformed into each

other by C_n -moves. Then Theorem 1.1 implies $a_k(L) = a_k(L')$, and $V^{(k)}(L) = V^{(k)}(L')$ for $k = 0, 1, \dots, n-1$. But much is not known for the changes of $a_n(L)$ and $V^{(n)}(L)$. In this paper, we shall study the difference between $a_n(L)$ and $a_n(L')$, and $V^{(n)}(L)$ and $V^{(n)}(L')$, and prove the following Theorems 1.3 and 1.5:

THEOREM 1.3. *Let n be an integer greater than 2 and L and L' oriented links which can be transformed into each other by C_n -moves. Then it holds that*

$$a_n(L) - a_n(L') \equiv 0 \pmod{2}.$$

REMARK 1.4. 1. In the case of knots, the same result has been obtained by Y. Ohyama and Y. Ogushi [11] using the notion of n -similarity.
 2. If $n = 2$, Theorem 1.3 holds for μ -component links ($\mu \geq 2$), but it does not hold for knots. A C_2 -move is a delta move defined by S.V. Matveev [8] and H. Murakami and Y. Nakanishi [9]. It is known that $|a_2(K) - a_2(K')| = 1$ for knots K and K' , where K is obtained from K' by a single delta move ([13, 16]).

THEOREM 1.5. *Let n be an integer greater than 2 and L and L' oriented links which can be transformed into each other by C_n -moves. Then it holds that*

$$V^{(n)}(L) - V^{(n)}(L') \equiv 0 \pmod{6 \cdot n!}.$$

REMARK 1.6. In the case of $n = 2, 3$ and 4 , B. Matsuzaka [7] and T. Tsukamoto [16] obtained more precise results for knots. They showed that, by a single C_n -move, $V^{(n)}(K)$ changes as follows:

THEOREM 1.7 ([16]). (1) *If a knot K is obtained from a knot K' by a C_2 -move, then it holds that*

$$|V^{(2)}(K) - V^{(2)}(K')| = 6.$$

(2) *If a knot K is obtained from a knot K' by a C_3 -move, then it holds that*

$$|V^{(3)}(K) - V^{(3)}(K')| = 0 \quad \text{or} \quad 36.$$

THEOREM 1.8 ([7]). *If a knot K is obtained from a knot K' by a C_4 -move, then it holds that*

$$|V^{(4)}(K) - V^{(4)}(K')| = 0, 144 \quad \text{or} \quad 288.$$

The author would like to thank Professors Shinji Fukuhara and Yoshiyuki Ohyama for their valuable advices and encouragements.

2. Vassiliev invariants and chord diagrams

Let v be an invariant for links which takes values in an abelian group. Then we can extend v to an invariant for singular links by the Vassiliev skein relation:

$$v(L_D) = v(L_+) - v(L_-),$$

where a singular link is an immersion of a disjoint union of oriented circles in S^3 which has only transversal double points as its singularities, and L_D , L_+ and L_- denote the diagrams of singular links which are identical except near one point as is shown in Figure 2.1. A singular link is considered as a flat vertex graph.

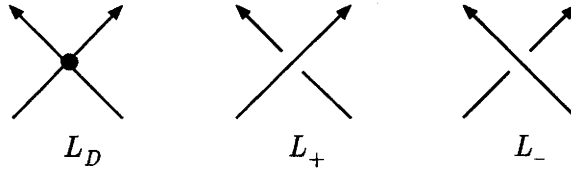


Fig. 2.1.

An invariant v is called a *Vassiliev invariant of order n* if $v(L) = 0$ for any singular link L with more than n double points and there exists a singular link L' with n double points such that $v(L') \neq 0$.

A *chord diagram of order n* is a disjoint union of counterclockwise oriented circles with n chords. A chord diagram of order n can be associated to a singular link with n double points by connecting the preimage of each double point with a chord. It is known that a Vassiliev invariant of order n is determined by a set of axioms and initial data called an *acutually table* containing values for the chord diagrams of order n and immersions that respect them of order less than n . Moreover any invariants of order n is determined, modulo the invariants of order less than n , by its values on chord diagrams of order n ([2]).

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. To do so, we suppose that L' is a link which is obtained from a link L by a C_n -move as shown in Figure 3.1. Note that we must consider all possibilities of the orientations for the strands. By $c_1, c_{21}, c_{22}, c_{31}, c_{32}, \dots, c_{n1}, c_{n2}$ we denote the crossings of L as indicated in Figure 3.1.

We start with the following lemma proved by Ohyama and Tsukamoto [12]:

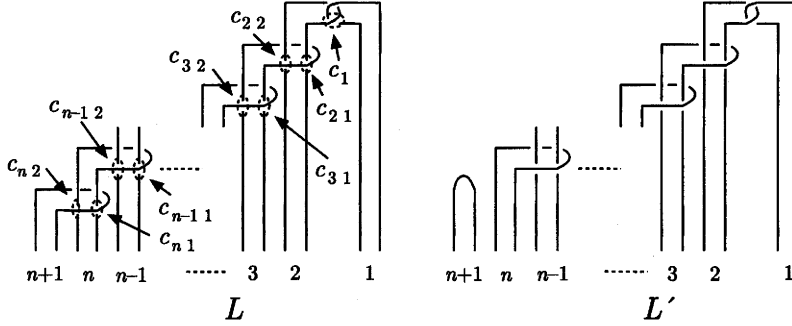


Fig. 3.1.

LEMMA 3.1 ([12]). *Let v be a Vassiliev invariant of order n , then*

$$v(L) - v(L') = \pm \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot v \left(L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix} \right),$$

where $L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix}$ is the singular link with n double points that is obtained from L by the following: Collapse the crossing to a double point at c_1 . If $i_j = 1$, collapse the crossing at c_{j1} and if $i_j = -1$, switch the crossing at c_{j1} and collapse the crossing to a double point at c_{j2} for $j = 2, 3, \dots, n$.

Using Lemma 3.1, we obtain the following formula:

$$a_n(L) - a_n(L') = \pm \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot a_n \left(L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix} \right).$$

From the Vassiliev skein relation and the definition of the Conway polynomial, it follows that

$$a_k(L_D) = a_k(L_+) - a_k(L_-) = a_{k-1}(L_0),$$

where L_D , L_+ and L_- are the same as stated in §2, and L_0 denotes the singular link obtained from L_D by smoothing the double point. Applying the above reduction n times to our formula, we obtain

$$a_n(L) - a_n(L') = \pm \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot a_0 \left(L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix} \right).$$

Here $L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix}$ is a non-singular link obtained from $L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix}$ by smoothing at the double points $c_1, c_{2i_2}, \dots, c_{ni_n}$.

Let us fix $i_j = \pm 1$ for $j = 2, 3, \dots, n-2$. We set

$$L_1 = L\left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \bar{1} & \bar{i}_2 & \cdots & \bar{i}_{n-2} & \bar{1} & \bar{1} \end{array}\right), \quad L_2 = L\left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \bar{1} & \bar{i}_2 & \cdots & \bar{i}_{n-2} & \bar{1} & -\bar{1} \end{array}\right),$$

$$L_3 = L\left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \bar{1} & \bar{i}_2 & \cdots & \bar{i}_{n-2} & -\bar{1} & \bar{1} \end{array}\right), \quad L_4 = L\left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \bar{1} & \bar{i}_2 & \cdots & \bar{i}_{n-2} & -\bar{1} & -\bar{1} \end{array}\right).$$

The diagrams of L_1, L_2, L_3 and L_4 differ only near the strands marked $n-1, n$ and $n+1$. Their difference depends only on relations of orientations of strands marked $n-1$ and n , and n and $n+1$. Hence it is enough to consider the following 4 cases:

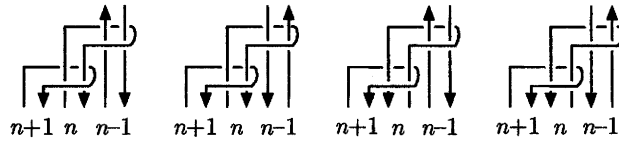


Fig. 3.2.

In Figure 3.3, we illustrate parts of diagrams of L_1, L_2, L_3 and L_4 near the strands $n-1, n$ and $n+1$ in each case in Figure 3.2.

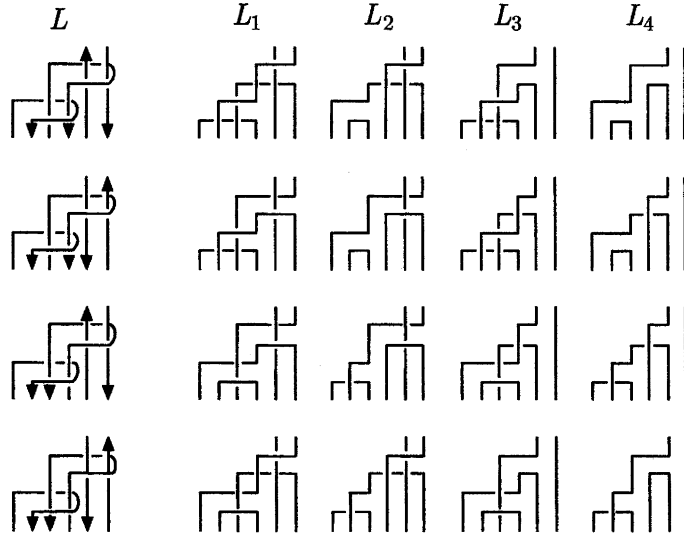


Fig. 3.3.

We consider all cases that strands $n-1$, n and $n+1$ can be connected outside the tangle in L_1 , L_2 , L_3 and L_4 shown in Figure 3.3. In Table 1 we show the all 96 connections of the strands $n-1$, n and $n+1$ for L_1 , L_2 , L_3 and L_4 . Note that L_1 , L_2 , L_3 and L_4 may have extra components which do not appear in the figures in Table 1. They have the same number of extra components for each connection. We denote the number of the extra components by m . From Table 1, we see that the number of knots included in $\{L_1, L_2, L_3, L_4\}$ is even in any case. It is known that the following formula holds:

$$a_0(L) = \begin{cases} 1 & \text{if } L \text{ is a knot} \\ 0 & \text{if } L \text{ is not a knot} \end{cases}.$$

Therefore

$$a_0(L_1) - a_0(L_2) - a_0(L_3) + a_0(L_4) = 0 \quad \text{or} \quad \pm 2.$$

Finally we obtain

$$\begin{aligned} a_n(L) - a_n(L') &= \pm \sum_{i_2, i_3, \dots, i_{n-2} = \pm 1} \prod_{j=2}^{n-2} i_j \cdot \{a_0(L_1) - a_0(L_2) - a_0(L_3) + a_0(L_4)\} \\ &\equiv 0 \pmod{2}. \end{aligned}$$

Hence we complete the proof.

4. Proof of Theorem 1.5

In this section, we consider Vassiliev invariants induced from the Jones polynomial. Similarly in the previous section, we have

$$V^{(n)}(L) - V^{(n)}(L') = \pm \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot V^{(n)} \left(L \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & i_2 & \cdots & i_n \end{pmatrix} \right).$$

T. Kanenobu and Y. Miyazawa proved the following reduction formula for $V^{(k)}$:

PROPOSITION 4.1 ([6]). *Let L^k be a singular link with k double points, L_-^{k-1} a singular link with $k-1$ double points obtained from L^k by changing a double point, say c , of L^k into a crossing point with a negative sign, and L_0^{k-1} the singular link with $k-1$ double points obtained from L^k by smoothing the double point c of L^k . Then*

$$V^{(k)}(L^k) = kV^{(k-1)}(L_0^{k-1}) + 2kV^{(k-1)}(L_-^{k-1}).$$

We apply Proposition 4.1 to our formula, and obtain

$$\begin{aligned} V^{(n)}(L) - V^{(n)}(L') \\ = \pm n \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^{n-1} i_j \cdot \left\{ V^{(n-1)} \left(L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & \overline{i_n} \end{array} \right) \right) \right. \\ \left. + 2V^{(n-1)} \left(L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & i_n^{-1} \end{array} \right) \right) \right\}. \end{aligned}$$

Here $L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & \overline{i_n} \end{array} \right)$ and $L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & i_n^{-1} \end{array} \right)$ denote singular links obtained from $L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & i_n \end{array} \right)$ by smoothing at c_{ni_n} , or changing the double point c_{ni_n} to a crossing with a negative sign, respectively. Then $L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & 1^{-1} \end{array} \right)$ and $L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & -1^{-1} \end{array} \right)$ correspond to the same chord diagram of order $n-1$ for $i_j = \pm 1$ ($j = 2, 3, \dots, n-1$). As stated in §2, the value of a Vassiliev invariant of order k for a singular link with k double points depends only on the chord diagram. Thus their values of a Vassiliev invariant of order $n-1$ coincide. Therefore we have

$$V^{(n)}(L) - V^{(n)}(L') = \pm n \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot V^{(n-1)} \left(L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-1 & n \\ 1 & i_2 & \cdots & i_{n-1} & \overline{i_n} \end{array} \right) \right).$$

Applying Proposition 4.1 to our formula n times, we obtain

$$\begin{aligned} V^{(n)}(L) - V^{(n)}(L') \\ = \pm n \cdot (n-1) \cdots 3 \cdot 2 \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot V^{(1)} \left(L \left(\begin{array}{cccc} 1 & 2 & \cdots & n \\ 1 & \overline{i_2} & \cdots & \overline{i_n} \end{array} \right) \right) \\ = \pm n! \sum_{i_2, i_3, \dots, i_n = \pm 1} \prod_{j=2}^n i_j \cdot \left\{ V^{(0)} \left(L \left(\begin{array}{cccc} 1 & 2 & \cdots & n \\ \overline{1} & \overline{i_2} & \cdots & \overline{i_n} \end{array} \right) \right) \right. \\ \left. + 2V^{(0)} \left(L \left(\begin{array}{cccc} 1 & 2 & \cdots & n \\ 1^{-1} & \overline{i_2} & \cdots & \overline{i_n} \end{array} \right) \right) \right\}. \end{aligned}$$

Fix $i_j = \pm 1$ ($j = 2, 3, \dots, n-2$). We set

$$\begin{aligned} L_1 &= L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \overline{1} & \overline{i_2} & \cdots & \overline{i_{n-2}} & \overline{1} & \overline{1} \end{array} \right), & L_2 &= L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \overline{1} & \overline{i_2} & \cdots & \overline{i_{n-2}} & \overline{1} & \overline{-1} \end{array} \right), \\ L_3 &= L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \overline{1} & \overline{i_2} & \cdots & \overline{i_{n-2}} & \overline{-1} & \overline{1} \end{array} \right), & L_4 &= L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ \overline{1} & \overline{i_2} & \cdots & \overline{i_{n-2}} & \overline{-1} & \overline{-1} \end{array} \right), \end{aligned}$$

$$L'_1 = L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ 1^{-1} & \frac{2}{i_2} & \cdots & \frac{n-2}{i_n-2} & \frac{n-1}{1} & \frac{n}{1} \end{array} \right), L'_2 = L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ 1^{-1} & \frac{2}{i_2} & \cdots & \frac{n-2}{i_n-2} & \frac{n-1}{1} & \frac{n}{-1} \end{array} \right),$$

$$L'_3 = L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ 1^{-1} & \frac{2}{i_2} & \cdots & \frac{n-2}{i_n-2} & \frac{n-1}{-1} & \frac{n}{1} \end{array} \right), L'_4 = L \left(\begin{array}{cccccc} 1 & 2 & \cdots & n-2 & n-1 & n \\ 1^{-1} & \frac{2}{i_2} & \cdots & \frac{n-2}{i_n-2} & \frac{n-1}{-1} & \frac{n}{-1} \end{array} \right).$$

We use Table 1 again. The value of $V^{(0)}(L)$ is determined by the number of components of L (we denote it by $\#L$), that is:

$$V^{(0)}(L) = (-2)^{\#L-1}.$$

Then in any case of Table 1,

$$\begin{aligned} & V^{(0)}(L_1) - V^{(0)}(L_2) - V^{(0)}(L_3) + V^{(0)}(L_4) \\ &= \begin{cases} 0 \\ \pm 2\{(-2)^2 - 1\}(-2)^m = \pm 6(-2)^m \\ \pm \{(-2)^3 - (-2)\}(-2)^m = \pm 6(-2)^m \end{cases}, \end{aligned}$$

and

$$\begin{aligned} & V^{(0)}(L'_1) - V^{(0)}(L'_2) - V^{(0)}(L'_3) + V^{(0)}(L'_4) \\ &= \begin{cases} 0 \\ \pm 2\{(-2)^2 - 1\}(-2)^m = \pm 6(-2)^m \\ \pm \{(-2)^3 - (-2)\}(-2)^m = \pm 6(-2)^m \end{cases}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & V^{(n)}(L) - V^{(n)}(L') \\ &= \pm n! \sum_{i_2, i_3, \dots, i_{n-2} = \pm 1} \prod_{j=2}^{n-2} i_j \cdot \{V^{(0)}(L_1) - V^{(0)}(L_2) - V^{(0)}(L_3) + V^{(0)}(L_4) \\ &\quad + 2\{V^{(0)}(L'_1) - V^{(0)}(L'_2) - V^{(0)}(L'_3) + V^{(0)}(L'_4)\}\} \\ &\equiv 0 \pmod{6 \cdot n!}. \end{aligned}$$

Hence we complete the proof of Theorem 1.5.

5. Examples

In this section, we give some examples of links L and L' such that L is obtained from L' by a C_n -move and $|a_n(L) - a_n(L')| = 0$ or 2 . Furthermore we observe Theorem 1.5 considering the examples.

Let K_n be an oriented knot as indicated in Figure 5.1. Then it is easy to see that K_{n-1} is obtained from K_n by a C_n -move. The Conway polynomial $\nabla_{K_n}(z)$ for K_n is $1 - z^2 + z^4 - \cdots + (-z^2)^{n-1}$. Therefore $\nabla_{K_n}(z) - \nabla_{K_{n-1}}(z) = (-z^2)^{n-1}$, hence $a_n(K_n) - a_n(K_{n-1}) = 0$ for $n \geq 3$.

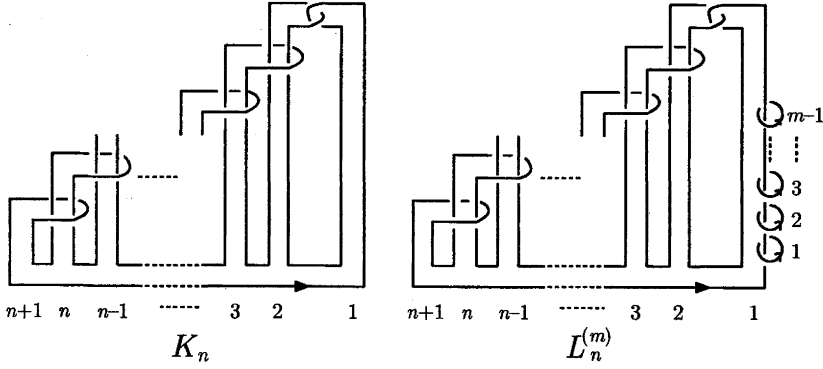


Fig. 5.1.

Next we deform K_n to an oriented m -component link $L_n^{(m)}$ as shown in Figure 5.1 for $m \geq 2$. Then it is obvious that $L_{n-1}^{(m)}$ is obtained from $L_n^{(m)}$ by a C_n -move. The Conway polynomial for $L_n^{(m)}$ is $\nabla_{L_n^{(m)}}(z) = z^{m-1} \nabla_{K_n}(z) = z^{m-1}(1 - z^2 + z^4 - \cdots + (-z^2)^{n-1})$. Therefore $a_n(L_n^{(m)}) - a_n(L_{n-1}^{(m)}) = 0$ for $n \geq 2$.

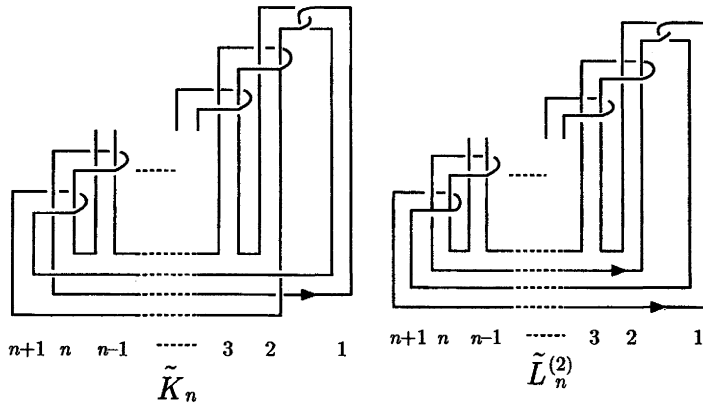


Fig. 5.2.

Let $\tilde{K}_n$ be an oriented knot illustrated in Figure 5.2. Then K_{n-1} in Figure 5.1 is obtained from $\tilde{K}_n$ by a C_n -move. By an easy computation, we have $a_{2n}(\tilde{K}_{2n}) = \begin{cases} 1 & n : \text{odd} \\ 3 & n : \text{even} \end{cases}$ and $a_{2n}(K_{2n-1}) = \begin{cases} -1 & n : \text{odd} \\ 1 & n : \text{even} \end{cases}$. Therefore $a_{2n}(\tilde{K}_{2n}) - a_{2n}(K_{2n-1}) = 2$.

We give the following example for the case of 2-component links: Let $\tilde{L}_n^{(2)}$ be an oriented 2-component link as shown in Figure 5.2. Then we obtain a Hopf link H with the linking number -1 from $\tilde{L}_n^{(2)}$ by a C_n -move. We have $a_n(\tilde{L}_n^{(2)}) = -2$ for an odd integer n , and $\nabla_H(z) = -z$ by an easy calculation. Hence $a_n(\tilde{L}_n^{(2)}) - a_n(H) = \begin{cases} -2 & n : \text{odd} \\ 0 & n : \text{even} \end{cases}$.

In the case of the Jones polynomial, we have the following results for $n \geq 3$:

$$|V^{(n)}(K_n) - V^{(n)}(K_{n-1})| = 6 \cdot n! 2^{n-3},$$

$$|V^{(n)}(\tilde{K}_n) - V^{(n)}(K_{n-1})| = \begin{cases} 6 \cdot n! \frac{1}{3} (2^{n-3} + 1) & n : \text{even} \\ 6 \cdot n! \frac{1}{3} (2^{n-3} - 1) & n : \text{odd} \end{cases}.$$

These are calculated using Ohyama and Tsukamoto's theorem [12]. Note that Theorem 1.5 is best possible since the greatest common divisor of the differences above is $6 \cdot n!$.

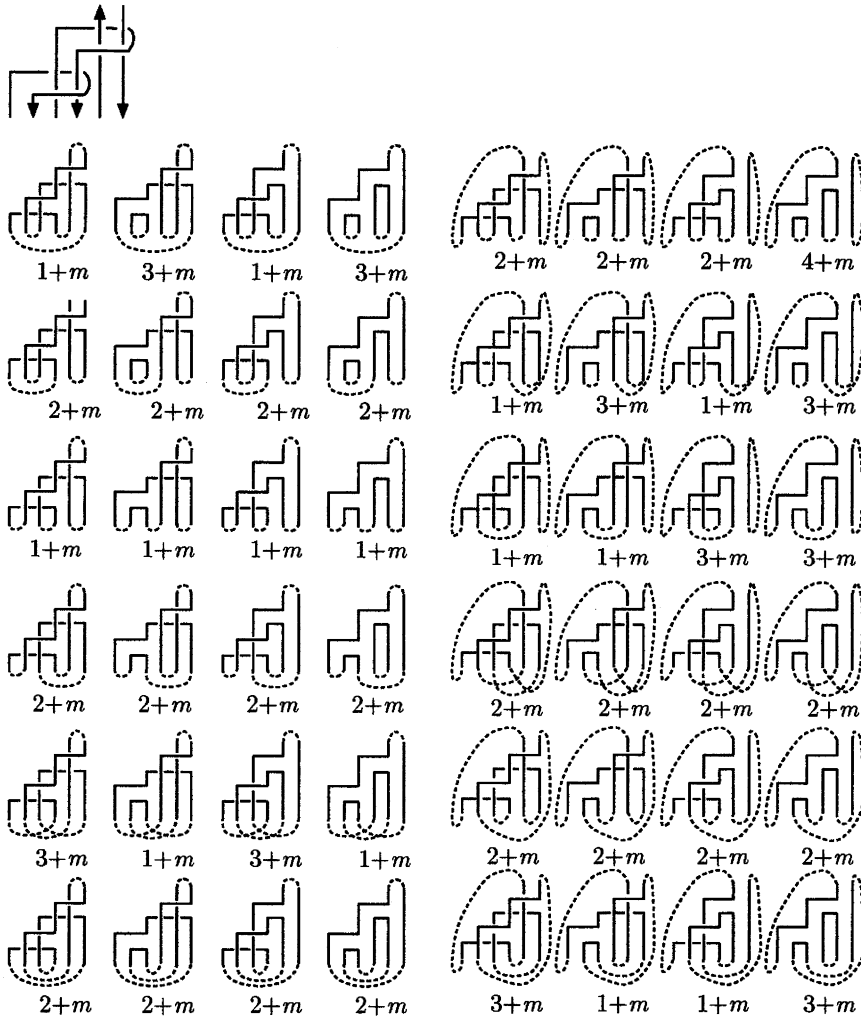


Table 1

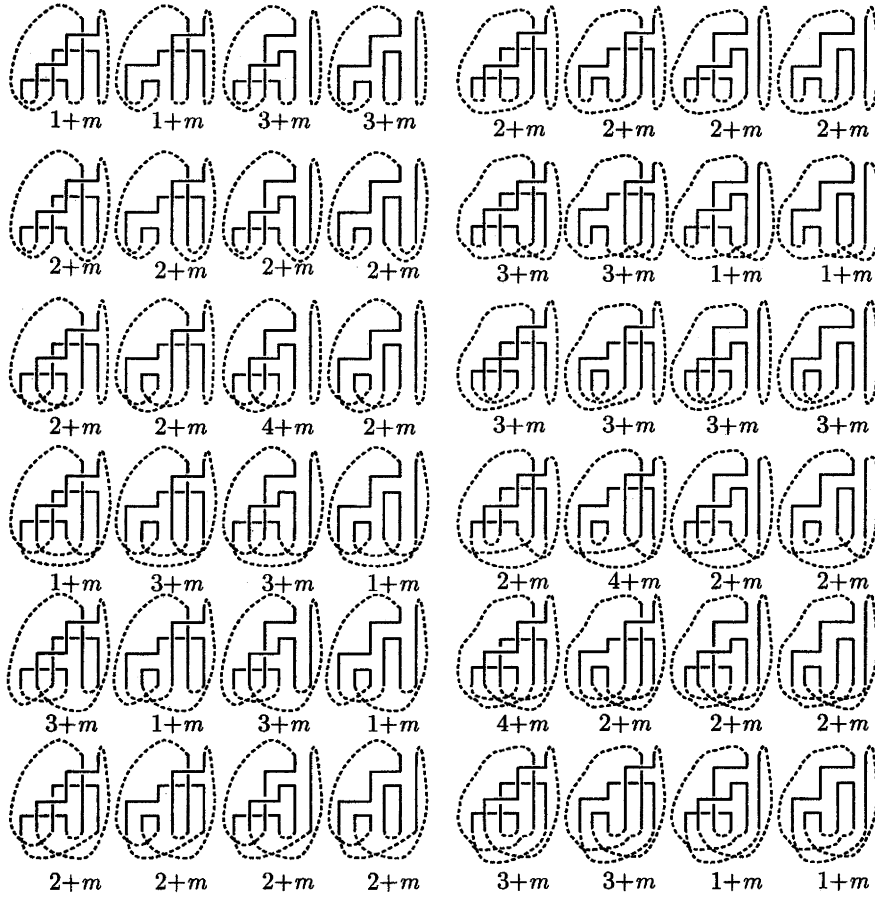


Table 1 (continued)

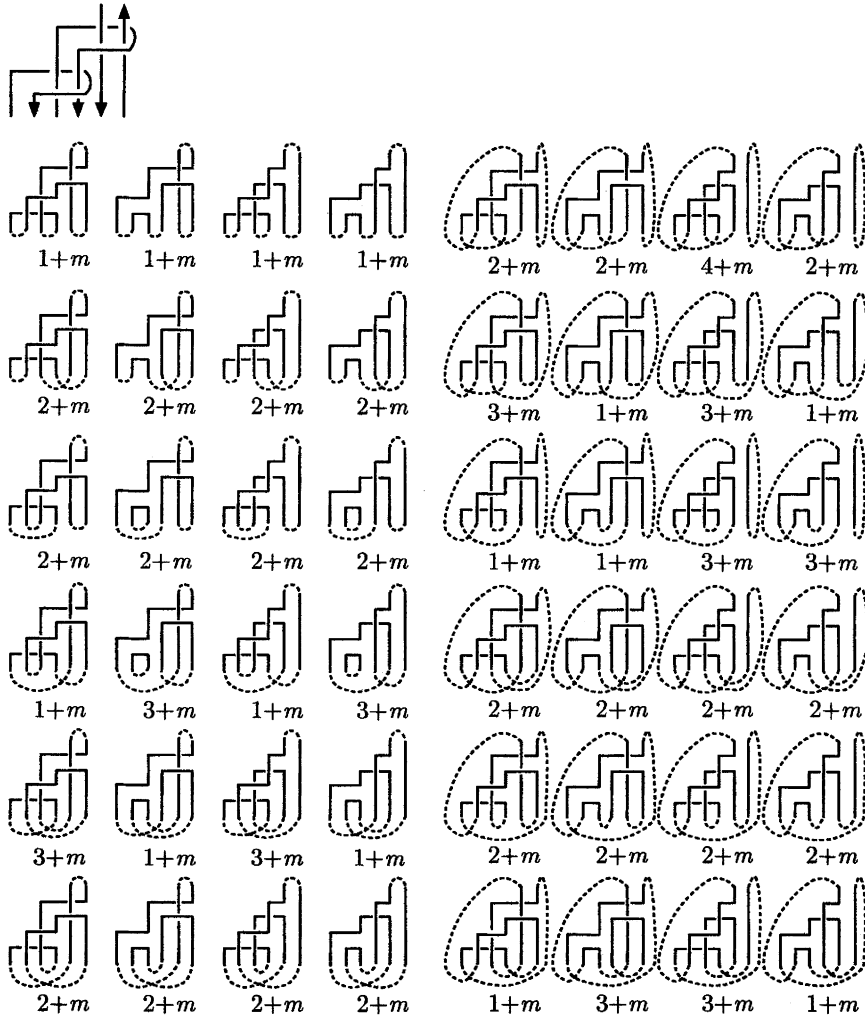


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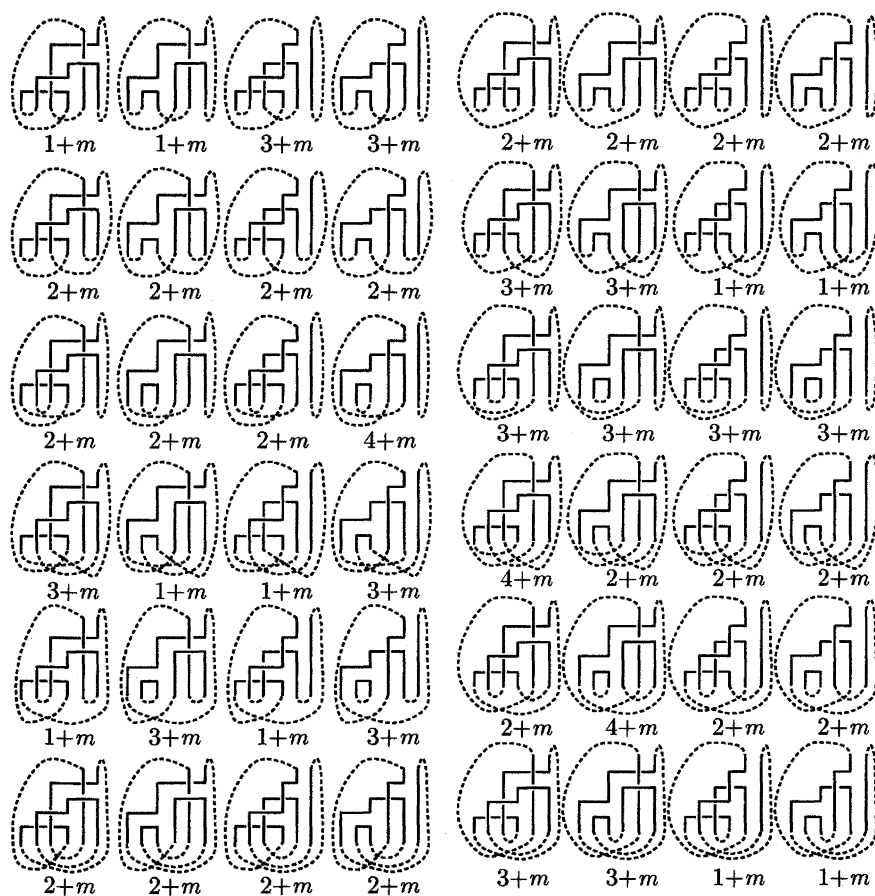
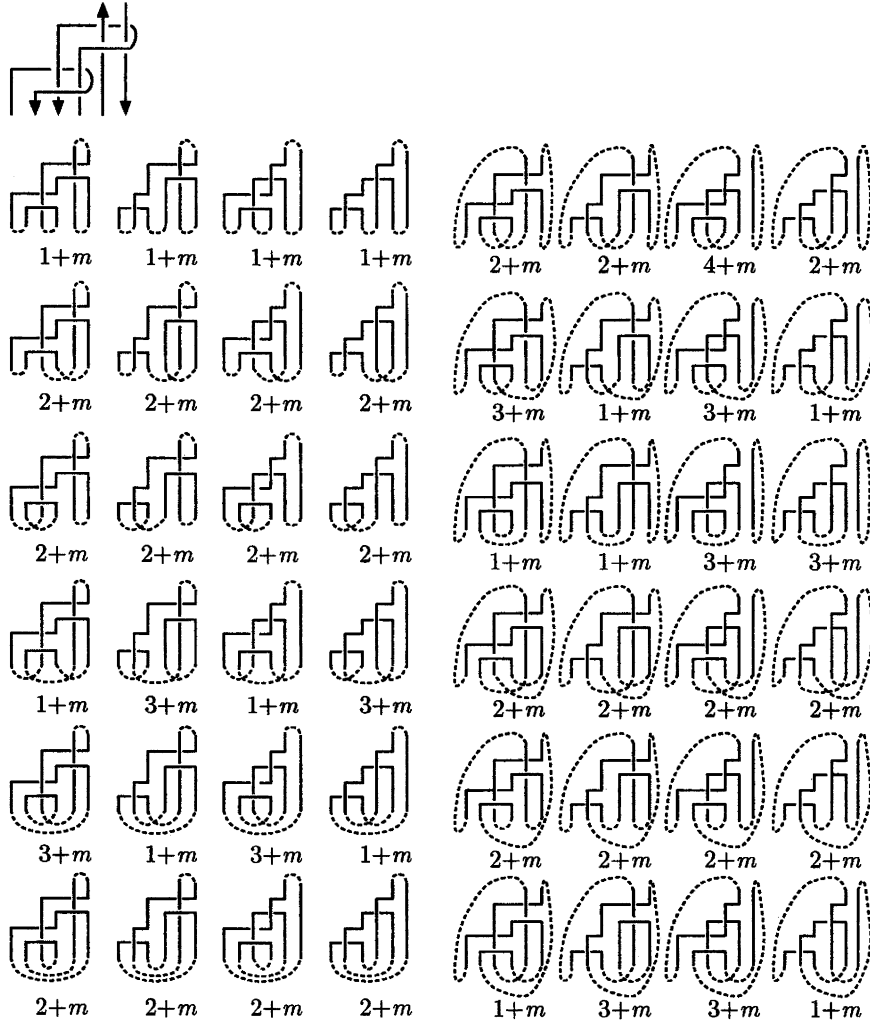


Table 1 (continued)



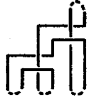
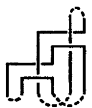
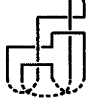
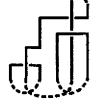
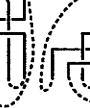
									
$1+m$	$1+m$	$1+m$	$1+m$	$2+m$	$2+m$	$4+m$	$2+m$		
									
$2+m$	$2+m$	$2+m$	$2+m$	$3+m$	$1+m$	$3+m$	$1+m$		
									
$2+m$	$2+m$	$2+m$	$2+m$	$1+m$	$1+m$	$3+m$	$3+m$		
									
$1+m$	$3+m$	$1+m$	$3+m$	$2+m$	$2+m$	$2+m$	$2+m$		
									
$3+m$	$1+m$	$3+m$	$1+m$	$2+m$	$2+m$	$2+m$	$2+m$		
									
$2+m$	$2+m$	$2+m$	$2+m$	$1+m$	$3+m$	$3+m$	$1+m$		

Table 1 (continued)

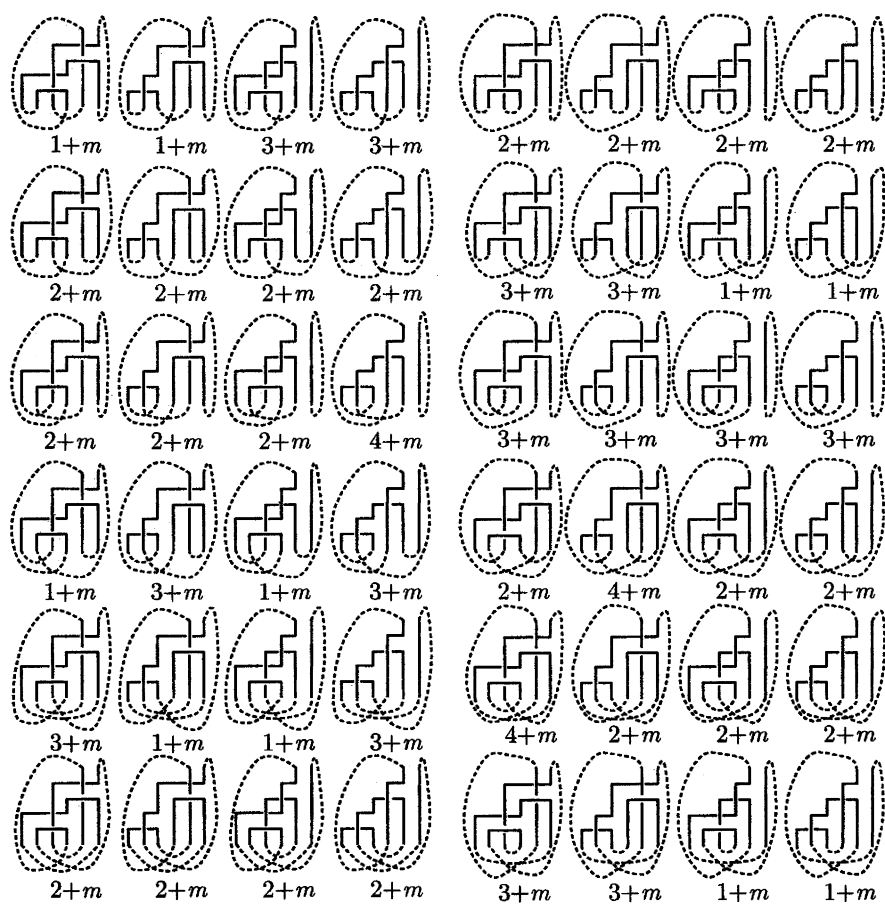


Table 1 (continued)

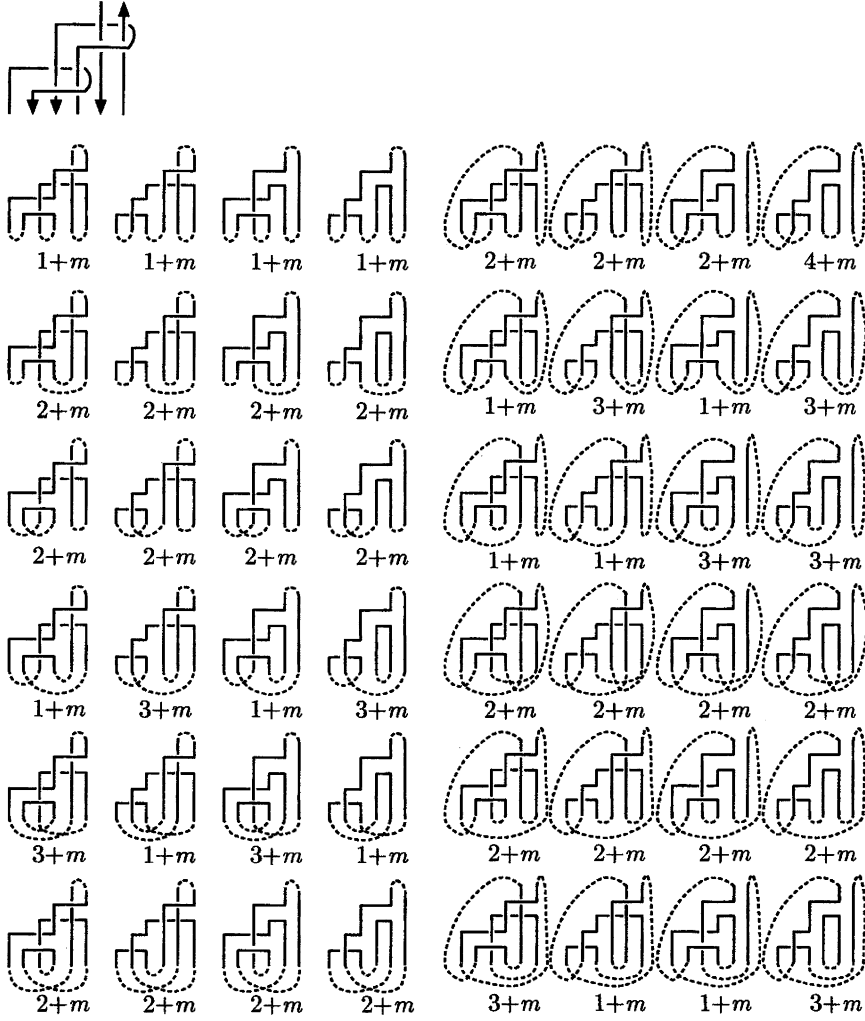


Table 1 (continued)

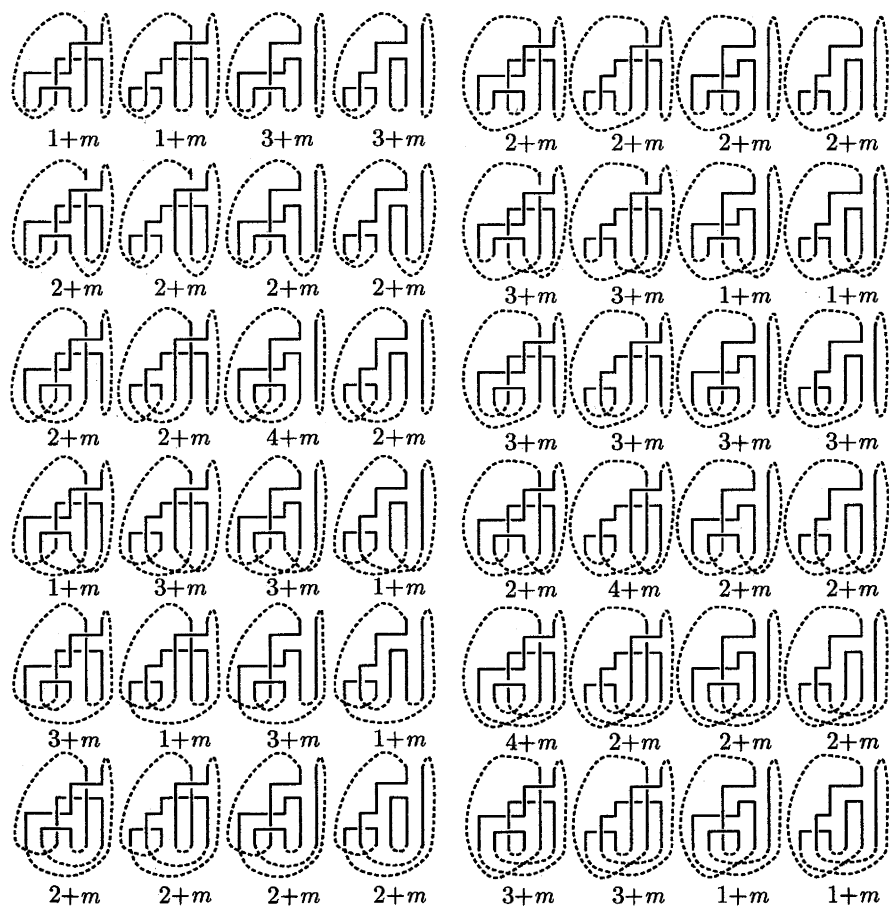


Table 1 (continued)

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