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PURE SPECTRUM OF A MONOID WITH ZERO

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1. Introduction.

In this paper S will denote a monoid with zero, that is, a semigroup with identity 1 and two-sided zero 0. Each S-system A is assumed to be right unitary (i.e. $A1 = A$) and centered (i.e. A has a fixed element θ such that $a0 = \theta x = \theta$ for all $a \in A$ and $x \in S$). A two-sided ideal I of S is called a pure ideal of S if, for each x in I, there is an element y in I such that $xy = x$. A pure ideal I is called purely prime if it is proper and if for any pure ideals I_1 and I_2: $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$.

Analogous to the construction of the pure spectrum of a ring developed in [1], we will construct the pure spectrum of S. For this purpose in section 2 we will establish some basic properties of pure and purely prime ideals of S. In section 4 we introduce the notion of quotient S-systems and investigate the properties of them. The notions of a presheaf and a sheaf of S-systems are introduced in section 5. We shall prove in sections 5 and 6 sheaf representation theorems for S.

For the definitions and notions used here we refer to [1] and [2].

2. Pure ideals of a monoid with zero.

A monoid with identity 1 is called a *monoid with zero* if it has an element 0, called *zero*, such that $0x = x0 = 0$ for all $x \in S$. In particular by (0) we denote the monoid with zero consisting of only one 0 such that $00 = 0$.

A mapping $f : S \to T$ of two monoids with zero is called a *homomorphism of monoids with zero*, if

(1) $f(xy) = f(x)f(y)$ for all $x, y \in S$,

(2) $f(1) = 1$ and $f(0) = 0$.

Now, let A, B be two subsets of S. Then we define a subset AB of S by

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

A non-empty subset J of S is a *(right) ideal* of S if $JS \subseteq J$. A right ideal J is *proper* if $J \neq S$. A non-empty subset I of S is a *two-sided ideal* if $IS \subseteq I$ and $SI \subseteq I$.

DEFINITION 2.1: Let I be a two-sided ideal. Then I is called *(right) pure*

if for any $x \in I$, there is an element $y \in I$ such that $xy = x$.

PROPOSITION 2.2. *Let I be a two-sided ideal of S. Then I is (right) pure if and only if*

$$J \cap I = JI \quad \text{for any (right) ideal } J \text{ of } S.$$

PROOF: Suppose I is a pure ideal of S. Since J is a right ideal of S, $JI \subseteq J$. On the other hand, I is a left ideal, so $JI \subseteq I$. Hence $JI \subseteq J \cap I$.

Now, let $x \in J \cap I$. Since I is pure, there exists an element y in I such that $xy = x$. Since $x \in J$ and $y \in I$, $xy \in JI$. This shows $x \in JI$ and $J \cap I \subseteq JI$. Hence $J \cap I = JI$.

Conversely, let us assume that $J \cap I = JI$ for any ideal J of S. We show that I is pure. Let x be any element of I. Since xS is a right ideal of S and I is a two-sided ideal, $xS \cap I = xSI \subseteq xI$. Since $x \in xS$ and $x \in I$, $x \in xS \cap I$. It follows $x \in xI$. Hence there exists y in I such that $x = xy$.

PROPOSITION 2.3.

(1) (0) *and S are pure ideals of S.*

(2) *Any union of pure ideals is a pure ideal.*

(3) *Any finite intersection of pure ideals is a pure ideal.*

PROOF: (1): (0) and S are obviously pure ideals of S.

(2): Let $(I_k)_{k \in K}$ be a family of pure ideals of S. Then $\bigcup_{k \in K} I_k$ is a two-sided ideal of S.

Suppose $x \in \bigcup_{k \in K} I_k$, then there exists $k \in K$ such that $x \in I_k$. Since I_k is pure, there exists $y \in I_k$ such that $x = xy$. It follows $y \in \bigcup_{k \in K} I_k$. Hence $\bigcup_{k \in K} I_k$ is a pure ideal of S.

(3): Let I_1 and I_2 be two pure ideals of S and $x \in I_1 \cap I_2$. Since $x \in I_1$ and I_1 is pure, there is an element $y_1 \in I_1$ such that $x = xy_1$. Similarly, there is an element $y_2 \in I_2$ such that $x = xy_2$. Thus we have $x = xy_2 = (xy_1)y_2 = x(y_1y_2)$. But $y_1y_2 \in I_1 \cap I_2$, so $I_1 \cap I_2$ is pure.

PROPOSITION 2.4. *Let I be any ideal of S. Then I contains a largest pure ideal. We call it the pure part of I and it is denoted by $P(I)$.*

PROOF: Let $P(I)$ be the union of all pure ideals contained in I. Such ideals exist, for example, (0). By the above proposition, $P(I)$ is a pure ideal. It is indeed the largest pure ideal contained in I.

PROPOSITION 2.5. *Let I, J be two ideals of S and $(I_k)_{k \in K}$ a family of ideals of S. Then*

(1) $P(I \cap J) = P(I) \cap P(J)$,

(2) $P(\bigcup_{k \in K} I_k) \supseteq \bigcup_{k \in K} P(I_k)$.

PROOF: (1): Since $P(I) \subseteq I$ and $P(J) \subseteq J$, $P(I) \cap P(J) \subseteq I \cap J$. But $P(I) \cap P(J)$ is pure by Proposition 2.3, so $P(I) \cap P(J) \subseteq P(I \cap J)$. On the other hand, $P(I \cap J) \subseteq I \cap J \subseteq I$ and $P(I \cap J)$ is pure, so $P(I \cap J) \subseteq P(I)$. Similarly, $P(I \cap J) \subseteq P(J)$. Hence $P(I \cap J) \subseteq P(I) \cap P(J)$. Thus $P(I \cap J) = P(I) \cap P(J)$.

(2): Since $P(I_k) \subseteq I_k$ and $P(I_k)$ is pure for all $k \in K$, $\bigcup_{k \in K} P(I_k) \subseteq \bigcup_{k \in K} I_k$ and $\bigcup_{k \in K} P(I_k)$ is pure. By the definition of P, we have $\bigcup_{k \in K} P(I_k) \subseteq P(\bigcup_{k \in K} I_k)$.

DEFINITION 2.6: Let I be a pure ideal of S. Then I is called *purely maximal* if I is maximal in the lattice of proper pure ideals.

DEFINITION 2.7: A proper pure ideal I of S is called *purely prime* if it satisfies the property that for any pure ideals I_1 and I_2 of S,

$$I_1 \cap I_2 \subseteq I \text{ implies } I_1 \subseteq I \text{ or } I_2 \subseteq I.$$

PROPOSITION 2.8. *Any purely maximal ideal is purely prime.*

PROOF: Suppose I is purely maximal and I_1, I_2 are pure ideals such that $I_1 \cap I_2 \subseteq I$. Now suppose $I_1 \not\subseteq I$. Then $I_1 \cup I$ is a pure ideal such that $I \subseteq I_1 \cup I$ and $I \neq I_1 \cup I \subseteq S$. Since I is maximal, $I_1 \cup I = S$. It follows that $I_2 = I_2 \cap S = I_2 \cap (I_1 \cup I) = (I_2 \cap I_1) \cup (I_2 \cap I) \subseteq I \cup I = I$. This shows I is purely prime.

PROPOSITION 2.9. *The pure part of any maximal ideal is purely prime.*

PROOF: Let M be a maximal ideal of S and $P(M)$ its pure part. Suppose $I_1 \cap I_2 \subseteq P(M)$ with I_1, I_2 pure. If $I_1 \subseteq M$, then $I_1 \subseteq P(M)$ and the result holds. If $I_1 \not\subseteq M$, then $I_1 \cup M = S$. Then we have $I_2 = I_2 \cap S = I_2 \cap (I_1 \cup M) = (I_2 \cap I_1) \cup (I_2 \cap M) \subseteq M \cup M = M$. Hence $I_2 \subseteq P(M)$, since I_2 is pure. This shows $P(M)$ is purely prime.

PROPOSITION 2.10. *Any proper pure ideal is contained in a maximal pure ideal.*

PROOF: Let I be a proper pure ideal of S. Consider the set, ordered by inclusion,

$$X = \{ J \mid J \text{ is a proper ideal and } I \subseteq J \}.$$

Then $X \neq \emptyset$ since $I \in X$. For any $J \in X$, $1 \notin J$ since J is proper.

Let $(J_k)_{k \in K}$ be any non-empty totally ordered subset of X. By Proposition 2.3 $\bigcup_{k \in K} J_k$ is a pure ideal such that $1 \notin \bigcup_{k \in K} J_k$ and $I \subseteq \bigcup_{k \in K} J_k$. Therefore $\bigcup_{k \in K} J_k \in X$. This shows that X is inductively ordered. Hence by Zorn's lemma, X contains a maximal element J such that $I \subseteq J$. Now let J' be any proper ideal such

that $J \subseteq J'$. Since $I \subseteq J \subseteq J'$, $J' \in X$. Since J is maximal in X, $J' = J$. This shows J is purely maximal.

PROPOSITION 2.11. *If I is a pure ideal of S and $a \notin I$, then there exists a purely prime ideal J such that $I \subseteq J$ and $a \notin J$.*

PROOF: Let us consider the set X ordered by inclusion:

$$X = \{J \mid J \text{ is a pure ideal, } I \subseteq J \text{ and } a \notin J\}.$$

Then $X \neq \emptyset$ as $I \in X$. Let $(J_k)_{k \in K}$ be any non-empty totally ordered subset of X. By Proposition 2.3, $\bigcup_{k \in K} J_k$ is a pure ideal such that $1 \notin \bigcup_{k \in K} J_k$, $I \subseteq J$ and $a \notin \bigcup_{k \in K} J_k$. This shows $\bigcup_{k \in K} J_k \in X$. Hence X is inductively ordered. Then by Zorn's lemma, X has a maximal element J such that J is pure, $I \subseteq J$ and $a \notin J$. We claim that J is purely prime. Suppose I_1 and I_2 are pure ideals such that $I_1 \not\subseteq J$ and $I_2 \not\subseteq J$. Since $I_k (k = 1, 2)$ and J are pure, $I_k \cup J$ is a pure ideal such that $J \subseteq I_k \cup J$. We then claim that $a \in I_k \cup J$ $(k = 1, 2)$. Because, if $a \notin I_k \cup J$, then by maximality of J, we have $I_k \cup J \subseteq J$. This contradicts the assumption $I_k \not\subseteq J$. Hence $a \in (I_1 \cup J) \cap (I_2 \cup J) = (I_1 \cap I_2) \cup J$. Since $a \notin J$, it follows that $a \in I_1 \cap I_2$ and so $I_1 \cap I_2 \not\subseteq J$. Hence by contraposition, we conclude that J is purely prime.

PROPOSITION 2.12. *Let I be a proper pure ideal of S. Then I is the intersection of the purely prime ideals of S containing I.*

PROOF: By Propositions 2.8 and 2.10, there are purely prime ideals containing a proper pure ideal I. Let $(J_k)_{k \in K}$ be the family consisting of all purely prime ideals containing I. Since $I \subseteq J_k$ for all $k \in K$, $I \subseteq \bigcap_{k \in K} J_k$. To show $I \supseteq \bigcap_{k \in K} J_k$ we show the complement:

$$I^C \subseteq \bigcup_{k \in K} J_k^C.$$

Let $a \notin I$. Then by Proposition 2.11, there exists a purely prime ideal J_k for some $k \in K$ such that $I \subseteq J_k$ and $a \notin J_k$. It follows $a \in J_k^C \subseteq \bigcup_{k \in K} J_k^C$. Hence $I^C \subseteq \bigcup_{k \in K} J_k^C$.

3. Pure spectrum of a monoid with zero.

In this section S will denote a monoid with zero.

DEFINITION 3.1.: We denote by $p(S)$ the lattice of pure ideals of S and by $pp(S)$ the set of purely prime ideals of S. For any pure ideal I of S, we define

$$\Theta_I = \{J \in pp(S) \mid I \not\subseteq J\} \quad \text{and} \quad \tau(pp(S)) = \{\Theta_I \mid I \in p(S)\}.$$

THEOREM 3.2. *The set* $\tau(pp(S))$ *forms a topology on the set* $pp(S)$. *Moreover the assignment*

$$I \mapsto \Theta_I$$

is an isomorphism between the lattice $p(S)$ *of pure ideals of* S *and the lattice of open subsets of* $pp(S)$.

PROOF: First, we show that $\tau(pp(S))$ forms a topology on the set $pp(S)$. Note that $\Theta_{(0)} = \{J \in pp(S) \mid (0) \not\subseteq J\} = \emptyset$. Thus $\Theta_{(0)}$ is the empty subset of $pp(S)$. On the other hand, $\Theta_S = \{J \in pp(S) \mid S \not\subseteq J\} = pp(S)$. This is true since purely prime ideals are proper. So $\Theta_S = pp(S)$ is an element of $\tau(pp(S))$.

Now let $\Theta_{I_1}, \Theta_{I_2} \in \tau(pp(S))$ with I_1, I_2 pure ideals of S. Then

$$\begin{aligned}\Theta_{I_1} \cap \Theta_{I_2} &= \{J \in pp(S) \mid I_1 \not\subseteq J \text{ and } I_2 \not\subseteq J\} \\ &= \{J \in pp(S) \mid I_1 \cap I_2 \not\subseteq J\} \\ &= \Theta_{I_1 \cap I_2}.\end{aligned}$$

This follows from the equivalence:

$$I_1 \cap I_2 \not\subseteq J \iff I_1 \not\subseteq J \text{ and } I_2 \not\subseteq J.$$

We prove the above equivalence. Since J is purely prime,

$$I_1 \cap I_2 \subseteq J \Rightarrow I_1 \subseteq J \text{ or } I_2 \subseteq J.$$

Hence $I_1 \not\subseteq J$ and $I_2 \not\subseteq J$ implies $I_1 \cap I_2 \not\subseteq J$. Conversely if $I_1 \subseteq J$ or $I_2 \subseteq J$ then $I_1 \cap I_2 \subseteq J$. This shows $I_1 \cap I_2 \not\subseteq J$ implies $I_1 \not\subseteq J$ and $I_2 \not\subseteq J$. Thus we have $\Theta_{I_1} \cap \Theta_{I_2} = \Theta_{I_1 \cap I_2}$.

Next, let us consider any family $(I_k)_{k \in K}$ of pure ideals. Since

$$\begin{aligned}\bigcup_{k \in K} \Theta_{I_k} &= \bigcup_{k \in K} \{J \in pp(S) \mid I_k \not\subseteq J\} \\ &= \{J \in pp(S) \mid \exists k \in K \text{ so that } I_k \not\subseteq J\} \\ &= \{J \in pp(S) \mid \bigcup_{k \in K} I_k \not\subseteq J\} \\ &= \Theta_{\bigcup_{k \in K} I_K}\end{aligned}$$

and since, by Proposition 2.3, $\bigcup_{k \in K} I_k$ is pure, it follows that $\bigcup_{k \in K} \Theta_{I_k}$ is contained in $\tau(pp(S))$. Thus the set $\tau(pp(S))$ of subsets Θ_I with I pure constitutes a topology on the set $pp(S)$.

Let $\varphi : p(S) \to \tau(pp(S))$ be the mapping defined by the assignment $I \mapsto \Theta_I$. Then it follows from the above that the prescription $\varphi(I) = \Theta_I$ preserves finite $\cap$ and arbitrary $\cup$. Thus φ is a homomorphism of lattices.

To conclude the proof we must show that φ is bijective. In fact, we need to

prove the equivalence:

$$I_1 = I_2 \Leftrightarrow \Theta_{I_1} = \Theta_{I_2} \text{ for any pure ideals } I_1 \text{ and } I_2.$$

Suppose $\Theta_{I_1} = \Theta_{I_2}$. If $I_1 \neq I_2$, then we may assume that there exists an element $x \in I_1$ and $x \notin I_2$. Then, by Proposition 2.11, there exists a purely prime ideal J such that $I_2 \subseteq J$ and $x \notin J$. Hence $I_1 \not\subseteq J$ and so $J \in \Theta_{I_1}$. By the assumption $\Theta_{I_1} = \Theta_{I_2}$, we have $J \in \Theta_{I_2}$. This means that $I_2 \not\subseteq J$ which contradicts to $I_2 \subseteq J$. Thus we have $I_1 = I_2$.

DEFINITION 3.3: The *pure spectrum* of S is the set $pp(S)$ of purely prime ideals provided with the topology $\tau(pp(S))$ given in the above theorem. We denote the toplogical space by $Spp(S)$.

PROPOSITION 3.4.

(1) *$Spp(S)$ is a compact space.*

(2) *Θ_I is open and closed in $Spp(S)$ if and only if $\Theta_I = \emptyset$ or $pp(S)$.*

PROOF: (1): Let $\bigcup_{k\in K} \Theta_{I_k} = pp(S)$ be any open covering of $pp(S)$. Then $\bigcup_{k\in K} I_k = S$. Since $1 \in S = \bigcup_{k\in K} I_k$, there exists I_k containing 1 for some $k \in K$. This shows $I_k = S$ and so $pp(S) = \Theta_{I_k}$ (finite covering). Hence $Spp(S)$ is compact.

(2): Let Θ_I, with I pure, be an element of $\tau(pp(S))$ which is open and closed. Then there exists Θ_J, with J pure, such that $\Theta_I \cup \Theta_J = pp(S)$ and $\Theta_I \cap \Theta_J = \emptyset$. Since $1 \in S = I \cup J$, it follows that

$$I = S \text{ and } J = (0) \quad \text{or} \quad I = (0) \text{ and } J = S.$$

Thus $\Theta_I = \emptyset$ or $\Theta_I = pp(S)$.

4. S-systems.

In this section S will denote a fixed monoid with zero.

DEFINITION 4.1: An *S-system* A, denoted by A_S, is a set A with a function $A \times S \to A$ such that if ax denotes the image of (a, x) for $a \in A$ and $x \in S$, then the following conditions hold:

(1) $(ax)y = a(xy)$ for $a \in A$ and $x, y \in S$.

(2) A contains a fixed element θ, called *zero* of A, such that $\theta x = \theta$ for all $x \in S$.

(3) For all $a \in A$, $a0 = \theta$ and $a1 = a$.

If there is no confusion, we will denote simply by A.

A *sub S-system* B of A_S, denoted by $B_S \subseteq A_S$, is a subset of A such that $bx \in B$ for all $b \in B$ and $x \in S$. For two S-systems A_S and B_S, a mapping f:

$A_S \to B_S$ is an *S-homomorphism* if

$$f(ax) = f(a)x \quad \text{for all } a \in A_S \text{ and } x \in S.$$

We mean by $Hom_S(A_S, B_S)$ the set of all S-homomorphisms from A_S to B_S and by $End_S(A_S)$ the set of all S-endomorphisms of A_S. We shall denote by $\mathcal{M}_S$ the category of all S-systems and S-homomorphisms.

Now, we note that the given monoid S with zero itself and every right ideal I of S are also S-systems.

DEFINITION 4.2: Let S be a given monoid with zero and T a monoid with zero. Then T is called an *S-monoid with zero* if T has a structure of S-system such that $(xy)s = x(ys)$ for $x, y \in T$ and $s \in S$. For two such S-monoids T_1 and T_2 with zero, a homomorphism of monoids with zero $f: T_1 \to T_2$ is called a *homomorphism of S-monoids with zero* if f is an S-homomorphism.

Let S be a monoid with zero and T an S-monoid with zero. Then an S-system A is called a *T-S-system* if A is a T-system such that

$$(ax)s = a(xs) \quad \text{for all } a \in A,\ x \in T \text{ and } s \in S.$$

PROPOSITION 4.3. *Let S be a monoid with zero and A an S-system. Then the followings hold*:

(1) *$End_S(A)$ is a monoid with zero.*
(2) *For each right ideal I of S, I is an S-system.*
(3) *For each pure ideal I of S, $End_S(I)$ is an S-monoid with zero and $Hom_S(I, A)$ is an $End_S(I)$-S-system.*
(4) *Moreover if S is commutative, then $End_S(I)$ is also commutative for a pure ideal I of S.*

PROOF: (1): Let f, g be elements of $End_S(A)$. If we define $f \circ g$ by the composition: $f \circ g(a) = f(g(a))$ for all $a \in A$ and elements $\mathbf{1}, \mathbf{0}$ in $End_S(A)$ by $\mathbf{1}(a) = a$, $\mathbf{0}(a) = \theta$ for all $a \in A$, then $End_S(A)$ is a monoid with zero.

(2): Since $IS \subseteq I$ and $0 \in I$, I is an S-system.

(3): Let I be a pure ideal of S. For each $f \in Hom_S(I, A)$ and $s \in S$, since I is a left ideal, we may define fs by

$$(fs)(x) = f(sx) \quad \text{for all } x \in I.$$

For the action, the element $\mathbf{0}$, defined by $\mathbf{0}(x) = \theta$ for all $x \in I$, is a zero element of $Hom_S(I, A)$. Thus $Hom_S(I, A)$ is an S-system. In particular, together with above (1) and (2), $End_S(I)$ is an S-monoid with zero.

The S-system $Hom_S(I, A)$ can be made into an $End_S(I)$-S-system with the action defined by the composition $f \circ \varphi$ for $f \in Hom_S(I, A)$ and $\varphi \in End_S(I)$.

(4): Let now S be commutative and f, g elements in $End_S(I)$. If x is an

element of I then we can find $y \in I$ such that $x = xy$. Since S is commutative, $g(xy) = xg(y)$. Then $(f \circ g)(x) = f(g(x)) = f(g(xy)) = f(xg(y)) = f(x)g(y) = g(y)f(x) = g(yf(x)) = g(f(x)y) = g(f(xy)) = (g \circ f)(xy) = (g \circ f)(x)$. Thus $End_S(I)$ is commutative if S is commutative.

DEFINITION 4.4: Let A be an S-system and B a sub S-system of A. Now, we define a relation $\sim$ on A by

$$a \sim a' \Leftrightarrow a = a' \quad \text{or} \quad a, a' \in B.$$

Then $\sim$ is an equivalence relation on A and it is compatible with the action of S, that is,

$$a \sim a',\ x \in S \Rightarrow ax \sim a'x.$$

We denote by $[a]$ the equivalence class of $a \in A$. Thus $A/\sim$ is an S-system if we define the action of S by

$$[a]x = [ax] \quad \text{for } a \in A \text{ and } x \in S,$$

which is denoted by A/B, called the *S-quotient* of A by B. In particular, if I is a right ideal of S, then I is a sub S-system and so S/I is an S-system. Moreover, if I is a two-sided ideal of S, then the equivalence relation $\sim$ is a congruence on S. This is just the *Rees congruence* on S and S/I is the *Rees quotient* of S by a two-sided ideal I. If we define the multiplication by $[x][y] = [xy]$ for all $x, y \in S$, then S/I is a monoid with zero such that $([x][y])s = [x]([y]s)$ for $x, y, s \in S$.

Thus we have

PROPOSITION 4.5. *Let A be an S-system and B a sub S-system of A. Then the followings hold:*

(1) *A/B is an S-system.*
(2) *For every two-sided ideal I of S, the Rees quotient S/I is an S-monoid with zero.*

Let A be an S-system and I a right ideal of S. Consider the subset of A:

$$AI = \{ax \mid a \in A \text{ and } x \in I\}.$$

Then AI is a sub S-system of A.

PROPOSITION 4.6. *Let A be an S-system and I a pure ideal of S. Then*

$$AI = \{a \in A \mid a = ax \quad \textit{for some } x \in I\}.$$

PROOF: Clearly each $a \in A$ such that $a = ax$ for some $x \in I$ is in AI. Conversely, suppose $a = a'x' \in AI$. Since I is a pure ideal, we can find $y' \in I$ such that $x' = x'y'$. Thus $a = a'x' = a'x'y' = ay'$.

PROPOSITION 4.7. *Let I be a pure ideal of S. If A and B are two S-systems, then any S-homomorphism $AI \to B$ factors through BI.*

PROOF: Let $f: AI \to B$ be an S-homomorphism. If $a \in AI$, then by the above proposition, there is an $x \in I$ such that $a = ax$. Therefore, $f(a) = f(ax) = f(a)x \in BI$.

PROPOSITION 4.8. *Let I be a pure ideal of S and A an S-system. Then, for a family $(A_k)_{k\in K}$ of sub S-systems of A and sub S-systems B, C of A, we have*

(1) $AI \cap (\bigcup_{k\in K} A_k) = \bigcup_{k\in K} (AI \cap A_k)$,
(2) $B \cap (AI \cup C) = (B \cap AI) \cup (B \cap C)$,
(3) $AI \cup (B \cap C) = (AI \cup B) \cap (AI \cup C)$,
(4) $B \cup (AI \cap C) = (B \cup AI) \cap (B \cup C)$.

PROOF: This follows from the distributive properties of union and intersection of sets.

DEFINITION 4.9: For a pure ideal I of S and an S-system A, we define the set

$$CAI = \{a \in A \mid ax = \theta \quad \text{for all } x \in I\}.$$

Then CAI is a sub S-system of A.

PROPOSITION 4.10. *CAI is the largest sub S-system of A whose intersection with AI is zero.*

PROOF: Consider the sub S-system $K = \bigcup_{AI\cap T=(\theta)} T$ where all T are sub S-systems of A. Then by the above proposition,

$$AI \cap K = AI \cap \Big(\bigcup_{AI\cap T=(\theta)} T\Big) = \bigcup_{AI\cap T=(\theta)} (AI \cap T) = (\theta).$$

Thus K is the largest sub S-system of A whose intersection with AI is (θ).

We show that $K = CAI$. Let $k \in K$ and $x \in I$. Then $kx \in K \cap AI = (\theta)$; so $kx = \theta$ and $k \in CAI$.

Now suppose $a \in CAI$. We will show that $aS \cap AI = (\theta)$. If $x \in aS \cap AI$ then by Proposition 4.6, there exists an element $y \in I$ such that $xy = x$. On the other hand, since $x \in aS$, there exists an element $z \in S$ such that $x = az$. Hence $x = xy = azy$ with $zy \in I$. But then $x = \theta$ since $a \in CAI$. Hence $aS \subseteq K$ and this implies that $a \in K$. Thus $K = CAI$.

PROPOSITION 4.11. *Let I, J be two pure ideals of S, $(I_k)_{k\in K}$ a family of pure ideals of S and A an S-system. Then*

(1) $ACI \subseteq CAI$,
(2) $I \subseteq J \Rightarrow CAI \supseteq CAJ$,
(3) $CA(\bigcup_{k\in K} I_k) = \bigcap_{k\in K} CAI_k$,
(4) $CA(I \cap J) \supseteq CAI \cup CAJ$.

PROOF: (1): Note that $CI = CSI = \{r \in S \mid rx = 0 \text{ for all } x \in I\}$ and $ACI = \{ar \mid a \in A \text{ and } r \in CI\}$.

Let $ar \in ACI$. Then $(ar)x = a(rx) = a0 = 0$ for all $x \in I$. This shows $ar \in CAI$. Hence $ACI \subseteq CAI$.

(2): This is obvious.

(3):

$$\begin{aligned} CA(\bigcup_{k\in K} I_k) &= \{a \in A \mid ax = \theta \text{ for all } x \in \bigcup_{k\in K} I_k\} \\ &= \{a \in A \mid ax = \theta \text{ for } \forall k \in K \text{ and } \forall x \in I_k\} \\ &= \bigcap_{k\in K} \{a \in A \mid ax = \theta \text{ for } \forall x \in I_k\} \\ &= \bigcap_{k\in K} CAI_k. \end{aligned}$$

(4): The second relation implies $CA(I \cap J) \supseteq CAI$ and $CA(I \cap J) \supseteq CAJ$. This implies $CA(I \cap J) \supseteq CAI \cup CAJ$. This completes the proof.

Note that for a pure ideal I of S, $I = SI$ is an S-system and $CI = \{r \in S \mid rx = 0$ for all $x \in I\}$.

PROPOSITION 4.12. *Let I be a pure ideal of S. Then CI is a two-sided ideal of S and the left annihilator of I.*

PROOF: By definition

$$L - AnnI = \{r \in S \mid rx = 0 \text{ for all } x \in I\} = CI.$$

To show that CI is a two-sided ideal of S, let $r \in CI$ and $s \in S$. For every $x \in I$, we have $sx \in I$, $xs \in I$ and $(rs)x = r(sx) = 0$, $(sr)x = s(rx) = s0 = 0$. This shows $rs, sr \in CI$. Hence CI is a two-sided ideal of S.

5. First representation theorem.

DEFINITION 5.1.: Let X be a topological space and $\tau(X)$ the category of open sets of X and inclusions.

A *presheaf* $\mathcal{P}$ of S-systems on X is a contravariant functor from the category $\tau(X)$ to the category $\mathcal{M}_S$ of S-systems, that is, it consists of the data:

(a) for every open subset $U \subseteq X$, an S-system $\mathcal{P}(U)$, and
(b) for every inclusion $V \subseteq U$ of open subsets, an S-homomorphism

$$\rho_{UV}^{\mathcal{P}}: \mathcal{P}(U) \to \mathcal{P}(V)$$

subjects to the following conditions:

(1) $P(\emptyset) = (\theta)$, where $\emptyset$ is the empty set,

(2) $\rho_{UU}^{\mathcal{P}}: \mathcal{P}(U) \to \mathcal{P}(U)$ is the identity map, and

(3) if $W \subseteq V \subseteq U$ are three open subsets, then

$$\rho_{UW}^{\mathcal{P}} = \rho_{VW}^{\mathcal{P}} \circ \rho_{UV}^{\mathcal{P}}.$$

If $\mathcal{P}$ is a presheaf on X, $\mathcal{P}(U)$ is called the *section* of the presheaf $\mathcal{P}$ on the open set U and the map $\rho_{UV}^{\mathcal{P}}$ are called *restriction maps*, and often the notation $\alpha|_V$ is used instead of $\rho_{UV}^{\mathcal{P}}(\alpha)$ if $\alpha \in \mathcal{P}(U)$.

A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* if the following additional conditions are satisfied:

(4) if U is an open set, if $(V_k)_{k\in K}$ is an open covering of U, and if $\alpha|_{V_k} = \beta|_{V_k}$ for $\alpha, \beta \in \mathcal{F}(U)$ and all V_k, then $\alpha = \beta$.

(5) if U is an open set, if $(V_k)_{k\in K}$ is an open covering of U, and if we have elements $\alpha_k \in \mathcal{F}(V_k)$ for each $k \in K$, with properties that for each k, $l \in K$, $\alpha_k|_{V_k \cap V_l} = \alpha_l|_{V_k \cap V_l}$, then there is an element $\alpha \in \mathcal{F}(U)$ such that $\alpha|_{V_k} = \alpha_k$ for each $k \in K$.

If a presheaf $\mathcal{P}$ satisfies the above condition (4), $\mathcal{P}$ is said to *be separated*.

Let $\mathcal{P}$ be a presheaf (resp. sheaf) of S-systems on X. If each $\mathcal{P}(U)$ is an S-monoid with zero and the $\rho_{UV}^{\mathcal{P}}$ are homomorphism of S-monoids with zero, then $\mathcal{P}$ is called a *presheaf (resp. sheaf) of S-monoids with zero*.

Now, let $\mathcal{P}$ be a presheaf (resp. sheaf) of S-monoids with zero on X and suppose that Q is a presheaf (resp. sheaf) of S-systems on X such that each $Q(U)$ is a $\mathcal{P}(U)$-S-system and the maps ρ_{UV}^{Q}: $Q(U) \to Q(V)$ are *system homomorphisms*, that is, if $\alpha \in Q(U)$, $\beta \in \mathcal{P}(U)$ then

$$\rho_{UV}^{Q}(\alpha\beta) = \rho_{UV}^{Q}(\alpha)\rho_{UV}^{\mathcal{P}}(\beta).$$

Then Q is said to be a *presheaf (resp. sheaf) over the presheaf (resp. sheaf)* $\mathcal{P}$.

We now describe a sheaf $\mathcal{F}_S$ of S-monoids with zero on the pure spectrum $Spp(S)$ of S in such a way that S is isomorphic to the monoid with zero of global sections of this sheaf.

THEOREM 5.2. *Let S be a monoid with zero. For every pure ideal I of S, the assignment*

$$\Theta_I \mapsto End_S(I) = \mathcal{F}_S(I)$$

defines a sheaf $\mathcal{F}_S$ of S-monoids with zero on $Spp(S)$ whose monoid with zero of global sections is isomorphic to S.

If S is commutative, then $\mathcal{F}_S$ is a sheaf of commutative S-monoids with zero.

PROOF: First we prepare the data for the existence of a presheaf. By Proposition 4.3, $\mathcal{F}_S(I) = End_S(I)$ is an S-monoid with zero for every pure ideal I of S. We need to define a restriction map $\rho_{IJ}^{\mathcal{F}}$: $End_S(I) \to End_S(J)$ when $\Theta_J \subseteq \Theta_I$, that is, when $J \subseteq I$. By Proposition 4.7, this is just the usual restriction of an S-endomorphism $f: I \to I$ to the sub S-system J: $\rho_{IJ}^{\mathcal{F}}(f) = f|_J$. By definition, $\rho_{IJ}^{\mathcal{F}}$ is a homomorphism of S-monoids with zero and $\mathcal{F}_S$ satisfies the conditions for a presheaf. Thus we have described the presheaf $\mathcal{F}_S$.

In order to show that the presheaf $\mathcal{F}_S$ is a sheaf, we need to check the conditions (4) and (5) for a sheaf.

Let $I = \bigcup_{k\in K} I_k$ in $p(S)$ and let $f, g \in \mathcal{F}_S(I)$ such that $f|_{I_k} = g|_{I_k}$ for all $k \in K$. Then for each $x \in I$, we have $x \in I_k$ for some $k \in K$. Thus $f(x) = f|_{I_k}(x) = g|_{I_k}(x) = g(x)$. Hence $f = g$ and $\mathcal{F}_S$ is separated.

To check the condition (5), consider $I = \bigcup_{k\in K} I_k$ in $p(S)$ and a family $(f_k)_{k\in K}$ with $f_k \in End_S(I_k)$ such that $f_k|_{I_k\cap I_l} = f_l|_{I_k\cap I_l}$ for all $k, l \in K$. For S-endomorphisms $f_k: I_k \to I_k$ and $f_l: I_l \to I_l$ which coincide on $I_k \cap I_l$, we define a map $f: I_k \cup I_l \to I_k \cup I_l$ by

$$f(x) = \begin{cases} f_k(x) & \text{if } x \in I_k \\ f_l(x) & \text{if } x \in I_l. \end{cases}$$

Since f_k and f_l coincide on $I_k \cap I_l$, f is an S-homomorphism extension of f_k and f_l.

Now if I_m is any pure ideal in the family, then $I_m \cap (I_k \cup I_l) = (I_m \cap I_k) \cup (I_m \cap I_l)$ by Proposition 4.8. Thus if $x \in I_m \cap (I_k \cup I_l)$, then $x \in (I_m \cap I_k)$ or $x \in (I_m \cap I_l)$. Hence

$$f(x) = \begin{cases} f_k(x) = f_m(x) & \text{or} \\ f_l(x) = f_m(x). \end{cases}$$

Thus it follows that $f(x) = f_m(x)$ for $x \in I_m \cap (I_k \cup I_l)$. This implies that the family $(I_k)_{k\in K}$ is stable with respect to finite unions.

Now if $x \in \bigcup_{k\in K} I_k$, then $x \in I_k$ for some k in K. So we may define, with no ambiguity, the map f: $\bigcup_{k\in K} I_k \to \bigcup_{k\in K} I_k$ by $f(x) = f_k(x)$, since two different f_l agree on x when $f_l(x)$ make sense. The map f is clearly an S-homomorphism extending each f_k for all $k \in K$. This proves that $\mathcal{F}_S$ is a sheaf.

Next we show $\mathcal{F}_S(S) = End_S(S) \cong S$. The isomorphism may be defined as follows.

Define $h: End_S(S) \to S$ by $h(\alpha) = \alpha(1)$, where $\alpha \in End_S(S)$. It is easy to show that h is a homomorphism of monoids with zero.

Suppose $h(\alpha) = h(\beta)$. Then $\alpha(1) = \beta(1)$. Now for all $s \in S$, we have $\alpha(s) = \alpha(1s) = \alpha(1)s = \beta(1)s = \beta(1s) = \beta(s)$. Hence $\alpha = \beta$. Therefore h is injective.

To show that h is surjective, let $t \in S$ and define $\alpha_t : S \to S$ by $\alpha_t(s) = ts$ for all $s \in S$. Then α_t is clearly an S-homomorphism. Hence $\alpha_t \in End_S(S)$ and $h(\alpha_t) = \alpha_t(1) = t1 = t$. Thus h is also surjective.

Finally, if S is commutative, then $End_S(I)$ is a commutative S-monoid with zero by Proposition 4.3. This completes the proof.

We prove an analogous representation theorem for an S-system A.

THEOREM 5.3. *Let A be an S-system. For every pure ideal I of S, the assignment*

$$\Theta_I \mapsto Hom_S(I, A) = \mathcal{F}_A(I)$$

defines a sheaf $\mathcal{F}_A$ of S-systems over the sheaf $\mathcal{F}_S$ of S-monoids with zero on $Spp(S)$ defined by Theorem 5.2.

A is isomorphic to the S-system of global sections of $\mathcal{F}_A$.

PROOF: First we note that the set $\mathcal{F}_A(I) = Hom_S(I, A)$ is an $End_S(I)$-S-system for each pure ideal I by Proposition 4.3.

Suppose J is a pure sub-ideal of I, then any S-homomorphism $f: I \to A$ restricts to an S-homomorphism $f|_J : J \to A$. This produces the necessary restriction map $\rho_{IJ}^A : Hom_S(I, A) \to Hom_S(J, A) : \rho_{IJ}^A(f) = f|_J$. By construction, $\mathcal{F}_A(I)$ and ρ_{IJ}^A define a presheaf of S-systems on $Spp(S)$. Moreover, if $\varphi \in End_S(I) = \mathcal{F}_S(I)$, then $(f \circ \varphi)|_J = f|_J \circ \varphi|_J$ which implies that the restriction map is a system homomorphism.

By methods similar to those of the above theorem, we may verify that $\mathcal{F}_A$ is in fact a sheaf of S-systems on $Spp(S)$. Thus $\mathcal{F}_A$ is a sheaf over $\mathcal{F}_S$.

Finally, we show that $\mathcal{F}_A(S) = Hom_S(S, A)$ is isomorphic to A. Define h: $Hom_S(S, A) \to A$ by $h(\alpha) = \alpha(1)$; h is an S-homomorphism since $h(\alpha r) = \alpha r(1) = \alpha(r) = \alpha(1)r = h(\alpha)r$ for $r \in S$.

Suppose $h(\alpha) = h(\beta)$ for $\alpha, \beta \in Hom_S(S, A)$. Then $\alpha(1) = \beta(1)$. For $s \in S$, $\alpha(s) = \alpha(1s) = \alpha(1)s = \beta(1)s = \beta(s)$. This implies that $\alpha = \beta$.

In order to show that h is surjective, suppose $a \in A$. Then the map $\alpha_a : S \to A$ given by

$$\alpha_a(s) = as \quad \text{for } s \in S$$

is an S-homomorphism from S to A and $h(\alpha_a) = \alpha_a(1) = a1 = a$. Hence h is a surjection. Thus the S-system $\mathcal{F}_A(A) = Hom_S(S, A)$ of global sections of $\mathcal{F}_A$ is isomorphic to A.

6. Second respresentation theorem.

Let S be a monoid with zero and I a pure ideal of S. Then by Proposition 4.12

$$CI = \{ r \in S \mid rx = 0 \quad \text{for all } x \in I \}$$

is a two-sided ideal of S and so S/CI is a monoid with zero. The congruence $\sim_I$ on S is defined by

$$r \sim_I s \iff r = s \quad \text{or } r, s \in CI.$$

We denote the equivalence class of $r \in S$ by $[r]_I$ or simply $[r]$. Define the action of S on S/CI by

$$[r]s = [rs] \quad \text{for} \quad [r] \in S/CI \text{ and } s \in S.$$

Then S/CI is a right S-system and so S/CI is an S-monoid with zero.

THEOREM 6.1. *Let S be a monoid with zero. For any pure ideal I, the assignment*

$$\Theta_I \mapsto S/CI = \mathcal{P}_S(I)$$

defines a separated presheaf $\mathcal{P}_S$ of S-monoids with zero of $Spp(S)$.

PROOF: If J is a pure sub-ideal of I, then we may define a mapping $\rho^{\mathcal{P}}_{IJ}: S/CI \to S/CJ$ since $CI \subseteq CJ$ by Proposition 4.11. If $[r] \in S/CI$ then we denote the image $\rho^{\mathcal{P}}_{IJ}([r])$ by $[r]|_J \in S/CJ$. Since, by definitions, $[rs]_J = [r]_J[s]_J = [r]_J s$, the map $\rho^{\mathcal{P}}_{IJ}$ is an S-homomorphism and a homomorphism of monoids with zero. Thus we have a presheaf $\mathcal{P}_S$ of S-monoids with zero on $Spp(S)$.

In order to show that $\mathcal{P}_S$ is separated, let $I = \bigcup_{k\in K} I_k \in p(S)$ and $[r], [s] \in S/CI$ such that $[r]|_{I_k} = [s]|_{I_k} \in S/CI_k$ for all $k \in K$. If $r = s$ then $[r] = [s] \in S/CI$ and the result holds. If $r \neq s$ then $r, s \in CI_k$ for all $k \in K$. Hence $r, s \in \bigcap_{k\in K} CI_k = C(\bigcup_{k\in K} I_k) = CI$ by Proposition 4.11. This means $r \sim_I s$, that is, $[r] = [s] \in S/CI$. Thus $\mathcal{P}_S$ is separated.

In order to construct a sheaf $\mathcal{F}_S$ associated to the presheaf $\mathcal{P}_S$, we need some necessary definitions.

DEFINITION 6.2: Let $(I_k)_{k\in K}$ be a family of pure ideals I_k of S. Then $(I_k)_{k\in K}$ is called a *pure covering* of I if $I = \bigcup_{k\in K} I_k$.

A family of elements

$$\alpha = ([r_k] \in S/CI_k)_{k\in K}$$

is called a *compatible family* of elements for a pure covering $(I_k)_{k\in K}$ of I if for any $k, l \in K$

$$[r_k]|_{I_k \cap I_l} = [r_l]|_{I_k \cap I_l} \in S/C(I_k \cap I_l).$$

Let $(I_k)_{k\in K}$, $(J_l)_{l\in L}$ be pure coverings of I and $\alpha = ([r_k] \in S/CI_k)_{k\in K}$, $\beta = ([s_l] \in S/CJ_l)_{l\in L}$ two such compatible families. Then they are *equivalent*, denoted by $\alpha \sim \beta$, if for any $k \in K$ and $l \in L$,

$$[r_k]|_{I_k \cap J_l} = [s_l]|_{I_k \cap J_l} \in S/C(I_k \cap J_l).$$

PROPOSITION 6.3. *The relation* $\sim$ *is an equivalence relation on the set of compatible families for a pure ideal* I *of* S.

PROOF: (1): $\alpha \sim \alpha$ and (2): $\alpha \sim \beta \Rightarrow \beta \sim \alpha$ are trivial from the definition.

To show the transitivity, suppose that

$$\alpha = ([r_k] \in S/CI_k)_{k\in K} \sim \beta = ([s_l] \in S/CI_l)_{l\in L} \quad \text{and}$$
$$\beta = ([s_l] \in S/CI_l)_{l\in L} \sim \gamma = ([t_m] \in S/CI_m)_{m\in M}.$$

Then we show that, for any $k \in K$ and $m \in M$, $[r_k]\,|_{I_k\cap I_m} = [t_m]\,|_{I_k\cap I_m}$. Since $I_k \cap I_m = \bigcup_{l\in L}(I_k \cap I_m \cap I_l) \in p(I_k \cap I_m)$ and the presheaf $\mathcal{P}_S$ on $Spp(S)$ is separated, it is sufficient to show that for any $l \in L$

$$([r_k]\,|_{I_k\cap I_m})\,|_{I_k\cap I_m\cap I_l} = ([t_m]\,|_{I_k\cap I_m})\,|_{I_k\cap I_m\cap I_l}.$$

But this follows from the assumption:

$$[r_k]\,|_{I_k\cap I_l} = [s_l]\,|_{I_k\cap I_l} \quad \text{and} \quad [s_l]\,|_{I_l\cap I_m} = [t_m]\,|_{I_l\cap I_m}.$$

DEFINITION 6.4: We denote by $\mathcal{F}_S(I)$ the quotient by this equivalence relation $\sim$ on the set of all compatible families for a pure ideal I of S.

DEFINITION 6.5: Let $(J_m)_{m\in M}$ and $(I_k)_{k\in K}$ be pure coverings of I. If for each J_m $(m \in M)$ there exists I_k such that $J_m \subseteq I_k$, we say that $(J_m)_{m\in M}$ is *smaller* than $(I_k)_{k\in K}$ and denote $(J_m)_{m\in M} \leq (I_k)_{k\in K}$.

PROPOSITION 6.6. *Let* $\bar{\alpha} \in \mathcal{F}_S(I)$ *be any element and*

$$\alpha = ([r_k] \in S/CI_k)_{k\in K}$$

its representative family of elements. If $(J_m)_{m\in M} \leq (I_k)_{k\in K}$, *then the family*

$$\beta = ([r_k]\,|_{J_m} \in S/CJ_m)_{m\in M}$$

is equivalent to α, *where* $J_m \subseteq I_k$.

PROOF: First we show that β is compatible. For each $m, n \in M$, let $J_m \subseteq I_k$ and $J_n \subseteq I_l$. Then we need to show that $([r_k]\,|_{J_m})\,|_{J_m\cap J_n} = ([r_l]\,|_{J_n})\,|_{J_m\cap J_n}$. Since α is compatible, $[r_k]\,|_{I_k\cap I_l} = [r_l]\,|_{I_k\cap I_l}$. Since $J_m \cap J_n \subseteq I_k \cap I_l$, it follows $[r_k]\,|_{J_m\cap J_n} = [r_l]\,|_{J_m\cap J_n}$. Hence β is compatible.

To show $\alpha \sim \beta$, let $[r_k] \in S/CI_k$ and $[r_l]\,|_{J_m} \in S/CJ_m$ with $J_m \subseteq I_l$. From the compatibility of α, $[r_k]\,|_{I_k\cap I_l} = [r_l]\,|_{I_k\cap I_l}$ and so $[r_k]\,|_{I_k\cap I_l\cap J_m} = [r_l]\,|_{I_k\cap I_l\cap J_m}$. Since $J_m \subseteq I_l$, it follows

$$[r_k]\,|_{I_k\cap J_m} = [r_l]\,|_{I_k\cap J_m} = ([r_l]\,|_{J_m})\,|_{I_k\cap J_m}.$$

Hence $\alpha \sim \beta$.

As corollary we have

PROPOSITION 6.7. *Let* $\overline{\alpha_1}, \ldots, \overline{\alpha_n}$ *be any finite elements of* $\mathcal{F}_S(I)$. *Then there exists a pure covering* $(I_k)_{k\in K}$ *of* I *such that each* $\overline{\alpha_i}$ $(i = 1, 2, \ldots n)$ *has a representative family of elements valued in the same* S/CI_k *for all* $k \in K$:

$$\alpha_i = ([r_k^{(i)}] \in S/CI_k)_{k\in K}.$$

Now we can define a structure of an S-monoid with zero of $\mathcal{F}_S(I)$ as follows.

DEFINITION 6.8: Let $(I_k)_{k\in K}$ be a pure covering of I and $\overline{\alpha}, \overline{\beta} \in \mathcal{F}_S(I)$ with representative families $\alpha = ([r_k] \in S/CI_k)_{k\in K}$ and $\beta = ([s_k] \in S/CI_k)_{k\in K}$. We define

$$\alpha + \beta = ([r_k] + [s_k] \in S/CI_k)_{k\in K},$$

$$\alpha s = ([r_k]\, s \in S/CI_k)_{k\in K} \quad \text{for } s \in S,$$

$$0 = ([0] \in S/CI_k)_{k\in K} \quad \text{and} \quad 1 = ([1] \in S/CI_k)_{k\in K}.$$

Then we may define as $\overline{\alpha} + \overline{\beta} = \overline{\alpha + \beta}$, $\overline{\alpha} s = \overline{\alpha s}$ and $\overline{0}, \overline{1} \in \mathcal{F}_S(I)$.

Next, let $J \subseteq I$ in $p(S)$ and $\overline{\alpha} \in \mathcal{F}_S(I)$ with representative family $\alpha = ([r_k] \in S/CI_k)_{k\in K}$. We define a mapping

$$\rho_{IJ}^{\mathcal{F}}: \mathcal{F}_S(I) \to \mathcal{F}_S(J)$$

by $\rho_{IJ}^{\mathcal{F}}(\overline{\alpha}) = \overline{\alpha|_J}$ with $\alpha|_J = ([r_k]|_{I_k\cap J} \in S/C(I_k \cap J))_{k\in K}$. Then $\rho_{IJ}^{\mathcal{F}}$ is a homomorphism of S-monoids with zero.

Finally we note that there is an injective homomorphism of monoids with zero

$$\lambda_I: \mathcal{P}_S(I) = S/CI \to \mathcal{F}_S(I)$$

which is defined by sending $[r] \in S/CI$ to the family $([r] \in S/CI)$ for a single pure covering of I itself.

Thus we have constructed a presheaf $\mathcal{F}_S$ of S-monoids with zero on $Spp(S)$.

THEOREM 6.9. *The presheaf* $\mathcal{F}_S$ *associated to the presheaf* $\mathcal{P}_S$ *defined by* $\Theta_I \to S/CI = P_S(I)$ *is a sheaf of S-monoids with zero on* $Spp(S)$; *its monoid with zero of global sections is isomorphic to S.*

PROOF: First we check the condition (4) for a sheaf. Let $\overline{\alpha}, \overline{\beta} \in \mathcal{F}_S(I)$ and $I = \bigcup_{m\in M} J_m$ and suppose that

$$\overline{\alpha}|_{J_m} = \overline{\beta}|_{J_m} \in \mathcal{F}_S(J_m) \quad \text{for all } m \in M.$$

Let $\alpha = ([r_k] \in S/CI_k)_{k\in K}$ and $\beta = ([s_l] \in S/CI_l)_{l\in L}$ be representative families of $\overline{\alpha}$ and $\overline{\beta}$. Since $\alpha|_{J_m} = ([r_k]|_{J_m} \in S/C(J_m \cap I_k))_{k\in K}$ is equivalent to $\beta|_{J_m} =$

$([s_l]\,|_{J_m} \in S/C\,(J_m \cap I_l))_{l\in L}$, we have

$$([r_k]\,|_{J_m})|_{(J_m\cap I_k)\cap(J_m\cap I_l)} = ([s_l]\,|_{J_m})|_{(J_m\cap I_k)\cap(J_m\cap I_l)}$$

for each $[r_k]\,|_{J_m}$ and $[s_l]\,|_{J_m}$. This means $[r_k]\,|_{J_m\cap I_k\cap I_l} = [s_l]\,|_{J_m\cap I_k\cap I_l}$. Since $I_k \cap I_l = \bigcup_{m\in M}(I_k \cap I_l \cap J_m)$ and $\mathcal{P}_S$ is separated, $[r_k]\,|_{I_k\cap I_l} = [s_l]\,|_{I_k\cap I_l}$. Thus $\alpha\sim\beta$ and so $\bar{\alpha}=\bar{\beta}$.

Next we check the condition (5) for a sheaf. Let $I = \bigcup_{m\in M} J_m \in p(S)$ and $(\overline{\alpha_m} \in F_S(J_m))_{m\in M}$ a family such that for any $m, n \in M$

$$\overline{\alpha_m}\,|_{J_m\cap J_n} = \overline{\alpha_n}\,|_{J_m\cap J_n}.$$

Let $\alpha_m = ([r_{mk}] \in S/CI_{mk})_{mk\in M_m}$ be a representative family of $\overline{\alpha_m}$ for a pure covering $(I_{mk})_{mk\in M_m}$ of J_m.

Consider

$$\alpha = ([r_{mk}] \in S/CI_{mk})_{mk\in\bigcup_{m\in M}M_m}.$$

Then we easily observe that α is also compatible and so $\bar{\alpha}\in\mathcal{F}_S(I)$. From the compatibility of α and the construction, we may see that $\alpha|_{J_m} \sim \alpha_m$. Thus $\mathcal{F}_S$ is indeed a sheaf.

Finally we need to show that the injection

$$\lambda_S : S = S/(0) = S/CS \to \mathcal{F}_S(S)$$

is surjective. Let $\alpha = ([r_k] \in S/CI_k)_{k\in K}$ be a representative family of any element $\bar{\alpha}$ of $\mathcal{F}_S(S)$. Then, by the compatibility of α, for any $k, l \in K$, the following equality holds:

$$[r_k]\,|_{I_k\cap I_l} = [r_l]\,|_{I_k\cap I_l}.$$

On the other hand, since $\bigcup_{k\in K} I_k = S \ni 1$, there exists $l\in K$ such that $1\in I_l = S$. Take $r = r_l \in S$, then the above equality shows that $\lambda_S(r) = ([r_l] \in S/CI_l)$ is equivalent to $([r_k] \in S/CI_k)_{k\in K}$. Thus λ_S is surjective.

We proceed now to construct an analogous sheaf representation $\mathcal{F}_A$ for any S-system A.

PROPOSITION 6.10. *Let A be an S-system. For any pure ideal I, the assignment*

$$\Theta_I \mapsto A/CAI = \mathcal{P}_A(I)$$

defines a separated presheaf $\mathcal{P}_A$ of S-systems over $\mathcal{P}_S$ on $Spp(S)$.

PROOF: Since for a pure ideal I of S, CAI is a sub S-system of A, $\mathcal{P}_A(I) = A/CAI$ is an S-system.

We may define an action of S/CI on A/CAI by $[a]\,[s] = [as]$ for $a\in A$

and $s \in S$. Then $\mathcal{P}_A(I)$ is an S/CI-S-system, since $([a][s])r = [as]r = [asr] = [a][sr] = [a]([s]r)$ for $r \in S$. If $J \subseteq I$ in $p(S)$, then $CAJ \supseteq CAI$ and thus there is a quotient S-homomorphism

$$\rho_{IJ}^{\mathcal{P}}: A/CAI \to A/CAJ.$$

Since $([a][s])|_J = [a]|_J[s]|_J$, $\rho_{IJ}^{\mathcal{P}}$ is a system homomorphism. Thus we have a presheaf $\mathcal{P}_A$ over $\mathcal{P}_S$.

To show that $\mathcal{P}_A$ is separated, consider $I = \bigcup_{k\in K} I_k$ in $p(S)$, and $[a], [b] \in A/CAI$ such that for any k, $[a]|_{I_k} = [b]|_{I_k} \in A/CAI_k$. If $a = b$, then $[a] = [b]$ and the result holds. If $a \neq b$, then $a, b \in CAI_k$ for all $k \in K$. Then it follows $a, b \in \bigcap_{k\in K} CAI_k = CAI$ and so $[a] = [b] \in A/CAI$. This shows $\mathcal{P}_A$ is separated.

Analogous to $\mathcal{F}_S$ associated to $\mathcal{P}_S$, we may construct a presheaf $\mathcal{F}_A$ associated to the separated preseheaf $\mathcal{P}_A$ as follows.

DEFINITION 6.11: An element in $\mathcal{F}_A(I)$ is represented by a compatible family $([a_k] \in A/CAI_k))_{k\in K}$ of elements for a pure covering $(I_k)_{k\in K}$ of I; two such families $([a_k] \in A/CAI_k)_{k\in K}$ and $([a_l] \in A/CAI_l)_{l\in L}$ are *equivalent* if for any $[a_k] \in A/CAI_k$ and $[a_l] \in A/CAI_l$,

$$[a_k]|_{I_k\cap I_l} = [a_l]|_{I_k\cap I_l} \in S/CA(I_k \cap I_l).$$

We need to define an action of $\mathcal{F}_S(I)$ on the S-system $\mathcal{F}_A(I)$. An element in $\mathcal{F}_S(I)$ is represented by a compatible family $([r_l] \in S/CI_l)_{l\in L}$ for a pure covering $(I_l)_{l\in L}$ of I. Define

$$\overline{([a_k])}_{k\in K}\,\overline{([r_l])}_{l\in L} = \overline{([a_m r_m])}_{m\in M}$$

where $(I_m)_{m\in M}$ is a smaller pure covering common to $(I_k)_{k\in K}$ and $(I_l)_{l\in L}$.

For $J \subseteq I$ in $p(S)$, the assignment

$$([a_k] \in A/CAI_k)_{k\in K} \mapsto ([a_k]|_{I_k\cap J} \in A/CA(I_k \cap J))_{k\in K}$$

defines the restriction map

$$\rho_{IJ}^{\mathcal{F}}: \mathcal{F}_A(I) \to \mathcal{F}_A(J)$$

which is an S-homomorphism and a system homomorphism. Thus we have constructed a presheaf $\mathcal{F}_A$ over $\mathcal{F}_S$.

THEOREM 6.12. *The presheaf $\mathcal{F}_A$ associated to the presheaf $\mathcal{P}_A(I)$ defined by the assignment*

$$\Theta_I \mapsto A/CAI = \mathcal{P}_A(I)$$

is a sheaf of S-systems over the sheaf $\mathcal{F}_S$ of Theorem 6.9; its S-system of global sections is isomorphic to A.

PROOF: Analogous to the proof of Theorem 6.9, we may prove that the presheaf $\mathcal{F}_A$ is indeed a sheaf.

Consider the S-homomorphism

$$\lambda_I : A/CAI = \mathcal{P}_A(I) \to \mathcal{F}_A(I)$$

which sends $[a] \in A/CAI$ on $([a] \in A/CAI) \in \mathcal{F}_A(I)$ for the single pure covering of I itself. Clearly the mapping λ_I is injective. In particular we have an injection

$$\lambda_S : A = A/(\theta) = A/CAS \to \mathcal{F}_A(S).$$

We shall prove that this is a surjection and thus an isomorphism.

Consider an element $\bar{\alpha}$ in $\mathcal{F}_A(S)$ represented by a family $\alpha = ([a_k] \in A/CAI_k)_{k\in K}$ where $(I_k)_{k\in K}$ is a pure covering of S. Since $1 \in S = \bigcup_{k\in K} I_k$, there exists $l \in K$ such that $1 \in I_l = S$. Now take $a = a_l \in S$. Then we need to prove that $\lambda_S(a) = ([a_l] \in A/CAI_l)$ is equivalent to $\alpha = ([a_k] \in A/CAI_k)_{k\in K}$. But this follows from the compatibility of α. This completes the proof.

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