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THE INDEPENDENCE OF REGULAR IDENTITIES IN GROUPOIDS

By Lajos KLUKOVITS*

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Let $f = g$ be a groupoid identity in the variables $x_1, \dots, x_n$ such that each x_i ($i = 1, \dots, n$) occurs at least in one of f and g . We say that the conditions of this identity $f = g$ are independent over a set A if for each $(a_1, \dots, a_n) \in A^n$ there exists a binary operation $\star$ on A such that in the groupoid $\underline{A} = \langle A, \star \rangle$ we have

$$f_{\underline{A}}(a_{i_1}, \dots, a_{i_k}) \neq g_{\underline{A}}(a_{j_1}, \dots, a_{j_l}),$$

but for any $(b_1, \dots, b_n) \in A^n - \{(a_1, \dots, a_n)\}$

$$f_{\underline{A}}(b_{i_1}, \dots, b_{i_k}) = g_{\underline{A}}(b_{j_1}, \dots, b_{j_l}).$$

It is easy to see, that the conditions of the identity above are non-independent over a set A , if and only if there exists a proper subset $B \subset A^n$ such that in every groupoid $\underline{A} = \langle A, \circ \rangle$, if for every $(b_1, \dots, b_n) \in B$

$$f_{\underline{A}}(b_{i_1}, \dots, b_{i_k}) = g_{\underline{A}}(b_{j_1}, \dots, b_{j_l}),$$

then the identity $f = g$ holds in $\underline{A} = \langle A, \circ \rangle$.

The investigations of identities from independence point of view originated in a problem of L. Rédei in which he asked whether the associativity conditions are independent. This original problem was solved by G. Szász [5]:

THEOREM. *The associativity conditions are independent over a set A , iff $\text{card } A \geq 4$.*

Later R. Wiegandt and J. Wiesenbauer [7], [8] considered the distributivity of two binary operations from the same point of view.

In [2] and [3] we investigated two other groupoid identities: the mediality $((xy)(zu) = (xz)(yu))$ and the self-distributivity $(x(yz) = (xy)(xz))$ and $(xy)z = (xz)(yz)$ and proved the following result.

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THEOREM. The $\left\{ \begin{array}{l} \text{mediality} \\ \text{self-distributivity} \end{array} \right\}$ conditions are independent over a set A , iff $\text{card } A \geq 4$.

We have to remark, that in the case of mediality we used a weaker form of independence similarly to that of G. Szász in the case of the associativity conditions of commutative groupoids [6]. Later we will call this form of independence weak independence.

The aim of this paper is to give a generalization of these results for groupoid identities having the same set of variables on both sides.

1. Preliminaries. Let $F(V)$ denote the free groupoid with the countable free generating set $V = \{x_1, x_2, \dots\}$. A groupoid identity (briefly identity in the following) is an unordered pair of elements of $F(V)$. The length $\lambda(f)$ of the term $f \in F(V)$ is defined as follows

- (i) $\lambda(f) = 1$, if $f \in V$;
- (ii) $\lambda(f) = \lambda(f_1) + \lambda(f_2)$, if $f = f_1 f_2$, where $f_1, f_2 \in F(V)$.

The set of variables (the elements of V) occurring in the term $f \in F(V)$ is denoted by $\sigma(f)$, and we say that the identity $f = g$ is *regular* [1] (or *homotypical* [4]) if $\sigma(f) = \sigma(g)$ and it is an identity in n variables if $\text{card } (\sigma(f) \cup \sigma(g)) = n$.

For any $f(x_1, \dots, x_n) \in F(V)$ and a set A the symbol $f(a_1, \dots, a_n)$, where $a_1, \dots, a_n \in A$, is called an *expression* over the set A . Thus the expressions over a set A are the elements of the free groupoid $F(A)$ freely generated by the set A . If $\underline{A} = \langle A, \circ \rangle$ is a groupoid, $f(x_1, \dots, x_n) \in F(V)$ is a term and $a_1, \dots, a_n \in A$, then the element $f_{\underline{A}}(a_1, \dots, a_n)$ — where $f_{\underline{A}}$ is the term function on the groupoid $\underline{A}$ induced by the term f — is called the value of the expression $f(a_1, \dots, a_n)$ in the groupoid $\underline{A} = \langle A, \circ \rangle$.

To classify the identities we introduce two equivalence relations on $F(V)$. For the first relation let ζ denote the homomorphism $F(V) \rightarrow F(x_1)$, which is the extension of the map $V \rightarrow \{x_1\}$. If two terms $f, g \in F(V)$ satisfy $\zeta(f) = \zeta(g)$, i.e., $(f, g) \in \ker \zeta$, then these terms have the same structure of parentheses thus they are of the same length. In this case we will say that these terms f and g are *(PA)-equivalent*.

Observe, that if the terms $f_1, f_2 \in F(V)$ are *(PA)-equivalent*, then the $f_1 = f_2$ can be given by a pair of maps $\{1, \dots, n\} \rightarrow V$ where $n = \lambda(f_1) = \lambda(f_2)$. Thus $f_i(k)$ denotes the variable standing on the k -th place of f_i (the parentheses and operation symbols are not counted). These maps can be denoted by f_1 and f_2 without any confusion.

Let A be any non-empty set, n a natural number and $Eq(n)$ denote the set

of all equivalence relations on the set $\{1, 2, \dots, n\}$. We say that ordered n -tuple $(a_1, \dots, a_n) \in A^n$ is of *pattern* $\alpha \in Eq(n)$, iff for any $1 \leq i, j \leq n$ $(i, j) \in \alpha$ implies $a_i = a_j$ and vice versa. Thus, the n -tuples $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are of the same pattern, if for any $1 \leq i, j \leq n$ $a_i = a_j$ then $b_i = b_j$ and conversely.

We can introduce a similar notion to terms. We say that two (PA) -equivalent terms $f, g \in F(V)$ are (SP) -equivalent, if there exists an automorphism $\varphi \in \text{Aut } F(V)$ such that $\varphi(f) = g$.

It is easy to see, that the (SP) -equivalent terms differ in naming their variables only, moreover, this second equivalence relation can be described with the help of the maps introduced above. Namely, f and g are (SP) -equivalent iff $\ker f = \ker g$.

We mention, that the terms appearing in the identities of associativity or self-distributivity are non (PA) -equivalent, but in the case of mediality they are (SP) -equivalent.

Since the expressions over a set A are elements of the free groupoid $F(A)$, we can speak about their length, (PA) -equivalence and (SP) -equivalence as in the case of terms.

We can describe the (SP) -equivalence of terms via permutations.

PROPOSITION 1. *Let the n -ary terms $f, g \in F(V)$ be (PA) -equivalent terms with $\sigma(f) = \sigma(g)$. These terms are (SP) -equivalent, iff there exists a permutation $\pi \in S(\sigma(f))$ such that*

$$g(x_1, \dots, x_n) = f(x_1 \pi, \dots, x_n \pi),$$

where $S(\sigma(f))$ denotes the full symmetric group over the set $\sigma(f)$.

PROOF. First suppose, that f and g are (SP) -equivalent, i.e., $\ker f = \ker g$. If for some $k \in (\lambda(f)) = \{1, 2, \dots, \lambda(f)\}$ we have $f(k) = x_i$ and $g(k) = x_j$, then put $x_i \pi = x_j$. Since $\ker f = \ker g$ implies, that for any $r, s \in (\lambda(f))$ $f(r) = f(s)$, iff $g(r) = g(s)$, the correspondence π defined above is a permutation of the set $\sigma(f)$, and $g(x_1, \dots, x_n) = f(x_1 \pi, \dots, x_n \pi)$.

The sufficiency is obvious.

REMARK. *It is easy to see, that with the help of the above permutation $\pi \in S(\sigma(f))$ we can define a permutation $\pi' \in S(m) (= S(\{1, 2, \dots, m\}))$ — where $m = \text{card } \sigma(f)$ — such that for any $1 \leq i, j \leq m$ $(i, j) \in \pi'$, iff $x_i \pi = x_j$. Without any confusion we can denote both permutations by the same letter in the sequel.*

Investigating the associativity conditions G. Szász in [5] introduced a very useful notion which has the following form in our general setting. Let $f(x_{i_1}, \dots, x_{i_k}) = g(x_{j_1}, \dots, x_{j_l})$ be a groupoid identity with the set of variables $\{x_1, \dots, x_n\}$, and A be an arbitrary (non-void) groupoid. We say that the ordered n -tuple over the set A

$(a_1, \dots, a_n)$ is isolated in the groupoid $\underline{A}$ with respect to the identity $f = g$, if

$$f_{\underline{A}}(a_{i_1}, \dots, a_{i_k}) \neq g_{\underline{A}}(a_{j_1}, \dots, a_{j_l}),$$

but for each $(b_1, \dots, b_n) \in A^n - \{(a_1, \dots, a_n)\}$ we have

$$f_{\underline{A}}(b_{i_1}, \dots, b_{i_k}) = g_{\underline{A}}(b_{j_1}, \dots, b_{j_l}).$$

Using the notion of isolatedness we obtain, that the conditions of the identity in n variables $f = g$ are independent over a set A , if for each $(a_1, \dots, a_n) \in A^n$ there exists a binary operation $\circ$ on A such that this ordered n -tuple is isolated in the groupoid $\langle A, \circ \rangle$ with respect to the identity $f = g$.

The reader can easily prove the following.

PROPOSITION 2. *Let $f = g$ be a regular groupoid identity in n variables, A any (non-void) set, moreover, $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ ordered n -tuples over A , which are of the same pattern. Then $(a_1, \dots, a_n)$ is isolated in some groupoid $\langle A, \circ \rangle$ with respect to the identity $f = g$, iff there exists a binary operation $\star$ on A such that $(b_1, \dots, b_n)$ is isolated in the groupoid $\langle A, \star \rangle$ with respect to the identity $f = g$.*

PROOF. Let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in A^n$ be n -tuples of the same pattern. We have a permutation $\pi \in S(A)$ such that $a_i \pi = b_i$ for $i = 1, 2, \dots, n$. Suppose, that $(a_1, \dots, a_n)$ is isolated in the groupoid $\langle A, \circ \rangle$ with respect to the identity $f = g$. Define the binary operation $\star$ on A as

$$u \star v = (u \pi^{-1} \circ v \pi^{-1}) \pi$$

for any $u, v \in A$. Then $(b_1, \dots, b_n)$ is isolated in the groupoid $\langle A, \star \rangle$ with respect to the identity $f = g$.

In the following we always suppose, that all terms are groupoid terms and the length of them is at least two, all identities are regular, and the base set A has at least two elements.

2. The case of non-(PA)-equivalent terms. In this section we will prove that the conditions of those regular identities, in which the terms are not (PA) equivalent, are independent over sets having sufficiently many elements. Our way to establish this result is constructive and the following lemma will be fundamental in the sequel.

LEMMA 1. *Let h be a term in n variables, A a set of at least $n + \lambda(h)$ elements and $(a_1, \dots, a_n) \in A^n$. Then there exists a binary operation $\circ$ on A such that for any n -ary term h' with $\sigma(h') = \sigma(h)$ and $\lambda(h') \geq \lambda(h)$, if in the groupoid $A = \langle A, \circ \rangle$ the equality*

$$h_{\underline{A}}(a_1, \dots, a_n) = h'_{\underline{A}}(b_1, \dots, b_n)$$

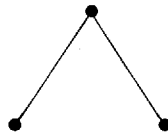
holds for some $(b_1, \dots, b_n) \in A^n$, then the terms h and h' are (PA)-equivalent, moreover, $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$.

PROOF. We associate the expression $h(a_1, \dots, a_n)$ with a labelled binary tree by the following procedure (in this construction we drop the assumption $\lambda(h) \geq 2$):

(i) if $h(a_1, \dots, a_n) = a_i$, then it is associated with the tree of one verted

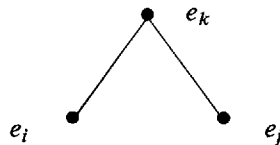


(ii) while, if $h(a_1, \dots, a_n) = h_1(a_{i_1}, \dots, a_{i_k}) \cdot h_2(a_{j_1}, \dots, a_{j_l})$, where $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_l\} = \{1, 2, \dots, n\}$, then it is associated with the tree derived from the partial tree

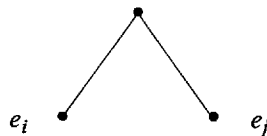


gluing to the left-and right edge the trees associated with expressions $h_1(a_{i_1}, \dots, a_{i_k})$ and $h_2(a_{j_1}, \dots, a_{j_l})$, respectively.

After this procedure we have a binary tree. The vertices of degree one (the vertices of the first level) are labelled by the elements $a_1, \dots, a_n$, and all the vertices of degree more than one (the vertices on the second, third, ... levels) are unlabelled. We will label all of the vertices using elements from the set $A' = A - \{a_1, \dots, a_n\}$ by the following procedure. Going on from the left to the right we label the vertices on the second level by different elements of the set A' , but if we have got a labelled subgraph



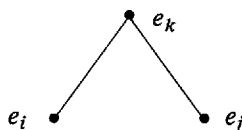
yet, and to the right from this there is a subgraph



then the unlabelled vertex of this subgraph get the label e_k , too. On the third,

fourth, ... levels we proceed on the same way taking into account the subgraphs on lower levels, too. We can suppose that the top vertex (the root of the tree) has the label a . It is easy to see, that we have used at most $n + \lambda(h) - 1$ elements in this procedure.

We can consider this labelled binary tree as a definition of a partial binary operation on the set A . Namely, if we have a subtree



then the product $e_i \circ e_j$ should be defined by e_k . Extend this partial operation to a total operation on A such that let each product undefined yet be equal to $b \in A$, where this element b has not been used as a label earlier. As $\text{card } A \geq n + \lambda(h)$ such element $b \in A$ exists.

It is clear, that in the groupoid $\underline{A} = \langle A, \circ \rangle$ $h_{\underline{A}}(a_1, \dots, a_n) = a$, moreover, for any expression over A $h'(b_1, \dots, b_n)$, if $h'(b_1, \dots, b_n) \neq h(a_1, \dots, a_n)$ and $\lambda(h) \leq \lambda(h')$, then $h'_{\underline{A}}(b_1, \dots, b_n) = b \neq a$. Obviously, $h'(b_1, \dots, b_n) = h(a_1, \dots, a_n)$ just in the case, when the terms h' and h are (SP)-equivalent, i.e., there is a permutation $\pi \in S(n)$ such that $h'(x_1, \dots, x_n) = h(x_{1\pi}, \dots, x_{n\pi})$, moreover, $(b_1, \dots, b_n) = (a_{1\pi}, \dots, a_{n\pi})$ which implies $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$ and this proves our lemma.

THEOREM 1. *If $f = g$ is a regular identity in n variables and the terms f, g are non-(PA)-equivalent ($\lambda(f), \lambda(g) \geq 2$), then the conditions of this identity are independent over any set A of at least $n + \min \{ \lambda(f), \lambda(g) \}$ elements.*

PROOF. Let A be a set, $\text{card } A \geq n + \min \{ \lambda(f), \lambda(g) \}$, and suppose that $\lambda(f) \leq \lambda(g)$. We will show that any $(a_1, \dots, a_n) \in A^n$ is isolated in a suitable groupoid over A with respect to the identity $f = g$.

Consider the expression $f(a_1, \dots, a_n)$ and construct the groupoid $\underline{A} = \langle A, \circ \rangle$ described in Lemma 1. As the terms f and g are non-(PA)-equivalent there is no n -tuple $(b_1, \dots, b_n) \in A^n$ such that $f(a_1, \dots, a_n) = g(b_1, \dots, b_n)$ holds in the free groupoid $F(A)$. By Lemma 1 this fact means, that the ordered n -tuple $(a_1, \dots, a_n)$ is isolated in the groupoid $\underline{A} = \langle A, \circ \rangle$ with respect to the identity $f = g$.

3. The case of (PA)-equivalent terms. We can separate the (regular) identities containing (PA)-equivalent terms into two classes via (SP)-equivalence. It is easy to see, that if in the regular identity $f = g$ the terms f and g are (SP)-equivalent, then the conditions of this identity are non-independent over any set A . Namely, for any

$a \in A$ we have $f(a, \dots, a) = g(a, \dots, a)$ in the free groupoid $F(A)$.

The following theorem says more about this case.

THEOREM 2. *Let f and g be n -ary (PA)-equivalent terms with $\sigma(f) = \sigma(g)$ and A an arbitrary set. If f and g are (SP)-equivalent, then every n -tuple $(a_1, \dots, a_n) \in A^n$ is non-isolated in any groupoid over A with respect to the identity $f = g$.*

PROOF. By Proposition 2 and the Remark after it we have a permutation $\pi \in S(n)$ such that

$$g(x_1, \dots, x_n) = f(x_{1\pi}, \dots, x_{n\pi}).$$

Let $\underline{A}$ be an arbitrary groupoid and $(a_1, \dots, a_n) \in A^n$. We will show, that this n -tuple is non-isolated in A with respect to the identity $f = g$.

If we have the equality $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ in the free groupoid $F(A)$, then $f_{\underline{A}}(a_1, \dots, a_n) = g_{\underline{A}}(a_1, \dots, a_n)$, therefore we can suppose that $(a_1, \dots, a_n) \neq (a_{1\pi}, \dots, a_{n\pi})$.

Let the order of the permutation π be k ($k > 1$) in $S(n)$. There exists a minimal natural number $1 < m \leq k$ such that

$$(a_1, \dots, a_n) = (a_{1\pi^m}, \dots, a_{n\pi^m}),$$

The n -tuple $(a_1, \dots, a_n)$ is isolated in $\underline{A}$ (with respect to the identity $f = g$) only if

$$f_{\underline{A}}(a_{1\pi^t}, \dots, a_{n\pi^t}) = g_{\underline{A}}(a_{1\pi^t}, \dots, a_{n\pi^t})$$

holds for each $1 \leq t < m$. By our assumption, for every natural number r the equalities

$$g(a_{1\pi^r-1}, \dots, a_{n\pi^r-1}) = f(a_{1\pi^r}, \dots, a_{n\pi^r})$$

hold in $F(A)$, thus

$$\begin{aligned} g_{\underline{A}}(a_1, \dots, a_n) &= f_{\underline{A}}(a_{1\pi}, \dots, a_{n\pi}) = g_{\underline{A}}(a_{1\pi}, \dots, a_{n\pi}) \\ &= f_{\underline{A}}(a_{1\pi^2}, \dots, a_{n\pi^2}) = g_{\underline{A}}(a_{1\pi^2}, \dots, a_{n\pi^2}) \\ &= \dots \\ &= f_{\underline{A}}(a_{1\pi^{m-1}}, \dots, a_{n\pi^{m-1}}) = g_{\underline{A}}(a_{1\pi^{m-1}}, \dots, a_{n\pi^{m-1}}) \\ &= f_{\underline{A}}(a_{1\pi^m}, \dots, a_{n\pi^m}) = f_{\underline{A}}(a_1, \dots, a_n) \end{aligned}$$

and this shows us, that

$$f_{\underline{A}}(a_1, \dots, a_n) = g_{\underline{A}}(a_1, \dots, a_n),$$

i.e., the n -tuple $(a_1, \dots, a_n)$ is non-isolated in the groupoid $\underline{A}$ with respect to the identity $f = g$.

As we mentioned earlier, we used a weaker form of independence in the case of mediality (the terms in this identity are (SP) -equivalent). In the present general setting this has the following form.

Let $f = g$ be a regular identity in n variables and A be groupoid. The ordered n -tuple $(a_1, \dots, a_n) \in A^n$ is *weakly isolated* in A with respect to the identity $f = g$, if there exists a permutation $\sigma \in S(n)$ such that

$$\begin{aligned} f_A(a_1, \dots, a_n) &\neq g_A(a_1, \dots, a_n), \\ f_A(a_{1\sigma}, \dots, a_{n\sigma}) &\neq g_A(a_{1\sigma}, \dots, a_{n\sigma}), \end{aligned}$$

but for any $(b_1, \dots, b_n) \in A^n - [(a_1, \dots, a_n), (a_{1\sigma}, \dots, a_{n\sigma})]$ we have

$$f_A(b_1, \dots, b_n) = g_A(b_1, \dots, b_n).$$

It is obvious, that isolated n -tuples are weakly isolated. Moreover, if $(a_1, \dots, a_n)$ is weakly isolated, but non-isolated in some $\underline{A}$ with respect to the regular identity $f = g$, then the terms f and g are (PA) -equivalent.

The conditions of the regular identity in n variables $f = g$ are *weakly independent* over the set A , if for each $(a_1, \dots, a_n) \in A^n$ there is a groupoid $\underline{A}$ such that $(a_1, \dots, a_n)$ is weakly isolated in $\underline{A}$ with respect to $f = g$, or the equality $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ holds in the free groupoid $F(A)$.

By Theorems 1 and 2 (and their proofs) we have the following.

THEOREM 3. *If in the regular identity in n variables $f = g$ the terms f, g are (SP) -equivalent, then the conditions of it are weakly independent over any set of at least $n + \lambda(f)$ elements.*

Finally, let f and g be (PA) -equivalent, but non (SP) -equivalent terms with $\sigma(f) = \sigma(g) = \{x_1, \dots, x_n\}$. Since $\ker f \neq \ker g$ we have $\ker f \not\subseteq \ker g$ and $\ker g \not\subseteq \ker f$ by our assumption on the set of the variables. We will show, that there exists a natural number q such that for any set A of at least q elements and for any ordered n -tuple $(a_1, \dots, a_n) \in A^n$ we have the following three possibilities:

- (i) $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ holds in $F(A)$,
- (ii) $(a_1, \dots, a_n)$ is isolated in some $\underline{A}$ with respect to $f = g$,
- (iii) $(a_1, \dots, a_n)$ is non-isolated, but weakly isolated in some $\underline{A}$ with respect to the identity $f = g$.

We separate the above cases via equivalence relations on the set $\{1, 2, \dots, n\}$.

LEMMA 2. *Let f and g be (PA) -equivalent terms with $\sigma(f) = \sigma(g) = \{x_1, \dots, x_n\}$. There exists a minimal equivalence relation $\rho \in Eq(n)$ such that, for*

any set A whenever $(a_1, \dots, a_n) \in A^n$ is of pattern ρ , $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ holds in $F(A)$.

PROOF. Let $(i, j) \in \rho' (1 \leq i, j \leq n)$, if there exists an $r \in (\lambda(f)) = \{1, 2, \dots, \lambda(f)\}$ such that $f(r) = x_i$ and $g(r) = x_j$, or $f(r) = x_j$ and $g(r) = x_i$. After a routine calculation we have that the equivalence closure of this relation is the required equivalence ρ .

If is obvious, that if $\alpha \in Eq(n)$ and $\rho \subseteq \alpha$, then we have in the free groupoid $F(A)$

$$f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$$

for any $(a_1, \dots, a_n) \in A^n$ of pattern α .

On the other hand there can exist an equivalent relation $\beta \in Eq(n)$ such that if an n -tuple $(a_1, \dots, a_n) \in A^n$ is of pattern β , then $f(a_1, \dots, a_n)$ and $g(a_1, \dots, a_n)$ are different elements of $F(A)$, but they are of the same pattern, i.e., they are (SP)-equivalent.

LEMMA 3. Let f and g be terms as in Lemma 2. There exists a minimal equivalence relation $\chi \in Eq(n)$ such that whenever for any set A , $(a_1, \dots, a_n) \in A^n$ is of pattern χ , the expressions $f(a_1, \dots, a_n)$ and $g(a_1, \dots, a_n)$ are of the same pattern.

PROOF. First we construct an increasing sequence of relations on the set $\{1, 2, \dots, n\}$. For any $1 \leq i, j \leq n$ let $(i, j) \in \chi_1$, if there exist $r, s \in (\lambda(f))$ such that $(r, s) \in \ker f$, moreover, $x_i = g(r)$ and $x_j = g(s)$, or $(r, s) \in \ker g$, moreover, $x_i = f(r)$ and $x_j = f(s)$. For any $k > 1$, let $(i, j) \in \chi_k$, if $(i, j) \in \chi_{k-1}$, or there exist $r, s \in (\lambda(f))$ such that $x_i = f(r)$ and $x_j = f(s)$, moreover, $g(r) = x_l$ and $g(s) = x_m$, where $(l, m) \in \chi_{k-1}$, or $x_i = g(r)$, $x_j = g(s)$, moreover, $f(r) = x_l$ and $f(s) = x_m$, where $(l, m) \in \chi_{k-1}$.

It is easy to see, that

- (i) the relations $\chi_i (i = 1, 2, \dots)$ are reflexive and symmetric,
- (ii) $\chi_1 \subseteq \chi_2 \subseteq \dots$,
- (iii) there exists a least natural number m such that $\chi_m = \chi_{m+1}$.

Let χ denote the transitive closure of χ_m defined in (iii). It is clear from the above construction that whenever $(a_1, \dots, a_n)$ is an n -tuple of pattern $\alpha \supseteq \chi$, every time the expressions $f(a_1, \dots, a_n)$ and $g(a_1, \dots, a_n)$ are of the same pattern.

The minimality of χ can be proved in a routine way, therefore we omit it.

LEMMA 4. If $\rho, \chi \in Eq(n)$ are the equivalence relations defined in the previous lemma, then $\chi \subseteq \rho$.

Combining Theorem 3 and Lemmas 3 and 4 we have.

LEMMA 5. Let f and g be n -ary (PA)-equivalent but non-(SP)-equivalent terms with $\sigma(f) = \sigma(g)$, and $(a_1, \dots, a_n) \in A^n$ an arbitrary n -tuple of pattern α , where $\chi \subseteq \alpha$ and $\rho \not\subseteq \alpha$. If $\text{card } A \geq n + \lambda(f)$, then there exists a binary operation $\star$ on A such that $(a_1, \dots, a_n)$ is weakly isolated in the groupoid $\langle A, \star \rangle$ with respect to the identity $f = g$.

LEMMA 6. Let f and g be terms as in the previous lemma and A an arbitrary set. If $(a_1, \dots, a_n) \in A^n$ is an n -tuple of pattern α , where $\chi \not\subseteq \alpha$, then there exists a natural number q such that whenever $\text{card } A \geq q$, there exists a binary operation $\star$ on A such that this n -tuple is isolated in the groupoid $\langle A, \star \rangle$ with respect to the identity $f = g$.

PROOF. Let $\alpha \in Eq(n)$, $\chi \not\subseteq \alpha$ and $(a_1, \dots, a_n) \in A^n$ an n -tuple of pattern α , moreover, $A_\alpha = \{a_1, \dots, a_n\}$. We have the following two possibilities:

- (i) The expression $f(a_1, \dots, a_n)$ is different from any expression of the form of $g(b_1, \dots, b_n)$, where $b_1, \dots, b_n \in A_\alpha$.
- (ii) There are n -tuples $(b_{11}, \dots, b_{1n}), (b_{21}, \dots, b_{2n}), \dots, (b_{k1}, \dots, b_{kn}) \in A_\alpha^n$ of patterns α_i ($i = 1, 2, \dots, k$), where $\chi \not\subseteq \alpha_i$ and we have the following equalities in the free groupoid $F(A)$

$$f(a_1, \dots, a_n) = g(b_{11}, \dots, b_{1n})$$

$$f(b_{i1}, \dots, b_{in}) = g(b_{i+1,1}, \dots, b_{i+1,n}), \quad i = 1, 2, \dots, k-1,$$

moreover, there is no $(d_1, \dots, d_n) \in A_\alpha^n$ such that $f(b_{k1}, \dots, b_{kn}) = g(d_1, \dots, d_n)$.

It is easy to see, that $(a_1, \dots, a_n) \notin \{(b_{11}, \dots, b_{1n}), \dots, (b_{k1}, \dots, b_{kn})\}$. We will consider only the case $k = 1$, because the general one can be treated similarly. Suppose that for some $(b_1, \dots, b_n), (c_1, \dots, c_n) \in A_\alpha^n$ we have the equalities in $F(A)$:

$$f(a_1, \dots, a_n) = g(b_1, \dots, b_n)$$

$$f(b_1, \dots, b_n) = g(c_1, \dots, c_n).$$

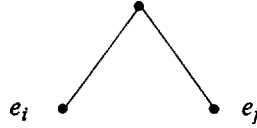
While the terms f and g are non-(SP)-equivalent and the pattern of $(a_1, \dots, a_n)$ is not less than or equal to χ , we can suppose that there are $i, j \in (\lambda(f))$, $i \neq j$, such that $f(i) = f(j) = x_m$, $g(i) = x_i$, $g(j) = x_j$ and $a_i \neq a_j$. The equality $f(a_1, \dots, a_n) = g(b_1, \dots, b_n)$ implies $b_i = b_j = a_m$. On the i -th and j -th places in the expression $f(b_1, \dots, b_n)$ -the parentheses and operation symbols are not counted-the element b_m stands. $f(b_1, \dots, b_n) = g(c_1, \dots, c_n)$ implies $c_i = c_j = b_m$, and on the i -th and j -th places in $f(c_1, \dots, c_n)$ the elements c_m stands, but on the same places in $g(a_1, \dots, a_n)$ there are different elements a_i and a_j . Therefore in $F(A)$ the equality $f(c_1, \dots, c_n) = g(a_1, \dots, a_n)$ is impossible.

It can be easily seen, that if in case (ii) $k = 0$, then we have case (i), therefore we will deal with case (ii) only allowing $k = 0$.

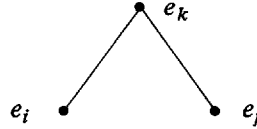
We shall use a generalization of the construction described in the proof of Lemma 1 which will provide us not only the operation but the natural number q .

Construct binary trees as in the proof of Lemma 1 corresponding to the expressions $f(a_1, \dots, a_n)$, $f(b_{11}, \dots, b_{1n})$, $\dots$, $f(b_{k1}, \dots, b_{kn})$ and place them side by side in one row in the above order. Obviously, in trees of the same form the vertices of degree one are labelled by the elements of A_α and all other vertices are unlabelled yet.

Suppose that the set A has enough elements for our procedure. Using different elements from the set $A - A_\alpha$ we label the vertices of degree three on the second level in each tree getting on from left to right such that if we have got a labelled subtree of the form of



and to the left of it there is a subtree



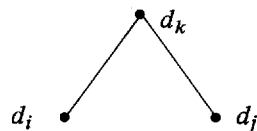
then this unlabelled vertex will have the label e_k , too. Following this instruction we label every vertex of degree three in each tree taking into account the subtrees on lower levels, too. The top vertices of the trees – the vertices of degree two – will be labelled by the same element, say $a (\in A)$, not used previously for labelling.

It is easy to see, that we need for this labelling procedure at most

$$q' = m + (k + 1) (\lambda(f) - 2) + 1$$

elements, where $m = \text{card } A_\alpha$.

Now let $q = q' + 1$ and suppose that $\text{card } A \geq q$. Observe, that by these $k + 1$ labelled binary trees – similarly to our previous construction – we can define a partial binary operation $\star$ on A in the following way. The product $d_i \star d_j$ is defined and equal to d_k , iff there exists a subtree of one of the trees of the form of



We can extend this partial operation to a total one by inserting to all empty places of the multiplication table the same element $b \in A$, which has not been used as a label. Such element must exist by our assumption on the cardinality of the set A .

It is clear from the construction, that if $f_A(d_1, \dots, d_n) \neq b$ for some $d_1, \dots, d_n \in A$ in the groupoid $\langle A, \star \rangle$, then $f(d_1, \dots, d_n) = a$, moreover, $(d_1, \dots, d_n) \in \{(a_1, \dots, a_n), (b_{11}, \dots, b_{1n}), \dots, (b_{k1}, \dots, b_{kn})\}$ and similarly, if $g_A(c_1, \dots, c_n) \neq b$, then $g_A(c_1, \dots, c_n) = a$, moreover, $(c_1, \dots, c_n) = (b_{j1}, \dots, b_{jk})$ for some $1 \leq j \leq k$. This means that the n -tuple $(a_1, \dots, a_n)$ is isolated in the groupoid $\langle A, \star \rangle$ with respect to the identity $f = g$, which proves our lemma.

We can summarize our last results in the following theorem.

THEOREM 4. *Let f and g be n -ary (PA)-equivalent but non (SP)-equivalent terms with $\sigma(f) = \sigma(g)$ and A an arbitrary set. For any $(a_1, \dots, a_n) \in A^n$ of pattern α , $\chi \not\subseteq \alpha$, let $k(a_1, \dots, a_n)$ denote the non-negative integer described in Lemma 6 and $t = \max \{k(a_1, \dots, a_n) \mid (a_1, \dots, a_n) \in A^n, \text{ the pattern of it is not less than or equal to } \chi\}$. The conditions of the identity $f = g$ are weakly independent over the set A , if $\text{card } A \geq n + (t + 1)(\lambda(f) - 2) + 2$.*

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