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(Citation)

Kobe Journal of Mathematics, 7(1):19-24

(Issue Date)

1990-06

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0003728>

(URL)

<https://hdl.handle.net/20.500.14094/E0003728>



ON DOUBLED SURFACES OF NONORIENTABLE SURFACES IN THE 4-SPHERE

Dedicated to Professor Fujitsugu HOSOKAWA
 on his sixtieth birthday

By Seichi KAMADA
 (Received July 8, 1989)

0. Introduction

Let F be a closed connected and nonorientable surface piecewise linearly and locally-flatly embedded in the 4-sphere S^4 with normal Euler number $e(F) = 0$. Since $e(F) = 0$, the tubular neighborhood $N(F)$ of F in S^4 which is a D^2 -bundle over F has a trivial I -subbundle. Let τ be a trivial I -subbundle of $N(F)$ and let τ^* be the orthogonal complement of τ in $N(F)$. We note that τ^* is a twisted I -subbundle of $N(F)$ and denote it by $F \tilde{\times} I \subset N(F)$. Then $F^* = F_\tau^* = F \tilde{\times} \partial I$ is a closed connected orientable surface embedded in S^4 . We call this surface a *doubled surface of F in S^4 (associated with τ)* (cf. [3]). A loop c on F is said to be *shrinkable* if there is a lift $\tilde{c}$ in $\partial N(F)$ which is null-homotopic in $S^4 - \overset{\circ}{N}(F)$.

THEOREM. *Suppose that there is a shrinkable, simple and orientation reversing loop c on F . Then for any doubled surface F^* , the complement $S^4 - F^*$ is homotopy equivalent to that of an unknotted orientable surface. Moreover, if F is a knot sum of a standard projective plane in S^4 and a surface in S^4 , then F^* is unknotted.*

For example, any Klein bottle in S^4 obtained from a 2-knot in S^4 by surgery along a trivial nonorientable 1-handle, which is generally knotted, satisfies the latter condition of this theorem (cf. [2; 4.6]). So such a Klein bottle in S^4 has only unknotted doubled surfaces.

Throughout this paper, we shall be working in the piecewise linear category. A *surface* is a closed connected 2-manifold; and if it is embedded in S^4 , the embedding is locally-flat. I stands for the interval $[-1, 1]$.

1. Some lemmas

Let K be a nonorientable surface of nonorientable genus n obtained from

a $2n$ -gon $D = |P_0 P'_0 P_1 P'_1 \cdots P_{n-1} P'_{n-1}|$ by gluing $|P_j P'_j|$ and $|P'_j P_{j+1}|$ for each j ($j = 0, 1, \dots, n-1$). And let Y be a D^2 -bundle over K with the euler number 0 obtained from the trivial D^2 -bundle $D \times D^2$ over D by identifying $(x, re^{i\theta})$ with $(x', re^{-i\theta})$ where $re^{i\theta} \in D^2$ and $x \in |P_j P'_j|$ and $x' \in |P'_j P_{j+1}|$ with $[x] = [x']$ in K . We abbreviate the element $[(x, re^{i\theta})]$ of Y as $[x, re^{i\theta}]$. ∂Y is a S^1 -bundle over K obtained from $D \times S^1$ by the above identification. We consider paths α_j ($j = 0, 1, \dots, n-1$) on $D \times S^1$ that run straight from $(P_j, 1)$ to $(P'_j, 1)$ and a loop β that starts from $(P_0, 1)$ and runs over $\{P_0\} \times S^1$ once in the positive direction of S^1 . They determine the elements of the fundamental group of ∂Y with base point $[P_0, 1]$; we denote them by the same symbols $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ and β . Then $\pi_1(\partial Y, [P_0, 1])$ has a presentation

$$(0.1) \quad \begin{aligned} &< \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta; \alpha_0^2 \alpha_1^2 \cdots \alpha_{n-1}^2 = 1, \\ &\alpha_j^{-1} \beta \alpha_j = \beta^{-1} (j = 0, 1, \dots, n-1) >. \end{aligned}$$

Let $\mathbf{R}(Y) = \{h | h \text{ is an orientation preserving auto-homeomorphism of } Y \text{ which acts on each fiber by rotation}\}$ and let $\mathbf{R}_0(Y) = \{h \in \mathbf{R}(Y) | \text{there exists a bundle map isotopy } \{h_t \in \mathbf{R}(Y)\} \text{ with } h_0 = id \text{ and } h_1 = h\}$. We note that $\mathbf{R}(Y)$ is an abelian group with group multiplication the composition of maps.

Here, we give some elements of $\mathbf{R}(Y)$. For each j ($j = 0, 1, \dots, n-1$), let Q_j and Q'_j be middle points of edges $|P_j P'_j|$ and $|P'_j P_{j+1}|$ respectively and let c_j be a simple proper arc on D connecting them. Let A_j be a regular neighborhood of c_j in D which is identified with $c_j \times [-1, 1]$ such that $(Q_j, s) \in \{Q_j\} \times [-1, 1]$ and $(Q'_j, -s) \in \{Q'_j\} \times [-1, 1]$ are the same point in K . Define a bundle map $h_j \in \mathbf{R}(Y)$ by

$$\begin{cases} h_j[x, re^{i\theta}] = [x, re^{i\theta}] & \text{for } x \in D - A_j, re^{i\theta} \in D^2, \\ h_j[(x, s), re^{i\theta}] = [(x, s), re^{i(\theta + \pi - \pi s)}] \\ \quad \text{for } (x, s) \in c_j \times [-1, 1] = A_j, re^{i\theta} \in D^2. \end{cases}$$

Note that $(h_j|_{\partial Y})_{\#}(\alpha_j) = \beta^{-1} \alpha_j = \alpha_j \beta$ and $(h_j|_{\partial Y})_{\#}(\alpha_i) = \alpha_i$ for $i \neq j$.

LEMMA 1. (1) The abelian group $\mathbf{R}(Y)/\mathbf{R}_0(Y)$ is generated by $[h_0], [h_1], \dots, [h_{n-1}]$ with defining relation $2[h_0] + 2[h_1] + \cdots + 2[h_{n-1}] = 0$.

(2) Let $(-1) \in \mathbf{R}(Y)$ be the antipodal map of D^2 -bundle Y . Then $[(-1)] = [h_0] + [h_1] + \cdots + [h_{n-1}]$.

PROOF. Let

$$\begin{aligned} \Pi &= \{f \in \text{Aut}(\pi_1(\partial Y, [P_0, 1])) | f(\beta) = \beta \text{ and } f(\alpha_j) \\ &= \alpha_j \beta^{n_j} (j = 0, 1, \dots, n-1) \text{ for some } n_j \in \mathbb{Z}\}. \end{aligned}$$

Π is an abelian subgroup of $\text{Aut}(\pi_1(\partial Y, [P_0, 1]))$. Since β has infinite order, Π is isomorphic to the free abelian group of rank n . Let $\mathbf{R}^*(Y) = \{h \in \mathbf{R}(Y) | h \text{ fixes the point } [P_0, 1]\}$. Consider a homomorphism $\varphi : \mathbf{R}^*(Y) \rightarrow \Pi$ defined by $\varphi(h) = (h|_{\partial Y})_{\#}$ for $h \in \mathbf{R}^*(Y)$. Then φ is an epimorphism, because $\varphi(h_0), \varphi(h_1), \dots, \varphi(h_{n-1})$ generate Π . Let i_β be the inner-automorphism of $\pi_1(\partial Y, [P_0, 1])$ by β . It is an element of Π with $i_\beta(\beta) = \beta$ and $i_\beta(\alpha_j) = \beta^{-1}\alpha_j\beta = \alpha_j\beta^2$.

ASSERTION. *There exists an isomorphism*

$$\bar{\varphi} : \mathbf{R}(Y)/\mathbf{R}_0(Y) \rightarrow \Pi / \langle i_\beta \rangle,$$

where $\langle i_\beta \rangle$ is the infinite cyclic normal subgroup of Π generated by i_β .

PROOF OF ASSERTION. $\bar{\varphi}$ is defined as follows: let $h \in \mathbf{R}^*(Y)$ be a representative of an element $[h]$ of $\mathbf{R}(Y)/\mathbf{R}_0(Y)$, then put $\bar{\varphi}([h]) = \varphi(h) \langle i_\beta \rangle$. We must show that $\bar{\varphi}$ is well-defined and injective. [Since φ is surjective, $\bar{\varphi}$ is so.] It is sufficient to show that the followings are equivalent for $h \in \mathbf{R}^*(Y)$:

- (I) h is an element of $\mathbf{R}_0(Y)$,
- (II) $\varphi(h)$ is an element of $\langle i_\beta \rangle$, equivalently $\varphi(h)(\alpha_j) = \alpha_j\beta^{2m}$ ($j = 0, 1, \dots, n-1$) for some integer m .

Suppose that $h \in \mathbf{R}^*(Y)$ is an element of $\mathbf{R}_0(Y)$. Since h is isotopic to the identity map by a bundle map isotopy which moves the base point $[P_0, 1]$ along the boundary of the fiber D_p^2 of $D \times D^2$ at P_0 , we see that $\varphi(h) = i_\beta^m$ for some integer m . Conversely, assume that $\varphi(h) = i_\beta^m$ for some m . It is easy to see that h is isotopic to a map $h' \in \mathbf{R}^*(Y)$ such that $\varphi(h') = id$ (, by a certain bundle map isotopy that turns D_p^2 m times in the negative direction). Since such a map h' is an element of $\mathbf{R}_0(Y)$, h is so. This completes the proof of Assertion.

We continue the proof of Lemma 1. (1) is a direct consequence of Assertion. Since the antipodal map (-1) is isotopic to a map $h \in \mathbf{R}^*(Y)$ with $\varphi(h)(\alpha_j) = \alpha_j\beta$ for each j ($j = 0, 1, \dots, n-1$), by Assertion, we obtain the result (2). This completes the proof of Lemma 1.

Let $T_0 : K \times I \rightarrow Y$ be a map defined by $T_0([x], t) = [x, t] \in Y$ for $x \in D$, $t \in I \subset D^2$. The image forms a trivial I -subbundle of Y . We call it the *standard trivial I -subbundle of Y* , and denote it by the symbol τ_0 . Let $(1/2)$ be a bundle map from Y into Y with $(1/2)[x, re^{i\theta}] = [x, (1/2)re^{i\theta}]$ for $x \in D$, $re^{i\theta} \in D^2$. For a trivial I -subbundle τ of Y , we define a genus $n-1$ orientable surface K_τ^{**} by $K_\tau^{**} = (1/2)(K \tilde{\times} \partial I)$ in $\tilde{Y}$, where $K \tilde{\times} \partial I$ is the boundary of the orthogonal complement $\tau^* = K \tilde{\times} I$ of τ . Let $h = h_0^{m_0} \circ h_1^{m_1} \circ \dots \circ h_{n-1}^{m_{n-1}} \in \mathbf{R}(Y)$ and consider

a surface $K_{h(\tau_0)}^{**}$ in Y associated with $h(\tau_0)$. The complement $Y - K_{h(\tau_0)}^{**}$ is a fiber bundle over K whose fiber D_0 is a disk with 2 holes. Let x and y be generators of the fundamental group of the fiber D_0 at $[P_0] \in K$ with base point $[P_0, 1]$ represented by small loops on D_0 encircling one hole and the other respectively with $xy = \beta$. Then $\pi_1(Y - K_{h(\tau_0)}^{**}, [P_0, 1])$ has a presentation

$$(1.1) \quad \begin{aligned} &< \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta, x, y; \\ &\alpha_0^2 \alpha_1^2 \cdots \alpha_{n-1}^2 = 1, \beta = xy, \\ &\alpha_j^{-1} x \alpha_j = \beta^{m_j} y^{-1} \beta^{-m_j} \quad (j = 0, 1, \dots, n-1), \\ &\alpha_j^{-1} y \alpha_j = \beta^{m_j} x^{-1} \beta^{-m_j} \quad (j = 0, 1, \dots, n-1) >. \end{aligned}$$

LEMMA 2. *Let τ be a trivial I -subbundle of Y and let K_τ^{**} be a surface in Y mentioned above. Then there exists an ambient isotopy of Y which carries K_τ^{**} to $K_{h(\tau_0)}^{**}$ and keeps ∂Y fixed, where $h = h_1^{m_1} \circ \cdots \circ h_{n-1}^{m_{n-1}}$ for some $m_j \in Z$ ($j = 1, \dots, n-1$).*

PROOF. There exists a bundle map isotopy of Y which carries τ to $h_0^{m_0} \circ \cdots \circ h_{n-1}^{m_{n-1}}(\tau_0)$ for some $m_j \in Z$ ($j = 0, 1, \dots, n-1$). By Lemma 1(1), we may assume $m_0 = 0$ or 1 according as m_0 is even or odd. If $m_0 = 1$, by Lemma 1(2), τ is carried to $h_1^{m_1+1} \circ \cdots \circ h_{n-1}^{m_{n-1}+1}(-\tau_0)$. Using the restriction of this bundle map isotopy to $(1/2)Y$, We have an ambient isotopy of Y which keeps ∂Y fixed and carries K_τ^{**} to $K_{h(\tau_0)}^{**}$ or $K_{h(-\tau_0)}^{**}$. But $K_{h(-\tau_0)}^{**}$ is the same surface as $K_{h(\tau_0)}^{**}$ in Y . This completes the proof.

LEMMA 3. *Let K be a nonorientable surface whose nonorientable genus ($= \text{rank}_{Z_2} H_1(F; Z_2)$) is even. Let c_1 and c_2 be simple orientation reversing loops on K . There is an auto-homeomorphism h of K such that $h \circ c_1 = c_2$.*

PROOF. Let A_1 and A_2 be the regular neighborhoods of c_1 and c_2 in K . Since c_1 and c_2 are orientation reversing loops on K , the boundary of each A_i is a simple loop on K , say d_i . Consider a compact surface $X_i = K - \dot{A}_i$ ($i = 1, 2$) with boundary d_i . It is a nonorientable surface. [If X_i is orientable, $2[c_i] = [d_i] = 0$ in $H_1(K; Z)$. This contradicts that c_i is an orientation reversing loop on K , since the nonorientable genus of K is even.] Hence X_1 and X_2 are homeomorphic. We have a homeomorphism $f : X_1 \rightarrow X_2$ such that $f \circ d_1 = d_2$, which extends to $h : K \rightarrow K$ such that $h \circ c_1 = c_2$. This completes the proof.

LEMMA 4. *Let $N(F)$ be the tubular neighborhood of a nonorientable surface F in S^4 with $e(F) = 0$ and let $\tilde{c}$ be a lift in $\partial N(F)$ of a simple orientation*

reversing loop c on F . Then there is an identification map $g : Y \longrightarrow N(F)$ such that $g \circ \alpha_0 = \tilde{c}$.

PROOF. Let $g : Y \longrightarrow N(F)$ be an identification. $g^{-1} \circ c$ is a simple orientation reversing loop on K . Since $e(F) = 0$, the nonorientable genus of F is even (cf. [4]). By lemma 3, there is an auto-homeomorphism f of K such that $f \circ p \circ \alpha_0 = g^{-1} \circ c$ where $p : \partial Y \longrightarrow K$ is the projection. The map f induces a fiber-preserving auto-homeomorphism $\tilde{f}$ of Y . The loop $(g \circ \tilde{f})^{-1} \circ \tilde{c}$ is a lift of the loop $p \circ \alpha_0$, so there is a bundle map $h \in R(Y)$ such that $h \circ \alpha_0 = (g \circ \tilde{f})^{-1} \circ \tilde{c}$. Then $g \circ \tilde{f} \circ h : Y \longrightarrow N(F)$ is a desired identification map. This completes the proof.

2. Proof of Theorem

PROOF OF THE FIRST PART OF THEOREM. Hosokawa/Kawauchi showed that if the fundamental group of the complement X of a closed connected orientable surface in S^4 is infinite cyclic, then X is homotopy equivalent to the complement of an unknotted closed connected orientable surface in S^4 [1; Corollary 2.11]. So it is sufficient to prove that $\pi_1(S^4 - F^*)$ is infinite cyclic. Let c be a simple orientation reversing loop such that a lift $\tilde{c}$ in $\partial N(F)$ is null-homotopic in $S^4 - \overset{\circ}{N}(F)$. Let $g : Y \longrightarrow N(F)$ be an identification of $N(F)$ such that $g \circ \alpha_0 = \tilde{c}$ (Lemma 4). And let $\bar{g} : \partial Y \longrightarrow \partial N(F)$ be the restriction of g and $i : \partial N(F) \longrightarrow S^4 - \overset{\circ}{N}(F)$ be the inclusion. For a given trivial I -subbundle τ of $N(F)$, let τ' be the corresponding trivial I -subbundle of Y (i.e. $g(\tau') = \tau$). Consider a surface $K_{\tau'}^{**} \subset Y$ mentioned before which is associated with τ' . Clearly the surface $F_{\tau'}^{**} = g(K_{\tau'}^{**})$ is ambient isotopic to F_{τ}^* in S^4 . So we shall show that $\pi_1(S^4 - F_{\tau'}^{**})$ is infinite cyclic. The group $\pi_1(S^4 - \overset{\circ}{N}(F), g([P_0, 1]))$ has a presentation $\langle x_1, x_2, \dots, x_s; r_1, \dots, r_t \rangle$, where x_1 is a meridian element and $x_2, \dots, x_s$ are conjugates of x_1 . By Lemma 2, $\pi_1(Y - K_{\tau'}^{**})$ has a presentation of the form (1.1) with $m_0 = 0$. We apply van Kampen's theorem for $S^4 - F_{\tau'}^{**} = (Y - K_{\tau'}^{**}) \cup_{\bar{g}} (S^4 - \overset{\circ}{N}(F))$. Since $(i \circ \bar{g})_{\#}(\alpha_0) = 1$ and $m_0 = 0$, we have $x = y^{-1}$ in $\pi_1(S^4 - F_{\tau'}^{**})$ by the third relation in (1.1). Then $x_1 = 1$ in $\pi_1(S^4 - F_{\tau'}^{**})$ because $\beta = xy$ and $(i \circ \bar{g})_{\#}(\beta) = x_1$ or x_1^{-1} . As the elements $x_2, \dots, x_s$ are conjugates of x_1 , they are 1 in $\pi_1(S^4 - F_{\tau'}^{**})$. Since the images of $\alpha_0, \dots, \alpha_{n-1}$ under the homomorphism $(i \circ \bar{g})_{\#}$ are words in $\{x_1, \dots, x_s\}$, $\alpha_0, \dots, \alpha_{n-1}$ are also 1 in $\pi_1(S^4 - F_{\tau'}^{**})$. Hence $\pi_1(S^4 - F_{\tau'}^{**})$ is generated by x , and therefore we see $\pi_1(S^4 - F_{\tau'}^{**}) \simeq Z$ (, since $H_1(S^4 - F_{\tau'}^{**}) \simeq Z$). This completes the proof of the first part of Theorem.

PROOF OF THE SECOND PART OF THEOREM: We assume that $F \subset S^4$ is a

knot sum of a surface $F_1 \subset S_1^4$ and a standard projective plane $F_2 \subset S_2^4$. Let D_i ($i = 1, 2$) be a 4-ball in S_i^4 such that $(D_i, D_i \cap F_i)$ is homeomorphic to the standard (4,2)-ball pair. We regard (S^4, F) as $(S_1^4 - \overset{\circ}{D}_1, F_1 - \overset{\circ}{D}_1) \cup_{\partial} (S_2^4 - \overset{\circ}{D}_2, F_2 - \overset{\circ}{D}_2)$. Let τ be a trivial I -subbundle of $N(F)$ and let τ^* be its orthogonal complement in $N(F)$. We decompose τ^* into τ_1^* and τ_2^* such that $\tau_i^* \subset S_i^4 - \overset{\circ}{D}_i$ is a twisted I -subbundle $(F_i - \overset{\circ}{D}_i) \widetilde{\times} I$ of $N(F_i - \overset{\circ}{D}_i; S_i^4 - \overset{\circ}{D}_i)$. τ_1^* is a genus $n - 1$ handle-body and τ_2^* is a solid torus, where n is the nonorientable genus of F . Let $B(\tau_i^*) = (F_i - \overset{\circ}{D}_i) \widetilde{\times} \partial I$ ($i = 1, 2$). Then we regard (S^4, F^*) as $(S_1^4 - \overset{\circ}{D}_1, B(\tau_1^*)) \cup_{\partial} (S_2^4 - \overset{\circ}{D}_2, B(\tau_2^*))$. Since $F_2 \subset S_2^4$ is a standard projective plane, we can move τ_2^* into ∂D_2 by an isotopy. Then $\tau_2^* \subset \partial D_2$ is a twisted I -bundle over a standard Möbius band in $S^3 = \partial D_2$ that is homeomorphic to a standard solid torus in S^3 . The complement $V = cl(\partial D_2 - \tau_2^*)$ is also a standard solid torus in $S^3 = \partial D_2$. τ_1^* and τ_2^* are identified on annuli $A_1 = cl(\partial \tau_1^* - B(\tau_1^*))$ and $A_2 = cl(\partial \tau_2^* - B(\tau_2^*))$. The core of A_2 is a $(\pm 2, 1)$ -loop on $\partial \tau_2^* \subset S^3$. Let $M \subset S^4$ be the union of τ_1^* and the solid torus V . Note that $\tau_1^* \cap V = A_2 \subset \partial V$. Since the core of A_2 is a $(1, \pm 2)$ -loop on $\partial V \subset S^3$, M is homeomorphic to the handle-body τ_1^* . Hence $F^* = \partial M$ is an unknotted genus $n - 1$ orientable surface in S^4 . This completes the proof.

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