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MINIMAL PRIME IDEALS IN SEMIGROUPS WITHOUT NILPOTENT ELEMENTS

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1. Introduction

Minimal prime ideals of commutative rings have been studied by many authors (see e.g. [1] and [2]). In [4], Koh characterized minimal prime ideals of noncommutative rings with no non zero nilpotent elements. He used this characterization to give a functional representation of rings without nilpotent elements, which extended to these non commutative rings, a well-known sheaf representation of commutative rings with identity (cf. [5], p.149). Minimal prime ideals of commutative semigroups have been investigated in [3]. The aim of this paper is to obtain a characterization of minimal prime ideals of (non commutative) semigroups without nilpotent elements, analogous to the one for the corresponding class of rings, found in [4]. We also define in this paper, the notion of symmetric ideals of a semigroup and establish some of their basic properties.

2. Preliminaries

Throughout this paper, S will denote a semigroup with identity 1, and a zero, 0. The word ideal will always mean a two-sided ideal. Let I be an ideal of S. I is prime if for any a, $b \in S$, $aSb \subseteq I$ implies that either $a \in I$ or $b \in I$. Equivalently, I is prime if and only if for any ideals A and B of S, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$. If I is a prime ideal, then the complement, M, of I is an m-system (that is, for any $a, b \in M$, there is some $x \in S$, so that $axb \in M$). A proper prime ideal I is called minimal prime if there is no prime ideal of S which is properly contained in I. An ideal I is called completely prime if for any $a, b \in S, ab \in I$ implies that either $a \in I$ or $b \in I$. Thus I is completely prime if and only if the complement of I is a subsemigroup of S. An ideal which is completely prime is prime, but the converse is not generally true. In the case of commutative semigroups, however, these concepts coincide. An ideal I is called completely semiprime if for $a \in S$ and n a positive integer, $a^n \in I$ implies that $a \in I$. Thus if S has no non zero nilpotent elements then the zero ideal of S is completely semiprime. We also recall that if I is any ideal of S, then the radical, $\sqrt{I}$, of I is the set $\{x \in S : x^n \in I$ for some positive integer $n\}$.

If S is a commutative semigroup, then it is well-known that $\sqrt{I} = \cap P$, where the intersection runs over all prime ideals P containing I. In general, however, the intersection of all prime ideals containing I is properly contained in $\sqrt{I}$.

3. Symmetric Ideals in Semigroups

DEFINITION 3.1. A right ideal I of S is called symmetric if $abs \in I$ implies that $asb \in I$, for all $a, b, s \in S$.

EXAMPLE 3.2. Let $S = \{0, a_1, a_2, a_3, 1\}$, and define the multiplication on S by the following table:

	0	a_1	a_2	a_3	1
0	0	0	0	0	0
a_1	0	0	a_1	a_1	a_1
a_2	0	0	a_2	a_2	a_2
a_3	0	0	a_3	a_3	a_3
1	0	a_1	a_2	a_3	1

Then S is a semigroup with identity 1 and zero element 0. In S, the ideal $I_1 = \{0, a_2\}$ is a non-symmetric right ideal. However, $I_2 = \{0, a_1\}$ and $I_3 = \{0, a_1, a_2, a_3\}$ are symmetric ideals.

PROPOSITION 3.3. Let I be a symmetric right ideal of S. Then the following statemetns are true:

(1) I is a two-sided ideal.

(2) For each $s \in S$, the set $s^{-1}I = \{t \in S : st \in I\}$ is a symmetric right ideal.

(3) If $a_1 \cdot a_2 \cdot \ldots \cdot a_n \in I$, then for any permutation p of the set $\{1, 2, \ldots, n\}$, also $a_{p(1)} \cdot a_{p(2)} \cdot \ldots \cdot a_{p(n)} \in I$.

PROOF. (1) If $a \in I$ and $t \in S$ then $at \in I$, since I is a right ideal. Hence by the symmetry condition on I, $ta \in I$. Thus I is a left, and hence, a two-sided ideal.

(2) First, we verify that $s^{-1}I$ is a right ideal. Let $a \in s^{-1}I$ and $t \in S$. Then $sa \in I$ and so $(sa)t = s(at) \in I$. Hence $at \in s^{-1}I$. Moreover, if $abt \in s^{-1}I$, for $a, b, t \in S$, then $sabt \in I$. Hence by the symmetry property of I, $satb \in I$. This implies that $atb \in s^{-1}I$, showing that $s^{-1}I$ is also symmetric.

(3) In order to prove (3), it is sufficient to show that $abc\ st \in I$ implies $asc\ bt \in I$, since any permutation is a product of transpositions. From $abc\ st \in I$, we obtain by successive applications of the symmetry on I, $a(st)(bc)$, $(as)c(tb)$, $(asc)bt$ all belong to I.

DEFINITION 3.4. Let Δ be a subsemigroup of S. If A is any subset of S, then we write $\Delta^{-1}A = \bigcup_{\delta\in\Delta} \delta^{-1}A = \{s \in S : \exists\delta \in \Delta \text{ with } \delta s \in A\}$.

PROPOSITION 3.5. Let Δ be a subsemigroup of S. If I is a symmetric right ideal of S, which is disjoint with Δ, then the set of all proper symmetric ideals J containing I such that $\Delta^{-1}J \subseteq J$ contains maximal elements and these are completely prime ideals.

PROOF. First, we verify that $\Delta^{-1}I$ is a right ideal. Let $a \in \Delta^{-1}I$ and $s \in S$. Then $\exists\delta \in \Delta$ such that $\delta a \in I$. Hence $(\delta a)s \in I$. Then we have $\delta(as) \in I$, showing that $as \in \delta^{-1}I \subseteq \Delta^{-1}I$. By Proposition 3.3, each $\delta^{-1}I$ is a symmetric ideal, so $\Delta^{-1}I$, being the union of symmetric ideals, is also a symmetric ideal. It can be easily verified that $I \subseteq \Delta^{-1}I$. Also, we note that $\Delta^{-1}I$ is proper since $1 \notin \Delta^{-1}I$ and that $\Delta^{-1}(\Delta^{-1}I) \subseteq \Delta^{-1}I$. Thus the set of all ideals with the stated properties is nonempty. By Zorn's Lemma, this set contains maximal elements. Let P be one such maximal element. We prove that P is completely prime. Let $a \notin P$. Then $a^{-1}P$ is a proper symmetric ideal containing P. Also, if $\delta \in \Delta$, then we show that $\delta^{-1}a^{-1}P \subseteq a^{-1}P$. Let $b \in \delta^{-1}a^{-1}P$, then $a\delta b \in P$. Hence $\delta ab \in P$ for any $\delta \in \Delta$. In particular, $ab \in P$, that is, $b \in a^{-1}P$. Thus $\delta^{-1}a^{-1}P \subseteq a^{-1}P$. Hence $a^{-1}P$ is the set under consideration. By maximality of P, $a^{-1}P \subseteq P$. This means that for any $t \notin P$, also $at \notin P$. Hence, by contrapositivity, we conclude that P is completely prime.

PROPOSITION 3.6. Let I be a proper symmetric right ideal of S. Then the following sets are equal:

(a) The intersection of all completely prime ideals containing I.

(b) The set of all $x \in S$ for which there is a natural number n, such that $x^n \in I$.

PROOF. (1) $(a) \subseteq (b)$: Suppose for all natural numbers $n, x^n \notin I$. Then the subsemigroup $\Delta = \{1, x, x^2, x^3, ...\}$ is disjoint with I. Hence by Proposition 3.5, there exists a completely prime ideal P containing I such that $\Delta^{-1}P \subseteq P$. This implies that $x \notin P$, for otherwise $1 \in \Delta^{-1}P \subseteq P$.

(2) $(b) \subseteq (a)$: Suppose P is a completely prime ideal containing I and $a \notin P$. Since P is a completely prime ideal, $a \cdot a \notin P$, continuing in this manner, we obtain for any natural number $n, a \cdot a \cdot ... \cdot a \notin P$. Since $I \subseteq P$, it follows that $a^n \notin I$, for any natural number $n \geq 1$.

COROLLARY ([3], Theorem 1.5). If S is a commutative semigroup, then $\sqrt{I}$ is the intersection of all prime ideals containing I.

Let π denote the set of all proper prime ideals of S. We endow π with a topology.

PROPOSITION 3.7. π is a topological space if as open sets we take all sets of the form $\tau(A) = \{P \in \pi : A \not\subseteq P\}$, where A is a right ideal of S.

PROOF.

$$\tau(0) = \{P \in \pi : (0) \not\subseteq P\} = \emptyset.$$

$$\tau(S) = \{P \in \pi : S \not\subseteq P\} = \pi$$

$$\begin{aligned} \bigcup_{i\in I} \tau(A_i) &= \{P \in \pi : \exists A_i \not\subseteq P\} \\ &= \{P \in \pi : \bigcup_{i\in I} A_i \not\subseteq P\} \\ &= \tau(\bigcup_{i\in I} A_i) \end{aligned}$$

and finally

$$\begin{aligned} \tau(A) \cap \tau(B) &= \{P \in \pi : A \not\subseteq P \quad \text{and} \quad B \not\subseteq P\} \\ &= \{P \in \pi : AB \not\subseteq P\} \\ &= \tau(AB). \end{aligned}$$

The space π will be called the *prime spectrum* of S.

4. Semigroups without Nilpotent Elements

LEMMA 4.1. Let K be a completely semiprime ideal of S. Then each of the following is true:

(i) If $ab \in K$ then $ba \in K$.
(ii) If $ab \in K$ and $x \in S$ then $axb \in K$.
(iii) If $ab^n \in K$, where n is a positive integer, then $ab \in K$.

PROOF.

(i) $ab \in K \Rightarrow (ba)^2 \in K \Rightarrow ba \in K$.
(ii) $ab \in K \Rightarrow ba \in K \Rightarrow bax \in K \Rightarrow (axb)^2 \in K \Rightarrow axb \in K$.
(iii) This is trivial for $n = 1$. Suppose $n > 1$. Then we have $ab^n \in K \Rightarrow (b^{n-1}ab)^2 \in K \Rightarrow b^{n-1}ab \in K \Rightarrow (b^{n-1}a)^2 \in K \Rightarrow b^{n-1}a \in K \Rightarrow ab^{n-1} \in K$. The proof can be finished by induction.

LEMMA 4.2. A completely semiprime ideal K is a symmetric ideal.

PROOF. Let $abs \in K$ for $a, b, s \in S$. Then a repeated application of Lemma 4.1 (ii) implies that $(asb)^2 \in K$. Hence $asb \in K$.

LEMMA 4.3. Let S be a semigroup with no non zero nilpotent elements. If $a^n y = 0$, for some integer n and $a, y \in S$ then $ay = 0$.

PROOF. We prove by using induction on n. The result is obviously true for $n = 1$. We assume $n > 1$ and let $a^n y = 0$. Then we have $(aya^{n-1})^2 = aya^{n-1} \cdot aya^{n-1} = 0$. This implies that $aya^{n-1} = 0$, since S has no non zero nilpotent elements. Again, $(ya^{n-1})^2 = ya^{n-1} \cdot ya^{n-1} = ya^{n-2}(aya^{n-1}) = 0$. Hence $ya^{n-1} = 0$. Since S has no non zero nilpotent elements, (0) is a completely semiprime ideal. Hence by Lemma 4.2, (0) is a symmetric ideal. Therefore, by Lemma 4.1 (i), $a^{n-1}y = 0$. Hence by induction, we conclude that $ay = 0$.

LEMMA 4.4. Let M be an m-system of S. If M does not intersect the completely semiprime ideal K, then there exists an ideal P which is maximal in the set of those completely semiprime ideals which contain K and do not intersect M. Any such ideal P is completely prime.

PROOF. The existence of such an ideal P is an immediate consequence of Zorn's Lemma. We show that P is completely prime. For each $a \in S$, $a \notin P$, let us define $a^{-1}P = \{b : b \in S \text{ and } ab \in P\}$. In view of Lemma 4.1 (ii), we see that $a^{-1}P$ is an ideal of S. Moreover $P \subseteq a^{-1}P$. We now verify that $a^{-1}P$ is a completely semiprime ideal. Suppose $t^n \in a^{-1}P$, where n is a positive integer. Then $at^n \in P$. Hence by Lemma 4.1 (iii), applied to the completely semiprime ideal P, we see that $at \in P$ and so $t \in a^{-1}P$. Hence $a^{-1}P$ is a completely semiprime ideal. Next, we show that $a^{-1}P = P$ for each $a \notin P$. We consider, separately, two cases:

CASE 1. $a \notin P$, $a \in M$. Suppose $a^{-1}P \cap M \neq \emptyset$. Let $b \in a^{-1}P \cap M$. Then $b \in a^{-1}P$. Hence $ab \in P$. Hence by Lemma 4.1 (ii) $axb \in P$ for each $x \in S$. On the other hand, since $a \in M$ and $b \in M$, and M is an m-system, so $\exists t \in S$ such that $atb \in M$. Hence $atb \in P \cap M$, contradicting the fact that $M \cap P = \emptyset$. Hence $a^{-1}P \cap M = \emptyset$. This implies that $a^{-1}P$ is a member of the set of ideals of which P is a maximal element, those ideals which contain K, are completely semiprime, and do not intersect M. Since $P \subseteq a^{-1}P$, we conclude that $a^{-1}P = P$.

CASE 2. $a \notin P$, $a \notin M$. Suppose $a^{-1}P \cap M \neq \emptyset$. Again we seek a contradiction. Let $c \in a^{-1}P \cap M$. Then $c \in a^{-1}P \Rightarrow ac \in P$. Hence by Lemma 4.1 (i), $ca \in P$. This means that $a \in c^{-1}P$. Since $c \in M$, we know that $c \notin P$, since $M \cap P = \emptyset$. Hence, by Case 1, we have $c^{-1}P = P$ and $a \in P$. But this contradicts the assumption that $a \notin P$, and we conclude that $a^{-1}P \cap M = \emptyset$. As in Case 1, it again follows that $a^{-1}P = P$. Now suppose $vu \in P$ with $u \notin P$. Then $u^{-1}P = P$ and $u \in u^{-1}P = P$. This completes the proof of the lemma.

PROPOSITION 4.5. Let S be a semigroup with no non zero nilpotent elements. If P is a prime ideal of S, then $O_p = \{s \in S : sa = 0, \text{ for some } a \notin P\}$ is an ideal, $O_p \subseteq P$, and the Rees factor semigroup S/O_p has no non zero nilpotent elements.

PROOF. First we verify that O_p is an ideal. Let $x \in O_p$ and $s \in S$. Then $\exists a \notin P$ such that $xa = 0$. Hence $sxa = 0$. This mean that $sx \in O_p$. Also, since S has no non zero nilpotent elements, (0) is a completely semiprime ideal. Hence (0) is a symmetric ideal by Lemma 4.2. Since $xa = 0$, $xas = 0$. Hence by symmetry property, $xsa = 0$. This implies that $xs \in O_p$. This proves that O_p is an ideal. We now prove that $O_P \subseteq P$. Let $x \in O_p$. Then $xy = 0$ for some $y \notin P$. But $xy = 0$ implies $xsy = 0$ $(s \in S)$ by Lemma 4.1 (ii). Hence $xSy \subseteq P$. Since P is prime and $y \notin P$, it follows that $x \in P$. Finally, we show that S/O_p contains no non zero nilpotent elements. Suppose $\exists a \in S$ such that $a^n \in O_p$ for some integer n. Then $\exists y \notin P$ such that $a^n y = 0$. Hence by Lemma 4.3, $ay = 0$. This implies that $a \in O_p$. Hence S/O_p has no non zero nilpotent elements.

THEOREM 4.6. Let S be a semigroup with no non zero nilpotent elements. An ideal P is a minimal prime ideal if and only if $P = O_p$.

PROOF. (Adapted from [4]) (1) Suppose P is a minimal prime ideal and assume that $P \neq O_P$. Since, by Proposition 4.5, we know that $O_p \subseteq P$, it follows that $\exists a \in P$ such that $a \notin O_p$. Let $M = S \backslash P$. Then M is an m-system (that is, for $a, b \in M$, $\exists x \in S$ such that $axb \in M$). Let $S = \{a, a^2, a^3, \ldots\}$, and $T = \{t \in S : t \neq 0, t = a^{i_0} x_0 a^{i_1} x_1 \ldots a^{i_n} x_n a^{i_{n+1}}$ for some non negative integer n, where $x_j \in M$ for $j = 0, 1, 2, \ldots, n$ and i_0, i_{n+1} are non negative integers and $i_1, \ldots, i_n$ are positive integers$\}$. By definition, $a^0 = 1$. We will prove that the set $K = M \cup S \cup T$ is an m-system. Clearly, $0 \notin K$. Let $x, y \in K$. If $x \in M$ and $y \in M$ then $\exists s \in S$ such that $xsy \in M$. In case $x \in M$ and $y \in S$, then $xy = xa^n$ for some positive integer n. If $xy = 0$ then $xa^n = 0$. Hence by Lemma 4.1 (iii), $xa = 0$. But then $a \in O_p$, which is impossible. Hence $xy \neq 0$. Thus there exists some $s \in S$ such that $xsy \in T$ and hence $xsy \in K$. Now suppose $x \in M$ and $y \in T$. Let $y = a^{i_0} x_0 a^{i_1} x_1 \ldots a^{i_n} x_n a^{i_{n+1}}$ for some $x_i \in M$ and $i = 0, \ldots, n$. Since M is an m-system with $x_0, \ldots, x_n \in M$ there exists $s_0, \ldots, s_n$ in S such that $xs_0x_0s_1x_1 \ldots s_nx_n \in M$. Let $t = xs_0x_0s_1x_1 \ldots s_nx_n$. If $ty \neq 0$ then $ty \in T$ and hence $xsy \in K$ for some $s \in S$. Now suppose $ty = 0$. Then we have $xs_0x_0s_1x_1 \ldots s_nx_n \cdot a^{i_0} x_0 a^{i_1} \ldots x_n a^{i_{n+1}} = 0$. Since S has no non zero nilpotent elements, (0) is a completely semiprime ideal, and hence, a symmetric ideal by Lemma 4.2. Hence by Proposition 3.3 (iii), we have $(a^{i_0} \cdot a^{i_1} \cdot \ldots a^{i_n} \cdot a^{i_{n+1}})(xs_0x_0s_1x_1 \ldots s_nx_n)(x_0x_1 \ldots x_n) = 0$. Let $i_0 + i_1 + i_2 + \ldots + i_n + i_{n+1} = i$. Then we have $a^i t = 0$ for some $t \in M$. Hence by Lemma 4.1 (iii), $at = 0$.

Since $t \notin P$, we have $a \in O_p$. But this is a contradiction because we assumed that $a \notin O_p$. We can similarly show that $x \in S \cup T$ also implies that K is an m-system. Now let A be an ideal of S which is maximal with respect to the property that $A \cap K = \emptyset$. Then by Lemma 4.4, A is a prime ideal and $A \subseteq P$ and $A \neq P$. This contradicts the minimality of P. Hence $P = O_p$.

(2) Conversely, suppose $P = O_p$ and assume P' is a prime ideal contained in P. Then for any $x \in P$, $\exists a \notin P$ such that $xa = 0 \in P'$. This implies that $x \in P'$. Hence $P' = P$, and P is a minimal prime ideal.

COROLLARY. Each element of a minimal prime ideal is a zero divisor.

PROOF. Let $x \in P$, where P is a minimal prime ideal. Then $x \in O_p$. Hence $\exists a \notin P$ such that $xa = 0$, and hence $ax = 0$. Since $a \notin P, a$ is a non zero element and so x is a zero divisor.

PROPOSITION 4.7. Let π_0 be the subspace of π which consists of all minimal prime ideals of S. Then π_0 is a Hausdorff space.

PROOF. Let $P_1, P_2 \in \pi_0$ with $P_1 \neq P_2$. Then from the above theorem, $P_1 = O_{p_1}$ and $P_2 = O_{p_2}$. Hence $O_{p_1} \nsubseteq P_2$. Let $x \in O_{p_1}$ so that $x \notin P_2$. Then $xs = 0$ for some $s \notin P_1$. It follows that $SxSsS = 0$. Hence $\tau(SxS) \cap \tau(SsS) = \emptyset$ and $P_1 \in \tau(SsS)$ and $P_2 \in \tau(SxS)$. Thus π_0 is a Hausdorff space.

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