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GEODESIC KNOTS IN A HYPERBOLIC 3-MANIFOLD

Dedicated to Professor Kunio Murasugi
on his 60th birthday

By Tsuyoshi SAKAI

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1. Introduction

A knot in a hyperbolic 3-manifold is called a geodesic knot if it is ambient isotopic to a simple closed geodesic (that is, a geodesic homeomorphic to a circle) (Sakai [1]).

In this note, we wish to consider the following:

PROBLEM *Characterize geodesic knots.*

According to R.H. Fox [2], a fundamental invariant of a knot k in general 3-manifold M is a group

$$G = Ker(i_* : \pi_1(M - k) \to \pi_1(M)),$$

where $i : M - k \to M$ is an inclusion (See also Brody [3]).

The following is our basic observation:

THEOREM 1. *Let M be a closed orientable hyperbolic 3-manifold and let γ be a simple closed geodesic in M. Then*

$$G = Ker(i_* : \pi_1(M - \gamma) \to \pi_1(M))$$

is a free group.

Theorem 2 below is a purely topological side of this note.

THEOREM 2. *Suppose that M is a closed orientable simple 3-manifold whose universal cover is R^3. Let α be a non-contractible knot in M satisfying the following two conditions:*

(1) *α is primitive (i.e. if α is freely homotopic to β^p for some loop β and an integer p, then the absolute value of p is equal to 1).*

(2)
$$G = Ker(i_* : \pi_1(M - \alpha) \to \pi_1(M))$$
is a free group.

Then α is a simple knot in M.

By Thurston's uniformization theorem [4][5] together with Theorems 1 and 2 above, we obtain:

THEOREM 3. *A geodesic knot in a closed orientable hyperbolic 3-manifold is a hyperbolic knot.*

In Kojima [6], link version of Theorem 3 is refered and a compact proof is given.

THEOREM 4. *Let α be a knot in a hyperbolic 3-manifold M satisfying the following three conditions:*

(1) *α is freely homotopic to a simple closed geodesic γ in M.*

(2) *$Ker(i_* : \pi_1(M - \alpha) \to \pi_1(M))$ is free.*

(3) *α is a satellite knot of γ (that is, there exists a tubular neighbourhood of γ in M that contains α in its interior).*

Then α is a geodesic knot (in fact, ambient isotopic to γ).

This is a corollary to the proof of Theorem 2.

Theorems 1, 3 and 4 give partial answers to our problem.

These results are reported in [1].

2. Proof

PROOF of THEOREM 1

Let $p : \widetilde{M} \to M$ be the universal cover of M. Then $p|\widetilde{M} - p^{-1}(\gamma) : \widetilde{M} - p^{-1}(\gamma) \to M - \gamma$ is the covering associated to $G = Keri_*$. Therefore $G = \pi_1(\widetilde{M} - p^{-1}(\gamma))$.

We consider $\widetilde{M} = H^3$ as Poincare disc model of hyperbolic 3-space. Note that $p^{-1}(\gamma)$ is a disjoint union of simple geodesics in $\widetilde{M} = H^3$. We will show that $\pi_1(H^3 - p^{-1}(\gamma))$ is a free group.

Let $\{\gamma_\lambda\}(\lambda \in \Lambda)$ denote the components of $p^{-1}(\gamma)$. Then we can construct a sequence $\{B_n\}$ $(n = 1, 2, \ldots)$ of 3-balls in H^3 with the following properties:

(1) For each n,
$$B_n = \{x \in H^3 | \, \|x\| \leq \varepsilon_n\}$$
where $\| \ \|$ is the Euclidean norm and $\{\varepsilon_n\}$ $(n = 1, 2, \ldots)$ is a sequence of numbers

such that

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n < \cdots < 1$$

and $\varepsilon_n \longrightarrow 1 (n \longrightarrow \infty)$.

(2) For each n, only finite numbers of γ_λ have non-empty intersections with B_n.

(3) If $\gamma_\lambda \cap intB_n = \emptyset$, then $\gamma_\lambda \cap B_n = \emptyset$.

By (2), Λ is countable, so we can number $\{\gamma_\lambda\}$ as

$$\{\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots\}$$

where $\gamma_k \cap B_n \neq \emptyset$ if and only if $1 \leq k \leq k(n)$. Of course inequalities

$$1 \leq k(1) \leq k(2) \leq \cdots$$

hold.

Let X_n denote $B_n \cap (H^3 - p^{-1}(\gamma))$ and let $i_n : X_n \to X_{n+1}$ denote an inclusion. Then

$$\pi_1(X_1) \to \pi_1(X_2) \to \cdots \to \pi_1(X_n) \to \cdots$$

is an inductive system.

We will show that $(i_n)_*$ is injective and $(i_n)_*(\pi_1(X_n))$ is a free factor of $\pi_1(X_{n+1})$. It is sufficient to construct a free basis

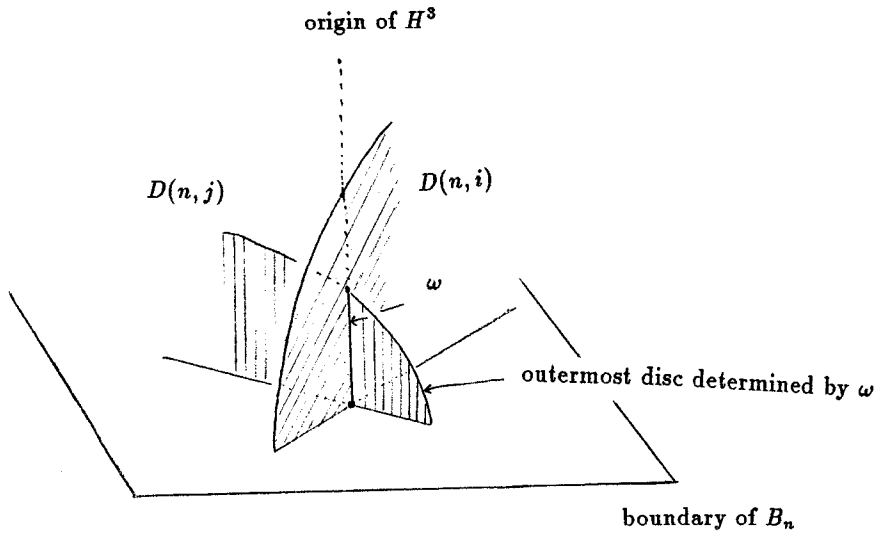
$$\{e(n,1), e(n,2), \ldots, e(n,k(n))\}$$

of $\pi_1(X_n)$ such that $(i_n)_*(e(n,k)) = e(n+1,k)$.

At first, we consider the plane determined by γ_k and the origin of H^3 for each k. Deviding this plane by γ_k, let D_k denote the component that does not contain the origin (If γ_k contains the origin, we choose any component.). And let $D(n,k)$ denote $B_n \cap D_k$ and let $\gamma(n,k)$ denote $B_n \cap \gamma_k$, where $1 \leq k \leq k(n)$.

We define a height function H on H^3 by $H(x) = 1 - \|x\|$. An arc ω in H^3 is called monotonic if the restriction of the height function H on ω is injective. For a monotonic arc ω, the maximal valule of H on ω is called the top height of ω.

Step 1. Construction of a free basis for $\pi_1(X_n)$. We will deform $\{D(n,k)\}$ to $\{\widetilde{D}(n,k)\}$ so that $\{\gamma(n,k)\}$ are unchanged and $\{\widetilde{D}(n,k)\}$ are mutually disjoint. Among the components of $\{D(n,i) \cap D(n,j)\}$ $(1 \leq i,j \leq k(n))$, let ω be one of arcs with minimal top heights (by construction, each arc is monotonic). By a height preserving isotopy in B_n using an outermost disc determined by ω, we deform $\{D(n,k)\}$ to $\{D'(n,k)\}$ so as to remove ω (see figure).


origin of H^3
$D(n,j)$
$D(n,i)$
ω
outermost disc determined by ω
boundary of B_n


a height preserving isotopy push along ω

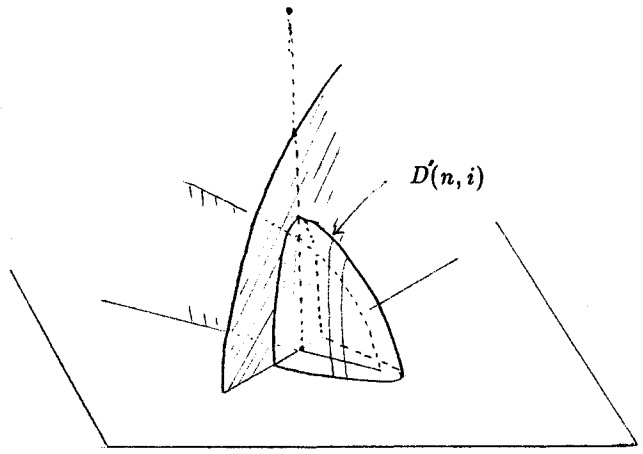

$D'(n,i)$

Fig.

By repeating this process, we obtain $\{\widetilde{D}(n,k)\}$. Now choosing a point $*$ in $B_n - \cup\{\widetilde{D}(n,k)|1 \leq k \leq k(n)\}$ and connecting $*$ to small meridians of $\{\gamma(n,k)\}$ by paths in $B_n - \cup_k \widetilde{D}(n,k)$, we construct loops $\{\alpha(n,1),\ldots,\alpha(n,k(n))\}$. Then the homotopy classes $\{e(n,1),\ldots,e(n,k(n))\}$ of these loops form a basis of $\pi_1(X_n)$.

Step 2. Construction of a free basis for $\pi_1(X_{n+1})$. At first, consider the components of $\{D(n+1,i)\cap D(n+1,j)\}$ $(1 \leq i,j \leq k(n+1))$ that are contained in $B_{n+1} - B_n$. By height preserving isotopies as in step 1 but with supports in $B_{n+1} - B_n$, we deform $\{D(n+1,k)\}$ to $\{D^\sharp(n+1,k)\}$ so as to remove the components of $\{D(n+1,k)\cap D(n+1,j)\}$ that are contained in $B_{n+1} - B_n$. Note that $D^\sharp(n+1,k)\cap B_n = D(n,k)$ and each component of $D^\sharp(n+1,i)\cap D^\sharp(n+1,j)$ contains one of the components of $D(n,i)\cap D(n,j)$ in one to one manner, when $1 \leq k,i,j \leq k(n)$. Next we deform $\{D^\sharp(n+1,k)\}$ to $\{\widetilde{D}(n+1,k)\}$ so as to remove the remaining intersections and $\widetilde{D}(n+1,k)\cap B_n = \widetilde{D}(n,k)$ for each k, $1 \leq k \leq k(n)$, where the deformations are natural extentions of isotopies in B_n that deform $\{D(n,k)\}$ to $\{\widetilde{D}(n,k)\}$. Then we choose $\alpha(n+1,k) = \alpha(n,k)$ for $1 \leq k \leq k(n)$ and for $k(n)+1 \leq k \leq k(n+1)$, as in step 1, we choose $\alpha(n+1,k)$ to be a loop obtained by connecting $*$ to a small meridian of $\gamma(n+1,k)$ by a path in $B_{n+1} - \cup\{\widetilde{D}(n+1,k)|1 \leq k \leq k(n+1)\}$. Then the homotopy classes $\{e(n+1,1),\ldots,e(n+1,k(n+1))\}$ of $\{\alpha(n+1,1),\ldots,\alpha(n+1,k(n+1))\}$ form a desired free basis for $\pi_1(X_{n+1})$.

Therefore, since $\pi_1(H^3 - p^{-1}(\gamma))$ is the inductive limit of the system $\pi_1(X_n)$, $\pi_1(H^3 - p^{-1}(\gamma))$ is a free group.

PROOF of THEOREM 2

For a given incompressible torus T in $M - \alpha$, we will prove that T bounds a solid torus V in M containing α in its interior and α is ambient isotopic to a core of V.

Let $p : \widetilde{M} \to M$ be the universal cover and $\widetilde{T}$ be a component of $p^{-1}(T)$. Then, as in the proof of Theorem 1, $G = Keri_* = \pi_1(\widetilde{M} - p^{-1}(\alpha))$ and $p|\widetilde{T} : \widetilde{T} \to T$ is a connected covering.

1. $\widetilde{T}$ is an open annulus.

In fact, since T is a torus, $\widetilde{T}$ is R^2 or $S^1 \times R$ or $S^1 \times S^1$. Since T is compressible in M, $\widetilde{T}$ is not R^3. Since $\widetilde{T}$ is incompressible in $\widetilde{M} - p^{-1}(\alpha)$, $\pi_1(\widetilde{M} - p^{-1}(\alpha)) \cong Keri_*$ is free and $\pi_1(T) \cong Z + Z$, $\widetilde{T} = S^1 \times R$.

2. T bounds a solid torus V in M containing α in its interior.

Since M is simple, T is compressible in M. Let D be a compressing disc of T. Since T is incompressible in $M - \alpha$, D must intersect with α. Let $\widetilde{D}$ be a

component of $p^{-1}(D)$ that touches with $\tilde{T}$. Since the restriction of $\widetilde{M} \to M$ to $\tilde{T} \to T$ is an infinite cyclic covering, we choose a covering translation t of $\widetilde{M}$ that restricts to a generator of the covering translation of $\tilde{T}$. Let A be an annulus on $\tilde{T}$ bounded by the circles $\partial\tilde{D}$ and $\partial(t\tilde{D})$. Then $\tilde{D} \cup A \cup (t\tilde{D})$ is a 2-sphere in $\widetilde{M} = R^3$, so bounds a 3-cell B^3. Then $V = p(B)$ is the desired one.

3. Let j be the inclusion $V - \alpha \to V$. Then

$$Ker(j_* : \pi_1(V - \alpha) \to \pi_1(V))$$

is free.

Considering the next diagram, by Van Kampen's theorem, we know

$$\pi_1(V - \alpha) \to \pi_1(M - \alpha)$$

is injective:

$$\begin{array}{ccc} \pi_1(T) & \longrightarrow & \pi_1(M - intV) \\ \downarrow & & \downarrow \\ \pi_1(V - \alpha) & \longrightarrow & \pi_1(M - \alpha) \end{array}$$

By the diagram

$$\begin{array}{ccc} \pi_1(M - \alpha) & \longrightarrow & \pi_1(M) \\ \uparrow & & \uparrow \\ \pi_1(V - \alpha) & \longrightarrow & \pi_1(V) \end{array}$$

$Kerj_*$ is contained in $Keri_*$. Since $Keri_*$ is free, $Kerj_*$ is also free.

4. Since α is primitive, α is freely homotopic to a core of V.
5. Therefore, by the following lemma, our proof is completed.

LEMMA. *Let V be a solid torus and α be a knot in int V. Suppose that α is homotopic to a core of V and $Ker(i_* : \pi_1(V - \alpha) \to \pi_1(V))$ is free. Then α is ambient isotopic to a core of V.*

(Proof) Let $\tilde{V} \to V$ be the universal cover and $\tilde{\alpha} = p^{-1}(\alpha)$. Since $\alpha \simeq$ (*core of* V), $H_1(\tilde{V} - \tilde{\alpha}) \cong Z$. Since $Keri_* \cong \pi_1(V - \alpha)$ is free, $Keri_* \cong Z$. So we obtain a group extension:

$$1 \to Keri_* \to \pi_1(V - \alpha) \to \pi_1(V) \to 1$$

Therefore, by Stallings' fibration theorem [7], the proof is completed.

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