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ON THE RIGHT ESSENTIAL SPECTRA OF QUASISIMILAR OPERATORS

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Abstract

In this article, we prove that if T and S are quasisimilar operators, then each nonempty closed-and-open subset of $\sigma_{re}(T)$ intersects $\sigma_N(T)\cap\sigma_N(S)$. As a simple corollary we give an affirmative answer to a question of Fialkow.

Let $L(X)$ denote the algebra of all operators acting on the complex, infinite dimensional Banach space X. Recall that $T\in L(X)$ and $S\in L(Y)$ are quasisimilar if there exist injective operators $A: X\to Y$ and $B: Y\to X$ with dense range such that $AT=SA$ and $TB=BS$.

For $T\in L(X)$, we write $\mathrm{nul}(T)=\mathrm{dimker}(T)$, $\mathrm{nul}(T^*)=\mathrm{dimker}(T^*)$ and $\mathrm{ind}(T)=\mathrm{nul}(T)-\mathrm{nul}(T^*)$ if at least one of $\mathrm{nul}(T)$ and $\mathrm{nul}(T^*)$ is finite.

Recall that $A\in L(X)$ is a Semi-Fredholm operator if $\mathrm{ran}(A)$ is closed and at least one of $\mathrm{nul}(A)$ and $\mathrm{nul}(A^*)$ is finite, and that $A\in L(X)$ is Fredholm operator if $\mathrm{ran}(A)$ is closed and both $\mathrm{nul}(A)$ and $\mathrm{nul}(A^*)$ are finite. For $T\in L(X)$, the essential spectrum $\sigma_e(T)=\{\lambda\in C: T-\lambda \text{ is not Fredholm}\}$, the right essential spectrum $\sigma_{re}(T)=\{\lambda\in C: \text{either } T-\lambda \text{ is not Semi-Fredholm or } \mathrm{ind}(T-\lambda)=-\infty\}$ and left essential spectrum $\sigma_{le}(T)=\{\lambda\in C: \text{either } T-\lambda \text{ is not Semi-Fredholm or } \mathrm{ind}(T-\lambda)=\infty\}$. Let

$$\sigma_{lre}(T)=\{\lambda\in C: T-\lambda \text{ is not Semi-Fredholm}\},$$

$$\psi_n(T)=\{\lambda\in C:\ \mathrm{ran}(T-\lambda) \text{ is closed and } \mathrm{ind}(T-\lambda)=n\}$$
$$(n=0,\pm1,\dots,\pm\infty)$$

$$H_n(T)=\{\lambda\in C:\ \mathrm{ind}(T-\lambda)=n\}\ (n=0,\pm1,\dots,\pm\infty),$$

$$H_{mn}(T)=\{\lambda\in C:\ \mathrm{nul}(T-\lambda)=m \text{ and } \mathrm{nul}(T-\lambda)^*=n\}$$
$$(m,n=0,1,\dots,\infty)$$

Let D° denote the interior of subset D of C and ∂D denote boundary of D. Let $\sigma_N(T)=\sigma_{le}(T)\backslash H_{\infty\infty}(T)^\circ$.

If $T\in L(X)$ and $S\in L(Y)$ are quasisimilar, then for each complex λ $\mathrm{nul}(T-\lambda)=\mathrm{nul}(S-\lambda)$ and $\mathrm{nul}(T-\lambda)^*=\mathrm{nul}(S-\lambda)^*$. It follows that $H_n(T)=H_n(S)$ and $H_{mn}(T)=H_{mn}(S)$.

LEMMA 1[1]. *If $T \in L(X)$ and $S \in L(Y)$ are quasisimilar, then each component of $\sigma_e(T)$ intersects $\sigma_e(S)$, and viceversa.*

THEOREM 2. *If $T \in L(X)$ and $S \in L(Y)$ are quasisimilar, then each nonempty closed-and-open subset of $\sigma_{re}(T)$ intersects $\sigma_N(T) \cap \sigma_N(S)$.*

PROOF. Let τ be any nonempty closed-and-open subset of $\sigma_{re}(T)$.

CASE 1. τ is not open subset of $\sigma_e(T)$. There exist $t \in \tau$ and a sequence $\{t_n\} \subset \sigma_e(T)\backslash\tau$ such that $\operatorname{Lim} t_n = t$. Since τ is an open subset of $\sigma_{re}(T)$, we may assume that $\{t_n\} \subset \sigma_e(T)\backslash\sigma_{re}(T)$. Now that $t \in \tau \subset \sigma_{re}(T)$, $t \notin \sigma_e(T)\backslash\sigma_{re}(T)$, because $t \in \partial(\sigma_e(T)\backslash\sigma_{re}(T))$. Since $\sigma_N(T) \cap \sigma_N(S)$ is closed and

$$\sigma_e(T)\backslash\sigma_{re}(T) = \psi_\infty(T) \subset H_\infty(T) = H_\infty(S) \subset \sigma_N(T) \cap \sigma_N(S),$$

we deduce that $t \in \partial(\sigma_e(T)\backslash\sigma_{re}(T)) \subset \sigma_N(T) \cap \sigma_N(S)$. Therefore

$$\tau \cap \sigma_N(T) \cap \sigma_N(S) \neq \phi.$$

CASE 2. τ is an open subset of $\sigma_e(T)$. Since τ is a closed subset of the closed set $\sigma_{re}(T)$, it is also a closed subset of $\sigma_e(T)$ and hence τ is a closed-and-open subset of $\sigma_e(T)$. A component say τ_1, of τ is also a component of $\sigma_e(T)$. Lemma 1 implies that $\tau_1 \cap \sigma_e(S) \neq \phi$. It follows that $\tau \cap \sigma_e(S) \neq \phi$.

Now suppose that $\tau \cap \sigma_N(T) \cap \sigma_N(S) = \phi$. Then

$$\begin{aligned} \tau \cap \sigma_e(S) &\subset (\sigma_{re}(T) \cap \sigma_e(s))\backslash(\sigma_N(T) \cap \sigma_N(S)) \\ &\subset (\sigma_{re}(T)\backslash\sigma_N(T)) \cup (\sigma_e(S)\backslash\sigma_N(S)) \\ &\subset D, \end{aligned}$$

where $D = [H_{-\infty}(S) \cup H_{\infty\infty}(S)]^\circ = [H_{-\infty}(T) \cup H_{\infty\infty}(T)]^\circ$, and $\tau \cap D \neq \phi$. We are now going to show that $\tau \cap D$ is closed-and-open in C. D is an open set and τ is a closed-and-open subset of $\sigma_{re}(T)$, hence $\forall\lambda \in \tau \cap D$ there exists $r > 0$ such that $O(\lambda, r) \subset D$ and $O(\lambda, \tau) \cap \sigma_{re}(T) \subset \tau$. Thus

$$O(\lambda, r) = O(\lambda, r) \cap D \subset O(\lambda, r) \cap \sigma_{re}(T) \subset \tau,$$

and we conclude that $O(\lambda, r) \subset \tau \cap D$. Therefore $\tau \cap D$ is open in C. $D \subset \sigma_e(S)$ and D is open, hence

$$\partial D \subset \sigma_e(S)\backslash D \subset \sigma_N(S).$$

Similarly $\partial D \subset \sigma_N(T)$. It follows that $\partial D \subset \sigma_N(T) \cap \sigma_N(S)$.

Suppose that $\{t_n\} \subset \tau \cap D$, $\lim_n t_n = t$. Since τ is closed.

We have

$$\begin{aligned} t \in \tau \cap (D \cup \partial D) &= (\tau \cap D) \cup (\tau \cap \partial D) \\ &\subset (\tau \cap D) \cup (\tau \cap \sigma_N(T) \cap \sigma_N(S)) \\ &= \tau \cap D. \end{aligned}$$

Hence $\tau \cap D$ is closed in C. We have now shown that $\tau \cap D$ is closed-and-open in C, which contradicts $\tau \cap D \neq \phi$, $\tau \cap D \neq C$. Therefore $\tau \cap \sigma_N(T) \cap \sigma_N(S) \neq \phi$.

THEOREM 3. *If $T \in L(X)$ and $S \in L(Y)$ are quasisimilar and Ω is a subset of C such that*

$$\sigma_{re}(T) \cap \Omega \neq \phi, \quad \textit{but} \quad \sigma_{re}(T) \cap \partial\Omega = \phi,$$

then

$$\Omega \cap \sigma_N(T) \cap \sigma_N(S) \neq \phi.$$

PROOF. Let λ be any point of $\sigma_{re}(T) \cap \Omega$, and let τ be the component of $\sigma_{re}(T)$ containing λ. We have $\tau \subset \Omega$ since $\sigma_{re}(T) \cap \partial\Omega = \phi$.

Since $\sigma_{re}(T)$ is a compact set of C, every component of $\sigma_{re}(T)$ is the intersection of all the closed-and-open subsets of $\sigma_{re}(T)$ containing it. Let $\tau = \bigcap_{\alpha \in I} \tau_\alpha$ (τ_α is closed-and-open in $\sigma_{re}(T)$ and contains τ). Theorem 2 implies that $\{\tau_\alpha \cap \sigma_N(T) \cap \sigma_N(S) : \alpha \in I\}$ is a family of closed subsets of $\sigma_N(T) \cap \sigma_N(S)$ have the finite intersection property. By compactness of $\sigma_N(T) \cap \sigma_N(S)$, it is clear that

$$\tau \cap \sigma_N(T) \cap \sigma_N(S) = \bigcap_{\alpha \in I} (\tau_\alpha \cap \sigma_N(T) \cap \sigma_N(S)) \neq \phi.$$

Therefore $\Omega \cap \sigma_N(T) \cap \sigma_N(S) \neq \phi$.

THEOREM 4. *If $T \in L(X)$ and $S \in L(Y)$ are quasisimilar and Ω is a subset of C such that*

$$\sigma_e(T) \cap \Omega \neq \phi, \quad \textit{but} \quad \sigma_e(T) \cap \partial\Omega = \phi,$$

then $\Omega \cap \sigma_F(T) \cap \sigma_F(S) \neq \phi$.

(*where* $\sigma_F(\cdot) = \sigma_{lre}(\cdot) \backslash F(\cdot)$, $F(\cdot) = [H_\infty(\cdot) \cup H_{-\infty}(\cdot) \cup H_{\infty\infty}(\cdot)]^\circ$)

PROOF. Since $\sigma_e(T) \cap \Omega \neq \phi$, but $\sigma_e(T) \cap \partial\Omega = \phi$, deduce that $\partial\sigma_e(T) \cap \Omega \neq \phi$. Let α be any point of $\partial\sigma_e(T) \cap \Omega$. Clearly, $\alpha \in \sigma_{lre}(T)$ and $F(T) = [H_\infty(T) \cup H_{-\infty}(T) \cup H_{\infty\infty}(T)]^\circ \subset \sigma_e(T)^\circ$, hence $\alpha \in \sigma_{lre}(T) \backslash F(T) = \sigma_F(T)$.

CASE 1. $\alpha \in \sigma_e(S)$. Suppose that $\alpha \notin \sigma_F(T) \cap \sigma_F(S)$, then

$$\alpha \in \sigma_e(S) \backslash \sigma_F(S) = F(S) = F(T) \subset \sigma_e(T)^\circ,$$

which contradicts $\alpha \in \partial\sigma_e(T)$. It follows that $\Omega \cap \sigma_F(T) \cap \sigma_F(S) \neq \phi$.

CASE 2. $\alpha \notin \sigma_e(S)$. Let τ be the component of $\sigma_e(T)$ containing α. By lemma 1, $\tau \cap \sigma_e(S) \neq \phi$. Let β be any point of $\tau \cap \sigma_e(S)$. Since that $\alpha \notin \sigma_e(S)$ and $\beta \in \sigma_e(S)$, we deduce that $\tau \cap \partial\sigma_e(S) \neq \phi$. Let λ be any point of $\tau \cap \partial\sigma_e(S)$, then $\lambda \in \sigma_F(S)$. Suppose that $\lambda \notin \sigma_F(T) \cap \sigma_F(S)$, we have

$$\lambda \in \sigma_e(T) \backslash \sigma_F(T) = F(T) = F(S) \subset \sigma_e(S)^\circ$$

which contradicts $\lambda \in \partial\sigma_e(S)$. Therefore $\lambda \in \sigma_F(T) \cap \sigma_F(S)$.

Suppose that $\lambda \notin \Omega$. Since $\alpha \in \tau \cap \Omega$ and $\lambda \in \tau$, we have $\tau \cap \partial\Omega \neq \phi$, which contradicts $\sigma_e(T) \cap \partial\Omega = \phi$. Therefore $\lambda \in \Omega$. It follows that $\Omega \cap \sigma_F(T) \cap \sigma_F(S) \neq \phi$.

COROLLARY 5. *If $T \in L(X)$ and $S \in L(Y)$ are quasisimilar and Ω is a subset of C such that*

$$\sigma_e(T) \cap \Omega \neq \phi, \quad \textit{but} \quad \sigma_e(T) \cap \partial\Omega = \phi,$$

then $\Omega \cap \partial(\sigma_F(T) \cap \sigma_F(S)) \neq \phi$.

PROOF. By theorem 4, $\Omega \cap \sigma_F(T) \cap \sigma_F(S) \neq \phi$. Let λ be any point of $\Omega \cap \sigma_F(T) \cap \sigma_F(S)$, and let τ be the component of Ω containing λ. Clear, $\partial\tau \subset \partial\Omega$. Suppose that $\tau \subset \sigma_F(T) \cap \sigma_F(S)$. Since $\sigma_F(T) \cap \sigma_F(S)$ is a closed set, $\partial\tau \subset \sigma_F(T) \cap \sigma_F(S) \subset \sigma_e(T)$. Thus, $\sigma_e(T) \cap \partial\tau \neq \phi$, which contradicts the $\sigma_e(T) \cap \partial\Omega = \phi$. Therefore $\tau \not\subset \sigma_F(T) \cap \sigma_F(S)$. It follows that $\tau \cap \partial(\sigma_F(T) \cap \sigma_F(S)) \neq \phi$. Hence $\Omega \cap \partial(\sigma_F(T) \cap \sigma_F(S)) \neq \phi$.

REMARKS.

(1) Let A and B be quasisimilar operators. In [2] Fialkow asked whether each nonempty closed-and-open subset of $\sigma_{re}(A)$ intersects $\sigma_{le}(B)$? Our theorem 2 has answered this question, affirmatively.

(2) Theorem 4 and corollary 5 strengthen the result of [1, theorem 5].

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