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Papageorgiou, Nikolaos S.

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SENSITIVITY ANALYSIS OF EVOLUTION INCLUSIONS AND ITS APPLICATIONS TO THE VARIATIONAL STABILITY OF OPTIMAL CONTROL PROBLEMS

By Nikolaos S. PAPAGEORGIOU*

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1. Introduction

In this paper, first we study the dependence of the solutions of evolution inclusions on the data determining them. Under mild hypotheses, we are able to show that the solution sets of the approximating problems converge in the Kuratowski and Hausdorff modes to the solution set of the limit problem. An analogous result is also presented for the attainable (reachable) sets. Then we use this general result about evolution inclusions, to study the variational stability of optimal control problems with semilinear dynamics. Such a sensitivity analysis is important from both the theoretical and applied viewpoints. It produces useful continuous dependence results, it suggests ways to solve parametric problems, it helps us determine the best mathematical model by telling us what tolerances are permitted in the specification of the data and finally it gives us robust computational procedures to solve numerically complicated problems.

Previous work in this direction was done by Chugunov [3], Lim [5], Stassinopoulos-Vinter [15] and Vasiliev [17], for differential inclusions in $\mathbf{R}^n$, Papageorgiou [7] for differential inclusions in Banach spaces and by Pieri [11], Stassinopoulos [14] and Zolezzi [20] for finite dimensional optimal control problems. Here we consider evolution inclusions, which model parabolic partial differential equations with multivalued terms. So our work goes beyond those mentioned above. In the last section we work in detail an example of a sequence of semilinear parabolic optimal control problems to which our results apply.

2. Preliminaries

Let (Ω, Σ, μ) be a σ -finite measure space and X a separable Banach space. By $P_{f(c)}(X)$ we will denote the family of nonempty, closed, (convex) subsets of X . A multifunction $F : \Omega \rightarrow P_f(X)$ is said to be measurable, if for all $x \in X, \omega \rightarrow d(x, F(\omega)) = \sup\{\|x - z\| : z \in F(\omega)\}$ is measurable. It turns out (see Wagner

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[18]), that this definition of measurability is equivalent to saying that there exist $f_n : \Omega \rightarrow X, n \geq 1$ measurable functions s.t. $F(\omega) = \text{cl}\{f_n(\omega)\}_{n \geq 1}$.

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting

$$h(A, B) = \max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)].$$

Recall that $(P_f(X), h)$ is a complete metric space. We say that a sequence $\{A_n\}_{n \geq 1} \subseteq P_f(X)$ converges in the Hausdorff metric to $A \in P_f(X)$, if $h(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$.

Another mode of convergence that we will need in the sequel, is the so-called Kuratowski convergence of sets. So let $\{A_n\}_{n \geq 1}$ be a sequence of nonempty subsets of X . We define

$$\underline{\lim} A_n = \{x \in X : x = s - \lim x_n, x_n \in A_n, n \geq 1\}$$

$$\text{and } \overline{\lim} A_n = \{x \in X : x = s - \lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \dots < n_k < \dots\}.$$

(here s -indicates the strong topology on X). It is clear from the above definitions that we always have $\underline{\lim} A_n \subseteq \overline{\lim} A_n$. We say that the A_n 's converge to A in the Kuratowski sense (denoted by $A_n \xrightarrow{K} A$) if and only if $\underline{\lim} A_n = \overline{\lim} A_n$.

If instead of $\overline{\lim} A_n$, we define $w - \overline{\lim} A_n = \{x \in X : x = w - \lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \dots < n_k < \dots\}$ (here w - denotes the weak topology on X), then we say that the A_n 's converge to A in the Kuratowski-Mosco sense (denoted by $A_n \xrightarrow{K-M} A$) if and only if $\underline{\lim} A_n = A = w - \overline{\lim} A_n$. Using this mode of set convergence, we can define a convergence for $\mathbf{R}$ -valued functions, in general different from the pointwise convergence. So if $\{f_n, f\}_{n \geq 1} \subseteq \mathbf{R}^X$ we say that $f_n \xrightarrow{\tau} f$ if and only if $\text{epi } f_n \xrightarrow{K-M} \text{epi } f$. From Mosco [6] we know that this definition is equivalent to saying that if $x_{n_k} \xrightarrow{w} x$, then $f(x) \leq \underline{\lim} f_{n_k}(x_{n_k})$ and for every $x \in X$, there exists $x_n \xrightarrow{s} x$ s.t. $f_n(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$.

Now let H be a separable Hilbert space and X a subspace of H carrying the structure of a separable, reflexive Banach space which embeds continuously and densely in H . Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$ with all embeddings being continuous and dense. We will also assume that they are compact. In order to have a concrete example in mind, consider the case in which $H = L^2(Z)$, $X = H_0^m(Z)$ and $X^* = H^{-m}(Z)$, where $m \in \mathbf{N}_+$ and Z is a bounded domain in $\mathbf{R}^n$ with smooth boundary. From the well-known Sobolev-Kondrachov embedding theorem, we know that $H_0^m(Z) \hookrightarrow L^2(Z) \hookrightarrow H^{-m}(Z)$, with all embeddings being continuous, dense and compact. Such a triple of space, is usually called "Gelfand triple". By $\langle \cdot, \cdot \rangle$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot, \cdot)$ the inner product in H . The two are compatible in the sense that $\langle \cdot, \cdot \rangle|_{X \times H} = (\cdot, \cdot)$. By $\|\cdot\|$ (resp. $|\cdot|, \|\cdot\|_*$), we will denote the norm of

the space X (resp. of H, X^*). Let $W(T) = \{x(\cdot) \in L^2(X) : \dot{x}(\cdot) \in L^2(X^*)\}$ where the derivative is understood in the distributional (generalized) sense. This is a separable reflexive Banach space with norm $\|x\|_{W(T)} = [\|x\|_{L^2(X)}^2 + \|\dot{x}\|_{L^2(X^*)}^2]^{1/2}$. It is well known (see Zhikov-Kozlov-Oleinik [19]), that $W(T) \hookrightarrow C(T, H)$ continuously i.e. every element of $W(T)$, after possible modification on a Lebesgue null subset of T , is equal to a continuous function from T into H . In fact if X is a Hilbert space, then $W(T) \hookrightarrow C(T, H)$ compactly. Following Zhikov-Kozlov-Oleinik [19], we make the following definition:

DEFINITION 2.1. *A sequence of operators $A_n : X \rightarrow X^*, n \geq 1$ G -converges to an operator $A : X \rightarrow X^*$ as $n \rightarrow \infty$, if the operators $A_n^{-1}, A^{-1} : X^* \rightarrow X$ are defined and for any $x^* \in X^*, A_n^{-1}x^* \xrightarrow{w} A^{-1}x^*$ in X (strongly in H) as $n \rightarrow \infty$.*

Finally by A_H we will denote the operator defined by

$$A_H x = Ax \quad \text{for } x \in D(A_H) = \{x \in X : Ax \in H\}.$$

3. Evolution inclusions

Let $T=[0, b]$ and (X, H, X^*) a Gelfant triple of spaces with X being a separable Hilbert space. We consider the following sequence of evolution inclusions:

$$\left\{ \begin{array}{l} \dot{x}_n(t) + A_n x_n(t) \in F_n(t, x_n(t)) \text{ a.e.} \\ x_n(0) = x_0^n \end{array} \right\} \quad (*)_n$$

and the limit multivalued Cauchy problem:

$$\left\{ \begin{array}{l} \dot{x}(t) + Ax(t) \in F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\} \quad (*).$$

We will need the following hypotheses on the data of the above problems:

$H(A)$: $A_n, A : X \rightarrow X^*, n \geq 1$ are linear continuous operators s.t.

- (1) $\langle A_n, x, x \rangle, \langle Ax, x \rangle \geq c\|x\|^2, c > 0$ (i.e. they are strongly monotone),
- (2) $A_n \xrightarrow{G} A$ as $n \rightarrow \infty$.

$H(F)$: $F_n, F : T \times H \rightarrow P_{fc}(H)$ $n \geq 1$ are multifunctions s.t.

- (1) $t \rightarrow F_n(t, x), n \geq 1$ are measurable,
- (2) $h(F_n(t, x), F_n(t, y)) \leq k_n(t)|x - y|$ a.e. and $\sup_{n \geq 1} k_n(\cdot) \in L_+^1$,
- (3) $F_n(t, x) \xrightarrow{h} F(t, x)$ a.e. as $n \rightarrow \infty$,
- (4) $|F_n(t, x)| = \sup\{|z| : z \in F_n(t, x)\} \leq a(t) + b(t)|x|$ a.e. with $a(\cdot), b(\cdot) \in L_+^1(T)$.

$$H_0 : x_0^n \xrightarrow{s} x_0 \text{ in } H.$$

By a solution of $(*)$ we understand a function $x(\cdot) \in W(T)$ s.t. $\dot{x}(t) + Ax(t) = f(t)$ a.e. with $f(\cdot) \in S_{F(\cdot, x(\cdot))}^1 = \{h(\cdot) \in L^1(H) : h(t) \in F(t, x(t)) \text{ a.e.}\}$ and $x(0) = x_0$. Similarly for the multivalued Cauchy problems $(*)_n$. We will denote the solution set of $(*)_n$ by P_n and that of $(*)$ by P . From [9] we know that $P_n, P \neq \emptyset$.

THEOREM 3.1. *If hypotheses $H(A)$, $H(F)$ and H_0 hold, then $P_n \xrightarrow{K} P$ and $P_n \xrightarrow{h} P$ as $n \rightarrow \infty$*

PROOF. Let $x(\cdot) \in P$. By definition we have:

$$\dot{x}(t) + Ax(t) = f(t) \text{ a.e., } x(0) = x_0, f \in S_{F(\cdot, x(\cdot))}^1.$$

Let $m_n(t) = \text{proj}(f(t), F_n(t, x(t)))$ and $f_n(t, z) = \text{proj}(m_n(t), F_n(t, z))$. Clearly for all $n \geq 1$ $f_n(t, x(t)) = m_n(t)$. Combining hypothesis $H(F)(2)$ and theorem 3.33 of Attouch [1], we see that $z \rightarrow f_n(t, z)$ is continuous. Also by definition $f_n(t, z) = \inf\{\|m_n(t) - y\| : y \in F_n(t, z)\} = \inf_{k \geq 1} \|m_n(t) - y_k^n(t)\|$, where $y_k^n : T \rightarrow H$ $k \geq 1$ are measurable maps s.t. $F_n(t, z) = \{y_k^n(t)\}_{k \geq 1}$. Such a dense sequence of selectors exists since by hypothesis $H(F)(1)$, $F_n(\cdot, z)$ $n \geq 1$ is measurable. So $t \rightarrow f_n(t, z)$ is measurable.

Now consider the following evolution equation:

$$\begin{cases} \dot{x}(t) + A_n x_n(t) = f_n(t, x_n(t)) \text{ a.e.} \\ x_n(0) = x_0^n \end{cases} \quad (*)'_n$$

We know (see [9]) that $(*)_n'$ has a solution $x_n(\cdot) \in W(T) \hookrightarrow C(T, H)$. Clearly $x_n(\cdot) \in P_n$. We claim that $x_n(\cdot) \rightarrow x(\cdot)$ in $C(T, H)$. Note that $A_{n,H}$ with domain $D(A_{n,H}) = \{x \in X : A_n x \in H\}$, being continuous, monotone, coercive, is maximal monotone (in fact it will be cyclically maximal monotone hence a subdifferential if $A_{n,H}$ is self-adjoint, see Barbu [2], proposition 2.7, p.60). So it generates a linear semigroup of contractions $\{S_n(t)\}_{t \in T} \subseteq \mathcal{L}(H)$, with respect to which the solution $x_n(\cdot)$ can be written as

$$x_n(t) = S_n(t)x_0^n + \int_0^t S_n(t-s)f_n(s, x_n(s))ds$$

(see Tanabe [16], proposition 5.5.1, p. 153).

Similarly if $\{S(t)\}_{t \in T}$ is the semigroup generated by A_H , we have that

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds, t \in T$$

Since $A_n \xrightarrow{G} A$, from theorem 3.62, p. 365 of Attouch [1] and theorem 4.2, p. 85 of Pazy [10], we have that $S_n(t)x \rightarrow S(t)x$ for all $x \in H$, uniformly on bounded t -intervals. Then because of hypothesis H_0 , from Kato [4], p. 151, we have that $S_n(t)x_0^n \xrightarrow{s} S(t)x_0$ in H . We have:

$$|x_n(t) - x(t)| \leq |S_n(t)x_0^n - S(t)x_0| + \left| \int_0^t S_n(t-s)f_n(s, x_n(s))ds - \int_0^t S(t-s)f(s)ds \right|$$

Working on the second term of the right hand side of the above inequality, we have:

$$\begin{aligned} & \left| \int_0^t S_n(t-s)f_n(s, x_n(s))ds - \int_0^t S(t-s)f(s)ds \right| \\ & \leq \left| \int_0^t S_n(t-s)f_n(s, x_n(s))ds - \int_0^t S_n(t-s)f_n(s, x(s))ds \right| \\ & \quad + \left| \int_0^t S_n(t-s)f_n(s, x(s))ds - \int_0^t S_n(t-s)f(s)ds \right| \\ & \quad + \left| \int_0^t S_n(t-s)f(s)ds - \int_0^t S(t-s)f(s)ds \right|. \end{aligned}$$

Observe that:

$$\begin{aligned} |f_n(t, x(t)) - f_n(t, x_n(t))| &= d(m_n(t), F_n(t, x_n(t))) \\ &\leq h(F_n(t, x(t)), F_n(t, x_n(t))) \leq k_n(t)|x_n(t) - x(t)| \text{ a.e.} \end{aligned}$$

Also we have:

$$|f_n(t, x(t)) - f(t)| \leq h(F_n(t, x(t)), F(t, x(t))) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

(see hypothesis $H(F)(3)$). Finally recall that $\|S_n(t)\|_{\mathcal{L}(H)} \leq 1$ $n \geq 1$. Hence we have that:

$$\begin{aligned} & \left| \int_0^t S_n(t-s)f_n(s, x_n(s))ds - \int_0^t S(t-s)f(s)ds \right| \\ & \leq \int_0^t k_n(s)|x_n(s) - x(s)|ds + \int_0^t h(F_n(s, x_n(s)), F(s, x(s)))ds \\ & \quad + \left| \int_0^t S_n(t-s)f(s)ds - \int_0^t S(t-s)f(s)ds \right| \end{aligned}$$

Therefore given $\varepsilon > 0$, we can find $n_0(\varepsilon)$ s.t. for $n \geq n_0(\varepsilon)$ we have

$$|X_n(t) - x(t)| \leq \varepsilon + \int_0^t k_n(s)|x_n(s) - x(s)|ds$$

Invoking Gronwall's inequality, we get that

$$\sup_{t \in T} |x_n(t) - x(t)| < \varepsilon(\exp M)$$

where $M = \|\sup_{n \geq 1} k_n(\cdot)\|_1$.

Since $\varepsilon > 0$ was arbitrary, we deduce that $x_n \rightarrow x$ in $C(T, H)$ and so we have that

$$P \subseteq \varprojlim P_n \quad (1)$$

Next let $x_n \rightarrow x$ in $C(T, H)$, $x_n(\cdot) \in P_n, n \geq 1$. We know that

$$x_n(t) = S_n(t)x_0^n + \int_0^t S_n(t-s)f_n(s)ds, \quad n \geq 1, \quad t \in T, \quad f_n(\cdot) \in S_{F_n(\cdot, x_n(\cdot))}^1 \quad (2)$$

Through a straightforward application of the Lusin-Yankov-Aumann selection theorem, we can find $f(\cdot) \in S_{F(\cdot, x(\cdot))}^1$ s.t. $\|f_n(t) - f(t)\| = d(f_n(t), F(t, x(t)))$. Observe that $d(f_n(t), F(t, x(t))) \leq h(F_n(t, x_n(t)), F(t, x(t))) \leq h(F_n(t, x_n(t)), F_n(t, x(t))) + h(F_n(t, x(t)), F(t, x(t))) \leq M_3(t)|x_n(t) - x(t)| + h(F_n(t, x(t)), F(t, x(t)))$ a.e. where $M_3(t) = \sup_{n \geq 1} k_n(t)$. So $\|f_n(t) - f(t)\| = d(f_n(t), F(t, x(t))) \rightarrow 0$ a.e. Then $S_n(t-s)f_n(s) \xrightarrow{s} S(t-s)f(s)$ a.e. in H and by passing to the limit as $n \rightarrow \infty$ in (2) we get that

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s)ds, \quad t \in T, \quad f(\cdot) \in S_{F(\cdot, x(\cdot))}^1.$$

Then theorem 5.5.1, p. 153 of Tanabe [16], tells us that $x(\cdot)$ solves the evolution equation $\dot{x}(t) + Ax(t) = f(t)$ a.e., $x(0) = x_0, f \in S_{F(\cdot, x(\cdot))}^1 \Rightarrow x(\cdot) \in P$ and so we have

$$\overline{\varprojlim P_n} \subseteq P \quad (3)$$

From (1) and (3) above, we deduce that $P_n \xrightarrow{K} P$. Also, since $\overline{\bigcup_{n \geq 1} P_n}$ is compact in $C(T, H)$ (see [9]; in fact some a priori estimates tell us that $\overline{\bigcup_{n \geq 1} P_n}$ is bounded in $W(T)$ and $W(T) \hookrightarrow C(T, H)$ compactly) and so $P_n \xrightarrow{h} P$ as $n \rightarrow \infty$. Q.E.D.

This leads us to the following convergence result concerning the attainable (reachable) sets of systems $(*)_n$ and $(*)$. The proof, which is based on theorem 3.1, is straightforward and so is omitted. So let $R_n(t) = \{x_n(t) : x_n(\cdot) \in P_n\}, n \geq 1$ and $R(t) = \{x(t) : x(\cdot) \in P\}, t \in T$.

THEOREM 3.2. *If the hypotheses of theorem 3.1 hold,*

then $R_n(t) \xrightarrow[h]{K} R(t)$ as $n \rightarrow \infty$ for all $t \in T$.

4. Optimal control problems

In this section we see how theorem 3.1 can be used to study the variational stability of a class of semilinear optimal control problems.

So consider the following sequence of optimal control problems:

$$\left\{ \begin{array}{l} \eta_n(x) \rightarrow \inf = m_n \\ \text{s.t. } \dot{x}_n(t) + A_n x_n(t) = B_n(t, x_n(t)) u_n(t) \text{ a.e.} \\ x_n(0) = x_0^n, \quad u_n(t) \in U_n(t) \text{ a.e.} \end{array} \right\} \quad (**)_n$$

and the limit optimal control problem

$$\left\{ \begin{array}{l} \eta(x) \rightarrow \inf = m \\ \text{s.t. } \dot{x}(t) + Ax(t) = B(t, x(t))u(t) \text{ a.e.} \\ x(0) = x_0, \quad u(t) \in U(t) \end{array} \right\} \quad (**)$$

We will need the following hypotheses on the data of these problems. Here Y is a separable Banach space modelling the control space.

$H(B)$: $B_n, B : T \times H \rightarrow \mathcal{L}(Y, H)$ are maps s.t.

- (1) $t \rightarrow B_n(t, x)u, n \geq 1$ is measurable,
- (2) $\|B_n(t, x)\|_{\mathcal{L}(Y, H)} \leq a(t) + b(t)|x|$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$,
- (3) $B_n(t, x)^* \xrightarrow{\text{on}} B(t, x)^*$ as $n \rightarrow \infty$ for all $(t, x) \in T \times H$ (here on denotes the operator norm)
- (4) $|B_n(t, x)^*h - B(t, x)^*h| \leq k(t)|x - y|$ for all $n \geq 1$ and all $h \in H$ s.t. $|h| \leq M$.

$H(U)$: $U_n : T \rightarrow P_{fc}(Y)$ are measurable multifunctions s.t.

- (1) $U_n(t) \xrightarrow{h} U(t)$ a.e.
- (2) $|U_n(t)| = \sup\{\|z\|_Y : z \in U_n(t)\} \leq M$ a.e. for all $n \geq 1$

$H(\eta)$: $\eta_n, \eta : C(T, H) \rightarrow \mathbf{R}$ are continuous, convex functions s.t. $\eta_n \xrightarrow{\tau} \eta$ and $\sup_{n \geq 1} |\eta_n(x)| < \infty$ for all $x \in C(T, H)$ s.t. $\|x\|_\infty \leq M_4$ where $|\bigcup_{n \geq 1} P_n| \leq M_4$ with P_n being the set of trajectories of $(**)_n$.

THEOREM 4.1. *If hypotheses $H(A)$, $H(B)$, $H(U)$, $H(\eta)$ and H_0 hold, then $m_n \rightarrow m$.*

PROOF. Let $F_n(t, x) = B_n(t, x)U_n(t)$ and $F(t, x) = B(t, x)U(t)$. Then note that for every $h \in H$ we have

$$\sup_{z \in F_n(t, x)} (h, z) = \sigma_{F_n(t, x)}(h) = \sigma_{U_n(t)}(B_n(t, x)^*h)$$

Note that the function $(t, v) \rightarrow \sigma_{U_n(t)}(v)$ defined on $T \times Y^*$ is a Caratheodory function (i.e. measurable in t , continuous in v), hence it is jointly measurable. Thus $t \rightarrow \sigma_{U_n(t)}(B_n(t, x)^*h)$ is measurable and so $t \rightarrow \sigma_{F_n(t, x)}(h)$ is measurable, which in turn implies that $t \rightarrow F_n(t, x)$ is measurable for all $n \geq 1$ and all $x \in H$ (see Wagner [18]). Also we have:

$$\begin{aligned} h(F_n(t, x), F(t, x)) &= h(B_n(t, x)U_n(t), B(t, x)U(t)) \\ &\leq h(B_n(t, x)U_n(t), B_n(t, x)U(t)) + h(B_n(t, x)U(t), B(t, x)U(t)) \\ &\leq \|B_n(t, x)\|_{\mathcal{L}(Y, H)} h(U_n(t), U(t)) + |U(t)| \cdot \|B_n(t, x)^* \\ &\quad - B(t, x)^*\|_{\mathcal{L}(H, Y^*)} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

Also for all $x, y \in H$ we have:

$$\begin{aligned} h(F_n(t, x), F_n(t, y)) &= \sup_{|h| \leq 1} |\sigma_{U_n(t)}(B_n(t, x)^*h) - \sigma_{U_n(t)}(B_n(t, y)^*h)| \\ &\leq M \cdot \|B_n(t, x)^* - B_n(t, y)^*\|_{\mathcal{L}(H, Y^*)} \\ &\leq Mk(t)|x - y| \text{ a.e.} \end{aligned}$$

Finally note that for all $n \geq 1$ we have

$$|F_n(t, x)| \leq a(t) + Mb(t)|x| \text{ a.e.}$$

So we have satisfied hypothesis $H(F)$ of section 3. Applying theorem 3.1 we get that $P_n \xrightarrow{K} P$, where P_n is the set of trajectories of $(*)_n, n \geq 1$ and P is the set of trajectories of $(*)$.

Let $x \in P$ s.t. $\eta(x) = m$. Such an trajectory exists since P is compact and $\eta(\cdot)$ is continuous. Then from the properties of τ -convergence (see Mosco[6]), we know that we can find $y_n(\cdot) \in C(T, H)$ s.t. $y_n \rightarrow x$ in $C(T, H)$ and $\eta_n(y_n) \rightarrow \eta(x)$. Also since $P_n \xrightarrow{K} P$, we can find $x_n \in P_n$ s.t. $x_n \rightarrow x$ in $C(T, H)$. Because of the pointwise boundedness hypothesis $H(\eta)$ on $\overset{\circ}{B}(M_4(0)) = \{y \in C(T, H) : \|y\|_\infty < M_4\}$, from Rockafellar [12], theorem 10.6, p. 88 we know that $\{\eta_n(\cdot)\}_{n \geq 1}$ is equi-Lipschitz on the set $\{x_n, y_n, x\}_{n \geq 1}$. So there exists $k > 0$ s.t.

$$\begin{aligned} |\eta_n(x_n) - \eta_n(y_n)| &\leq \widehat{k} \cdot \|x_n - y_n\|_\infty, \quad n \geq 1 \\ \implies |\eta_n(x_n) - \eta(y_n)| &\rightarrow 0 \text{ as } n \rightarrow \infty \\ \implies \eta_n(x_n) &\rightarrow \eta(x) = m \text{ as } n \rightarrow \infty \\ \implies \overline{\lim} m_n &\leq m \end{aligned} \tag{1}$$

Next for each $n \geq 1$, let $x_n(\cdot) \in P_n(x_0)$ s.t. $\eta_n(x_n) = m_n$. Recalling that $\overline{\{x_n(\cdot)\}_{n \geq 1}}$ is compact in $C(T, H)$ and by passing to a subsequence if necessary, we may assume that $x_n \rightarrow x \in P$ in $C(T, H)$. Then from the properties of τ -convergence, we have

$$m \leq \eta(x) \leq \lim \eta(x_n) = \lim m_n \quad (2)$$

Combining (1) and (2) above, we conclude that $m_n \rightarrow m$. Q.E.D.

Let $Q_n = \{\text{set of optimal trajectories of } (**)_n\}$ and $Q = \{\text{set of optimal trajectories of } (**)\}$. We have the following theorem relating them:

THEOREM 4.2. *If the hypotheses of theorem 4.1 hold, then $\overline{\lim} Q_n \subseteq Q$ in $C(T, H)$.*

PROOF. Let $x_n \in Q_n$ $n \geq 1$ and $x_n \rightarrow x$ in $C(T, H)$. Then $\eta_n(x_n) = m_n \Rightarrow \eta(x) \leq \underline{\lim} \eta_n(x_n) = \underline{\lim} m_n = m$ (theorem 4.1). Since $x \in P$ (theorem 3.1), we have $\eta(x) = m \Rightarrow x \in Q$. Q.E.D.

5. An example

In this section we present an example of a sequence of parabolic distributed parameter optimal control problems on which the results of the previous sections can be applied.

So let $T = [0, b]$ and Z a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial Z = \Gamma$. The approximating terminal cost control problems are the following:

$$\left\{ \begin{array}{l} \eta_n(x) = \int_Z \phi_n(z, x_n(b, z)) dz \rightarrow \inf = m_n \\ \text{s.t. } \frac{\partial x_n(t, z)}{\partial t} - \sum_{i,j=1}^N \frac{\partial}{\partial z_i} \left(a_{ij}^n(z) \frac{\partial x_n(t, z)}{\partial z_j} \right) \\ \quad = b_n(t, z, x_n(t, z)) u_n(t, z) \quad \text{on } TxZ \\ x_n(t, z) = 0 \quad \text{on } Tx\Gamma \\ x_n(0, z) = x_0^n(z) \quad \text{on } \{0\} \times Z, \int_Z u_n(t, z)^2 dz \leq r_n^2 \end{array} \right. \quad (***)$$

and the limit problem is

$$\left\{ \begin{array}{l} \eta_n(x) = \int_Z \phi(z, x(b, z)) dz \rightarrow inf = m_n \\ s.t. \frac{\partial x(t, z)}{\partial t} - \sum_{i,j=1}^N \frac{\partial}{\partial z_i} \left(a_{ij}(z) \frac{\partial x(t, z)}{\partial z_j} \right) \\ \quad = b_n(t, z, x(t, z)) u(t, z) \quad \text{on } T \times Z \\ x(t, z) = 0 \quad \text{on } T \times \Gamma \\ x(0, z) = x_0(z) \quad \text{on } \{0\} \times Z, \int_Z u(t, z)^2 dz \leq r^2 \end{array} \right\} \quad (***)$$

We will need the following hypotheses on the data

$$\begin{aligned} H(a): \quad & \{a_{ij}^n, a_{ij}\} \in L^\infty(Z) \quad a_{ij}^n = a_{ji}^n \text{ and } \lambda_0 |\theta|^2 \\ & \leq \sum_{i,j=1}^N a_{ij}^n(z) \theta_i \theta_j, \quad \sum_{i,j=1}^N a_{ij}(z) \theta_i \theta_j, \quad a_{ij}^n = a_{ij}^n(z_1), \quad a_{ij} = a_{ij}(z_1) \\ & \leq \lambda_1 |\theta|^2 \quad \text{for all } \theta \in \mathbf{R}^N \quad \text{and with } \lambda_0, \lambda_1 > 0. \quad \text{Also } \frac{1}{a_{11}^n} \xrightarrow{w^*} \frac{1}{a_{11}} \\ & \frac{a_{il}^n}{a_{11}^n} \xrightarrow{w^*} \frac{a_{il}}{a_{11}}, \quad a_{ij}^n - \frac{a_{il}^n a_{ij}^n}{a_{11}^n} \xrightarrow{w^*} a_{ij} - \frac{a_{il} a_{ij}}{a_{11}} \quad \text{in } L^\infty(Z_1), \quad Z_1 = \text{proj}_1 Z. \end{aligned}$$

$H(b): \quad b_n, b: T \times Z \times \mathbf{R} \rightarrow \mathbf{R}$ are maps s.t.

- (1) $(t, z) \rightarrow b_n(t, z, x), \quad b(t, z, x), \quad n \geq 1$ are measurable,
- (2) $|b_n(t, z, x)| \geq a(t, z)$ a.e. with $a(\cdot, \cdot) \in L^1(T, L^\infty(Z))$,
- (3) $|b_n(t, z, x) - b_n(t, z, y)| \leq k(t, z) |x - y|$ a.e. $n \geq 1$ with $k(\cdot, \cdot) \in L^1(T, L^\infty(Z))$,
- (4) for every $x(\cdot) \in L^2(Z), \quad b_n(t, \cdot, x(\cdot)) \rightarrow b(t, \cdot, x(\cdot))$ in $L^\infty(Z)$.

$H(\phi): \quad \phi_n, \phi: Z \times \mathbf{R} \rightarrow \mathbf{R}$ are integrands s.t.

- (1) $\phi_n(\cdot, x), \phi(\cdot, x) \quad n \geq 1$ are measurable,
- (2) $\phi_n(z, \cdot), \phi(z, \cdot) \quad n \geq 1$ are continuous, convex,
- (3) for every $w_n \xrightarrow{s} w$ in $L^2(Z), \{\phi_n(\cdot, w_n(\cdot))\}_{n \geq 1}$ is uniformly integrable,
- (4) $\phi_n(z, \cdot) \xrightarrow{c} \phi(z, \cdot)$ a.e., where c - denotes the continuous convergence.

$H(r): \quad r_n \rightarrow r.$

$H_0: \quad x_0^n \xrightarrow{s} x_0$ in $L^2(Z).$

In this case the Gelfand triple consists of the spaces $X = H_0^1(Z), \quad H = L^2(Z)$ and $X^* = H^{-1}(Z)$. Clearly all embeddings are continuous, dense and compact (Sobolev-Kondrachov theorem). Also $Y = L^2(Z)$. Consider the Dirichlet forms

$\alpha_n : H_0^1(Z) \times H_0^1(Z) \rightarrow \mathbf{R}$ $n \geq 1$ defined by

$$\alpha_n(x, y) = \int_Z \sum_{i,j=1}^N a_{ij}^n(z) \frac{\partial x}{\partial z_i} \frac{\partial y}{\partial z_j} dz.$$

From hypothesis $H(a)$, we have $|\alpha_n(x, y)| \leq \widehat{M} \|x\|_{H_0^1(Z)} \|y\|_{H_0^1(Z)}$ $n \geq 1$. So we can define a continuous linear operator $A_n : X \rightarrow X^*$ by setting $\langle A_n x, y \rangle = \alpha_n(x, y)$. Similarly using the Dirichlet form $\alpha(x, y) = \int_Z \sum_{i,j=1}^N a_{ij}(z) \frac{\partial x}{\partial z_i} \frac{\partial y}{\partial z_j} dz$, we can define $A : X \rightarrow X^*$ by setting $\langle Ax, y \rangle = \alpha(x, y)$. Also from hypothesis $H(a)$, we have $\langle A_n x, x \rangle \geq \lambda \|\nabla x\|_{L^2(Z)}$ and invoking Poincaré's inequality, we get $\langle A_n x, x \rangle \geq c \|x\|_{H_0^1(Z)}$ $c > 0$. Similarly $\langle Ax, x \rangle \geq c \|x\|_{H_0^1(Z)}$. Also from lemma 3, p. 289 of Sokolowski [13], we have $A_n \xrightarrow{G} A$. Hence we have verified hypothesis $H(A)$.

Next let $B_n : T \times L^2(Z) \rightarrow \mathcal{L}(L^2(Z))$ be defined by $(B_n(t, x)v)(z) = b_n(t, z, x(z))v(z)$. If $\beta(\cdot) \in L^2(Z)$, we have $(B_n(t, x)v, \beta)_{L^2(Z)} = \int_Z b_n(t, z, x(z))v(z)\beta(z)dz \Rightarrow t \rightarrow (B_n(t, x)v, \beta)_{L^2(Z)} = \int_Z b_n(t, z, x(z))v(z)\beta(z)dz \Rightarrow t \rightarrow (B_n(t, x)v, \beta)$ is measurable (Fubini's theorem). So $t \rightarrow B_n(t, x)v$ is weakly measurable and since $L^2(Z)$ is separable, Pettis' theorem tells us that $t \rightarrow B_n(t, x)v$ $n \geq 1$ is measurable. Also note that $\|B_n(t, x)\|_{\mathcal{L}(L^2(Z))} \leq \widehat{a}(t) + \widehat{b}(t)\|x\|_{L^2(Z)}$ a.e., where $\widehat{a}(t) = \|a(t, \cdot)\|_{L^1(Z)}$ and $\widehat{b}(t) = \|b(t, \cdot)\|_{L^\infty(Z)}$. Observe that $B_n(t, x) = B(t, x)^*$ and $B(t, x) = B(t, x)^*$. Furthermore for $v \in Y$ s.t. $\|v\|_Y \leq r_0 = \sup_{n \geq 1} r_n$ we have

$$\begin{aligned} & \|B_n(t, x)v - B_n(t, y)v\|^2 \\ &= \int_Z |b_n(t, z, x(z))v(z) - b(t, z, x(z))v(z)|^2 dz \\ &\leq \|k(t, \cdot)\|_{L^\infty(Z)}^2 r_0^2 \int_Z |x(z) - y(z)|^2 dz \\ &= k(t)^2 \|x - y\|_{L^2(Z)}^2 (\widehat{k}(t) = \|k(t, \cdot)\|_{L^\infty(Z)} r_0) \end{aligned}$$

Also $\|B_n(t, x)v - B(t, x)v\|_{L^2(Z)}^2 = \int_Z |b_n(t, z, x(z))v(z) - b(t, z, x(z))v(z)|^2 dz \leq \|b_n(t, \cdot, x(\cdot)) - b(t, \cdot, x(\cdot))\|_{L^\infty(Z)}^2 \|v\|_{L^2(Z)}^2 \Rightarrow B_n(t, x) \xrightarrow{on} B(t, x)$ in $\mathcal{L}(L^2(Z))$. So we have verified hypothesis $H(B)$.

Next let $U_n = \{u \in L^2(Z) : \|u\|_{L^2(Z)} \leq r_n\}$ and $U = \{u \in L^2(Z) : \|u\|_{L^2(Z)} \leq r\}$. Since by hypothesis $H(r)$, $r_n \rightarrow r$, we get $U_n \xrightarrow{h} U$. Thus hypothesis $H(U)$ has been verified.

Finally note that because of hypothesis $H(\phi)$, $\eta_n, \eta : C(T, H) \rightarrow \mathbf{R}$ $n \geq 1$ are continuous, convex functions and $\eta_n \xrightarrow{c} \eta$, thus $\eta_n \xrightarrow{\tau} \eta$. So hypothesis $H(\eta)$ has been verified.

Therefore we see that we have verified all hypotheses of theorem 4.1. Applying that theorem, we get:

THEOREM 5.1. *If hypotheses $H(a)$, $H(b)$, $H(\phi)$, $H(r)$ and H_0 hold, then $m_n \rightarrow m$.*

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Florida Institute of Technology
Department of Applied Mathematics
150 West University Boulevard
Melbourne, FL 32901-6988
U.S.A.

Permanent address:
National Technical University
Department of Mathematics
Zografou Campus
Athens 15773, Greece