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PDF issue: 2026-11-10

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(Citation)
Kobe Journal of Mathematics,8(2):173-190

(Issue Date)
1991-12

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0003762

(URL)
https://hdl.handle.net/20.500.14094/E0003762

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KOBE J. MATH.,
8 (1991) 173—190

CHARACTERIZATIONS OF MONOIDS BY p-INJECTIVE AND NORMAL S-SYSTEMS

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(Received December, 20, 1990)

Introduction

In this paper S will denote a monoid. We will deal with S-systems as representations of S.

In section 1, we discuss relations between p-injective S-systems and S-divisible S-systems. We also construct an S-divisible S-system $\mathbf{Q}(A)$ from an S-system A under some conditions.

In section 2, we characterize regular and von Neumann regular S-systems in terms of pM-injectivity and deduce characterizations of pp-monoids and von Neumann regular monoids, analogous to the corresponding characterizations of (semi) hereditary monoids found in [3].

In section 3, we study weakly regular monoids. As a generalization of these monoids, we also introduce the notion of normal S-systems and characterize weakly regular monoids by the property that each S-system is normal.

1. p-injective S-systems and S-divisible S-systems

In this paper S will denote a *monoid*, that is, a semigroup with identity 1. We denote the category consisting of all monoids and their homomorphisms by **Mon**.

A *right S-system* M_S is a set M on which S acts unitarily from the right, that is, $(as)t = a(st)$ and $a1 = a$ for $1, s, t \in S$ and $a \in M$.

Let A and B be two right S-systems. A mapping $f : A \to B$ is called an *S-homomorphism* if $f(as) = f(a)s$ for $a \in A$ and $s \in S$. We denote the set of all S-homomorphisms from A to B by $Hom_S(A, B)$. The right S-systems together with the S-homomorphisms form a category which we denote by $\mathbf{Act}_S$. In the category $\mathbf{Act}_S$, epimorphisms are surjective and monomorphisms are injective.

An S-system with one generating element is called *cyclic*, and a right (left) ideal of S with one generating element *principal*. A one element S-subsystem θ of M is called a *zero* of M.

DEFINITION 1.1. *Let M be a right S-system. An S-system Q is called pM-*

injective if each S-homomorphism from a cyclic S-subsystem $aS(a \in M)$ of M to Q extends to an S-homomorphism from M to Q. Q is called a p-injective S-system if Q is pS-injective ([11]). *An S-system all of whose quotient S-systems are pM-injective will be called a completely pM-injective S-system. Completely p-injective S-systems are defined analogously.*

DEFINITION 1.2. *Let S be a monoid and Q an S-system.*

An element x of Q is said to be S-divisible in Q if for every $a \in S$ there exists $y \in Q$ such that $x = ya$. Q is S-divisible if $Qa = Q$ for all $a \in S$. For a monoid S this means that S is a group, that is, a monoid S is a group if and only if S is S-divisible. If every quotient S-system $\overline{Q}$ of Q is S-divisible, Q will be called a completely S-divisible S-system.

Then we have

PROPOSITION 1.1.
If Q is (completely) S-divisible, then Q is (completely) p-injective.

PROOF. Suppose that Q is S-divisible. To show the p-injectivity, let $aS(a \in S)$ be any principal ideal of S and $\varphi : aS \to Q$ an S-homomorphism. Then φ is determined by the element $\varphi(a) = x \in Q$, that is, $\varphi(as) = xs$ for all $s \in S$. Since Q is S-divisible, there exists an element $y \in Q$ such that $x = ya$. Define $\psi : S \to Q$ by $\psi(1) = y$, that is, $\psi(s) = ys$ for all $s \in S$. Then we have $\psi(as) = yas = xs = \varphi(as)$ for $s \in S$. This shows ψ is an extension of φ. Thus Q is p-injective.

DEFINITION 1.3. *Let A and Q be S-systems.*

A is a retract of Q if there exist S-homomorphisms $q : A \to Q$ and $p : Q \to A$ such that $p \circ q = 1_A$. Then we call p a retraction and q a coretraction.

Then we have

PROPOSITION 1.2.
If A is a retract of an S-divisible S-system Q, then A is S-divisible.

PROOF. Let p be the retraction and q the coretraction such that $p \circ q = 1_A$. To show that A is S-divisible, let $x \in A$ and $a \in S$. Then $q(x) \in Q$. Since Q is S-divisible, there exists $y \in Q$ such that $q(x) = ya$. Then $x = p \circ q(x) = p(ya) = p(y)a$ and $p(y) \in A$. This shows A is S-divisible.

DEFINITION 1.4. *Let A be a right S-system.*

A is right S-cancellative if A has the following property:

$$xs = x's \quad for \quad x, x' \in A \quad and \quad s \in S \Longrightarrow x = x'.$$

Thus S is right cancellative if S is right S-cancellative.
Dually, A is left A-cancellative if A has the following property:

$$xs = xs' \quad for \quad x \in A \quad and \quad s, s' \in S \Longrightarrow s = s'.$$

Thus S is left cancellative if S is left S-cancellative.

PROPOSITION 1.3. *For a left cancellative monoid S, the following assertions are equivalent:*

(1) *Q is a (completely) p-injective right S-system,*

(2) *Q is a (completely) S-divisible right S-system.*

PROOF. (2) $\Rightarrow$ (1): This follows from Proposition 1.1.

(1) $\Rightarrow$ (2): Let $x \in Q$ and $a \in S$. Define a map $\varphi : aS \to Q$ by $\varphi(as) = xs$ for all $s \in S$. Since S is left cancellative, this is well-defined and an S-homomorphism. Since Q is p-injective, there exists an extension ψ from S to Q. Then $x = \varphi(a) = \psi(a) = \psi(1)a$ and $\psi(1) \in Q$. This shows Q is S-divisible.

Next we will construct an S-divisible S-system $\mathbf{Q}(A)$ from a right S-system A under some conditions.

For a set $A \times S = \{(x, a) \mid x \in A \text{ and } a \in S\}$, we define an S-action on $A \times S$ by $(x, a)s = (xs, a)$ for all $s \in S$. Then $A \times S$ together with the S-action is a right S-system which we denote by $Q(A)$.

Consider a relation $\equiv$ on $Q(A)$ defined by

$$(x, a) \equiv (x', a') \iff xa' = x'a.$$

Then we have

LEMMA 1.4.

If S is a commutative monoid and A is a right S-cancellative S-system, then the above relation $\equiv$ is an S-congruence on $Q(A)$.

PROOF. By the definition, the relation $\equiv$ is reflexive and symmetric. To show the transitivity, suppose that

$$(x, a) \equiv (x', a') \quad \text{and} \quad (x', a') \equiv (x'', a'').$$

By the assumption, $xa' = x'a$ and $x'a'' = x''a'$. Since S is commutative, we have $xa''a' = xa'a'' = x'aa'' = x'a''a = x''a'a = x''aa'$ and so $xa''a' = x''aa'$. Since A is right S-cancellative, we have $xa'' = x''a$. This shows $(x, a) \equiv (x'', a'')$.

Finally the compatibility with S follows directly from the definition and the commutativity of S. Thus the relation $\equiv$ is an S-congruence on $Q(A)$.

By the above lemma, we may construct a quotient right S-system $Q(A)/\equiv$ which we will denote by $\mathbf{Q}(A)$. If we denote an element of $\mathbf{Q}(A)$ by $\overline{(x,a)}$, the S-action on $\mathbf{Q}(A)$ is defined by

$$\overline{(x,a)}s = \overline{(x,a)s} = \overline{(xs,a)} \quad \text{for} \quad s \in S.$$

PROPOSITION 1.5. *Let S be a commutative monoid and A a right S-cancellative S-system. Then $\mathbf{Q}(A)$ has following properties:*

(1) *$\mathbf{Q}(A)$ is S-divisible and A is considered as an S-subsystem of $\mathbf{Q}(A)$,*
(2) *$\mathbf{Q}(A)$ is right S-cancellative,*
(3) *for every $\overline{(x,a)} \in \mathbf{Q}(A)$, $\overline{(x,a)}a = \overline{(x,1)}$.*

PROOF. (1): Define $q : A \to \mathbf{Q}(A)$ by $q(x) = \overline{(x,1)}$. Then q is injective. Thus we may consider A as an S-subsystem of $\mathbf{Q}(A)$.

Now let $\overline{(x,a)} \in \mathbf{Q}(A)$ and $s \in S$. Since S is commutative, $xas = xsa$. This shows $(x,a) \equiv (xs,as)$ and so $\overline{(x,a)} = \overline{(xs,as)}s$. This means $\mathbf{Q}(A)$ is S-divisible.

(2): Suppose that $\overline{(x,a)}s = \overline{(x',a')}s$. Then $(xs,a) \equiv (x's,a')$ and so $xsa' = x'sa$. Since S is commutative, $xa's = x'as$. Since A is right S-cancellative, we have $xa' = x'a$. This means $\overline{(x,a)} = \overline{(x',a')}$. Hence $\mathbf{Q}(A)$ is right S-cancellative.

(3): For every $\overline{(x,a)} \in \mathbf{Q}(A)$, $\overline{(x,a)}a = \overline{(xa,a)} = \overline{(x,1)}$.

COROLLARY. *Let S be a commutative and cancellative monoid. Then $\mathbf{Q}(S)$ is S-divisible and $S \subseteq \mathbf{Q}(S)$.*

In this case, $\mathbf{Q}(S)$ is a commutative group with the following multiplication:

$$\overline{(b,a)} \cdot \overline{(b',a')} = \overline{(bb',aa')}.$$

REMARK. In this case, $\mathbf{Q}(S)$ is the well-known classical construction from a commutative and cancellative monoid S.

PROPOSITION 1.6. *Let S be commutative and A right S-cancellative. Then the following assertions are equivalent:*

(1) *A is S-divisible,*
(2) *A is a retract of $\mathbf{Q}(A)$.*

PROOF. (2) $\Rightarrow$ (1): This follows from Proposition 1.2 since $\mathbf{Q}(A)$ is S-divisible.

(1) $\Rightarrow$ (2): To define a retraction $p : \mathbf{Q}(A) \to A$, let $\overline{(x,a)} \in \mathbf{Q}(A)$. Since A is S-divisible, there exists $y \in A$ such that $x = ya$. Since A is right S-cancellative, such y is unique. Then we define p by $p\overline{(x,a)} = y$ for all $\overline{(x,a)} \in \mathbf{Q}(A)$. Now suppose $(x,a) \equiv (x',a')$ with $x = ya$ and $x' = y'a'$. Since $xa' = x'a$, $yaa' = y'a'a$ and so $yaa' = y'aa'$ by the commutativity of S. Since A is right S-cancellative, it follows $y = y'$. Thus the map p is well-defined. To see that p is an S-homomorphism, let $p\overline{(x,a)} = y$ with $x = ya$ and $p\overline{(xs,a)} = y'$ with $xs = y'a$. Then $yas = y'a$. Since S is commutative, we have $ysa = y'a$. Since A is right S-cancellative, it follows that $y' = ys$. Hence $p(\overline{(x,a)}s) = p\overline{(xs,a)} = y' = ys = p\overline{(x,a)}s$. This shows p is an S-homomorphism. Let $q : A \to \mathbf{Q}(A)$ be the inclusion defined by $q(x) = \overline{(x,1)}$. Then $p \circ q(x) = p\overline{(x,1)} = x$ because $x = x \cdot 1$. This shows A is a retraction of $\mathbf{Q}(A)$.

COROLLARY. *Let S be a commutative and cancellative monoid. Then the following assertions are equivalent*:

(1) *S is a commutative group,*
(2) *S is p-injective,*
(3) *S is S-divisible,*
(4) *S is a retract of $\mathbf{Q}(S)$.*

Finally we will prove the following universal property for $\mathbf{Q}(A)$.

THEOREM 1.

Let S be a commutative monoid and A a right S-cancellative S-system. Then there exist an S-system $\overline{A}$ and an S-homomorphism $f : A \to \overline{A}$ satisfying following conditions (1) $\sim$ (4) :

(1) *f is injective,*
(2) *Each element of $f(A)$ is S-divisible in $\overline{A}$,*
(3) *$\overline{A}$ is right S-cancellative,*
(4) *For each $y \in \overline{A}$, there exist $a \in S$ and $x \in A$ such that $ya = f(x)$.*

If $\overline{A}'$ and f' satisfy the conditions (1) $\sim$ (4), *then there exists a unique S-isomorphism $\varphi : \overline{A} \to \overline{A}'$ such that $f' = \varphi \circ f$.*

PROOF. Since $\mathbf{Q}(A)$ satisfies the conditions (1) $\sim$ (4), we need only to prove the last part.

To define a map $\varphi : \overline{A} \to \overline{A}'$, let y be any element of $\overline{A}$. By the condition (4), there exist $a \in S$ and $x \in A$ such that $ya = f(x)$. For $f'(x) \in \overline{A}'$ and $a \in S$, there exists $y' \in \overline{A}'$ such that $y'a = f'(x)$ by the condition (2) of $\overline{A}'$.

Now let $ya' = f(x')$ and $y''a' = f'(x')$ be another expression. Then $yaa' = f(x)a' = f(xa')$ and $ya'a = f(x')a = f(x'a)$. Hence we have $xa' = x'a$ by commutativity of S and injectivity of f. It follows that $y'aa' = f'(x)a' = f'(xa') =$

$f'(x'a) = f'(x')a = y''a'a$. By commutativity of S and right S-cancellativity of $\overline{A}'$ we have $y' = y''$. This shows that $y' \in \overline{A}'$ is uniquely determined from $y \in \overline{A}$ by the following rule:

$$ya = f(x) \quad \text{and} \quad y'a = f'(x).$$

Thus we may define a map $\varphi : \overline{A} \to \overline{A}'$ by

$$\varphi(y)a = f'(x) \quad \text{for all} \quad y \in \overline{A}.$$

To show $\varphi(ys) = \varphi(y)s$, let $\varphi(y) = y'$ and $\varphi(ys) = y''$. Since $ya = f(x)$, $ysa = yas = f(xs)$. Therefore $y''a = f'(xs) = f'(x)s = y'sa$. Since $\overline{A}'$ is right S-cancellative, $y'' = y's$. Thus we have an S-homomorphism $\varphi : \overline{A} \to \overline{A}'$. By the definition of φ we may easily check that φ is an S-isomorphism such that $f' = \varphi \circ f$.

Finally suppose $f' = \varphi' \circ f$ and $ya = f(x)$. Then $\varphi'(y)a = f'(x) = \varphi(y)a$. Since $\overline{A}'$ is right S-cancellative, we have $\varphi'(y) = \varphi(y)$. This shows the uniqueness of φ such that $f' = \varphi \circ f$.

REMARK. If $\overline{A}$ satisfies conditions (1) $\sim$ (4), then $\overline{A}$ is S-divisible and so p-injective.

To see this, suppose $y \in \overline{A}$ and $a \in S$. By the condition (4), there exist $b \in S$ and $x \in A$ such that $yb = f(x)$. By the condition (2), for $f(x) \in f(A)$ and $ab \in S$, there exists $z \in \overline{A}$ such that $f(x) = zab$. Hence $yb = zab$. By the condition (3), it follows that $y = za$. This shows $\overline{A}$ is S-divisible.

2. Characterizations of monoids by p-injective S-systems

First we will state the following definitions for a monoid S and S-systems.

DEFINITION 2.1. *An S-system M is called regular if for each $a \in M$ there exists an S-homomorphism $f \in Hom_S(aS, S)$ such that $a = af(a)$. A monoid S is called regular if S_S is regular as an S-system* ([6]).

An S-system M is called von Neumann regular if for each $a \in M$ there exists an S-homomorphism $g \in Hom_S(M, S)$ such that $a = ag(a)$ ([18]).

Thus if S_S is von Neumann regular, then for each $a \in S$ there exists $g \in Hom_S(S, S)$ such that $a = ag(a) = ag(a)a$ and $g(1) \in S$. Hence S is von Neumann regular in the familiar sense.

DEFINITION 2.2. *Let M and Q be right S-systems.*

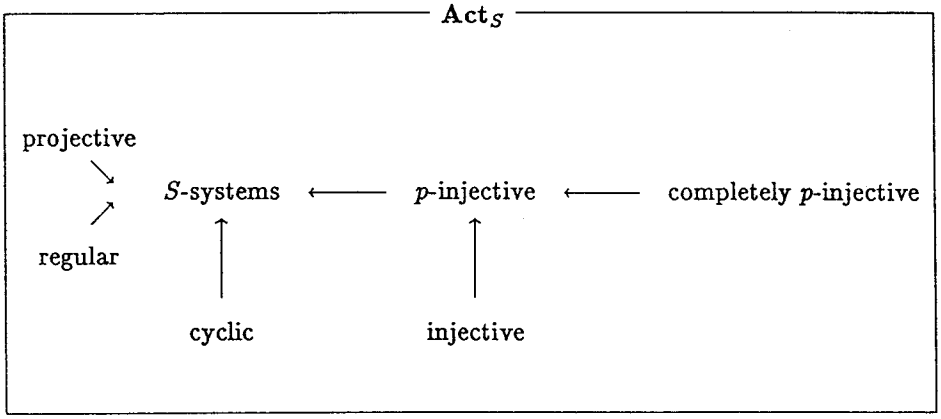
Q is M-projective if for each S-epimorphism $g : M \to \overline{M}$ and each S-homomorphism $h : Q \to \overline{M}$ there exists an S-homomorphism $k : Q \to M$ such that $g \circ k = h$. Thus Q is projective if Q is M-projective for each S-system M. We notice that every monoid S is always projective.

Dually, Q is M-injective if for each S-monomorphism $g : N \to M$ and each S-homomorphism $h : N \to Q$ there exists an S-homomorphism $k : M \to Q$ such that $k \circ g = h$. Thus Q is injective if Q is M-injective for each S-system M.

DEFINITION 2.3.

A right S-system M is called a right ppS-system if each cyclic S-subsystem aS of M with $a \in M$ is projective. S is called a right pp-monoid if all its principal ideals are projective as right S-systems.

From the abaove definitions we have the following diagram in the category $\mathbf{Act}_S$:



Next we state the following proposition due to T.L. Hách.

PROPOSITION 2.1. (*T.L. Hách*)

For an S-system M the following are equivalent:

(1) *M is a regular S-system,*

(2) *M is a ppS-system.*

The proof is found in [6].

COROLLARY. *For a monoid S the following are equivalent:*

(1) *S is regular.*

(2) *S is a pp-monoid.*

(3) *Every projective S-system is regular.*

The proof of (1) $\Leftrightarrow$ (3) is found also in [6].

For a right S-system M and $a \in M$, we may always define an S-epimorphism $\pi : S \to aS$ by $\pi(s) = as$ for all $s \in S$ and we have an inclusion $k : aS \to M$.

We notice the following are equivalent:

(1) *M is regular,*

(2) *for each $a \in M$, aS is a retract of S,*

(3) *for each $a \in M$, π is a retraction.*

For a monoid S, S is von Neumann regular if and only if the inclusion $k : aS \to S$ is a coretraction for each $a \in S$.

PROPOSITION 2.2. *For a projective S-system M the following assertions are equivalent:*

(1) *M is a regular S-system.*

(2) *Each pM-injective S-system is completely pM-injective.*

(3) *Each injective S-system is completely pM-injective.*

PROOF. (1) $\Rightarrow$ (2): Let A be a pM-injective S-system and $\overline{A}$ a quotient S-system of A. So, there is an S-epimorphism $p : A \to \overline{A}$. In order to prove that $\overline{A}$ is pM-injective, consider a cyclic S-subsystem aS with $a \in M$ and an S-homomorphism $f : aS \to \overline{A}$. Since M is a regular S-system, it follows that aS is projective by Proposition 2.1. Hence there exists an S-homomorphism $h : aS \to A$ such that $p \circ h = f$. Now since A is pM-injective, there exists an S-homomorphism $g : M \to A$ such that $g \circ k = h$, where $k : aS \to M$ is the inclusion map. Let $\mu = p \circ g$. Then $\mu : M \to \overline{A}$ is an S-homomorphism such that $\mu \circ k = p \circ g \circ k = p \circ h = f$. Hence $\overline{A}$ is pM-injective.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let aS be a cyclic S-subsystem of M and let $k : aS \to M$ be the inclusion map. Now consider an epimorphism $p : A \to \overline{A}$ and an S-homomorphism $f : aS \to \overline{A}$. In order to prove that aS is projective, we may assume, by Lemma 4 of [3], that A is injective, with no loss of generality. Now the rest of the proof is dual to that of (1) $\Rightarrow$ (2). This proves the proposition.

PROPOSITION 2.3.

For an S-system M the following assertions are equivalent:

(1) *M is von Neumann regular,*

(2) *M is regular and S is pM-injective.*

PROOF. (1) $\Rightarrow$ (2): Suppose M is von Neumann regular. It follows easily that M is regular. We show that S is pM-injective. Let aS ($a \in M$) be a cyclic

S-subsystem of M and let $f : aS \to S$ be an S-homomorphism. Since M is von Neumann regular and $a \in M$, there exists an S-homomorphism $g : M \to S$ such that $a = ag(a)$. Define $\overline{f} : M \to S$ by $\overline{f}(x) = f(a)g(x)$ for all $x \in M$. Clearly, $\overline{f}$ is an S-homomorphism which extends f. Hence S is pM-injective.

(2) $\Rightarrow$ (1): Suppose M is regular and S pM-injective. Then for every $a \in M$, there exists an S-homomorphism $f : aS \to S$ such that $a = af(a)$. Since S is pM-injective, so there exists an S-homomorphism $g : M \to S$, extending f. Hence $a = ag(a)$ and so M is von Neumann regular.

LEMMA 2.4. *Let Q be an M-projective S-system.*

If M_0 is either an S-homomorphic image or an S-subsystem of M, then Q is M_0-projective.

PROOF. The result is almost obvious in case M_0 is an S-homomorphic image of M. So we assume that M_0 is an S-subsystem of M. In order to show that Q is M_0-projective, consider an S-epimorphism $\varphi_0 : M_0 \to \overline{M}_0$ and an S-homomorphism $g : Q \to \overline{M}_0$.

Let ρ be the relation on M defined by $\rho = \ker \varphi_0 \cup i_M$, where $\ker \varphi_0$ is the usual kernel S-congruence and i_M is the identity relation on M. Set $\overline{M} = M/\rho$ and let $\varphi : M \to \overline{M}$ be the natural map. We can identify $\overline{M}_0$ with the S-subsystem $M_0/\ker \varphi_0$ of the S-system $\overline{M}$. Thus the natural map $\varphi : M \to \overline{M}$ is an extension of φ_0. By the M-projectivity of Q, there exists an S-homomorphism $f : Q \to M$ such that $\varphi \circ f = g$. But $\varphi(f(Q)) = g(Q) \subseteq \overline{M}_0 = M_0/\ker \varphi_0$. Hence $f(Q) \subseteq M_0$. Thus f can be regarded as an S-homomorphism from Q to M_0. This proves that Q is M_0-projective.

Berthiaume [1] has shown that each S-system M has a unique essential extension $E = E(M)$ called the *injective hull* of M.

LEMMA 2.5. *Let M be a projective S-system and $E = E(M)$ be the injective hull of M. If E is completely pM-injective, then each cyclic S-subsystem of M is M-projective.*

PROOF. Let aS ($a \in M$) be a cyclic S-subsystem of M, and let $k : aS \to M$ be the inclusion map. Consider an epimorphism $p : E \to \overline{E}$ and an S-homomorphism $\alpha : aS \to \overline{E}$. Since E is completely pM-injective, $\overline{E}$ is pM-injective. Hence there exists an S-homomorphism $\beta : M \to \overline{E}$ such that $\beta \circ k = \alpha$. Since M is a projective S-system, there exists an S-homomorphism $\varphi : M \to E$ such that $p \circ \varphi = \beta$. Let $\theta = \varphi \circ k$. Then $p \circ \theta = p \circ \varphi \circ k = \beta \circ k = \alpha$. Thus aS is E-projective. Hence, by Lemma 2.4, aS is M-projective.

To summarize, we may now state the following characterization theorems for

a monoid.

THEOREM 2.
For a monoid S, the following assertions are equivalent:
(1) *S is regular.*
(2) *S is a pp-monoid.*
(3) *$E = E(S)$ is completely p-injective.*
(4) *Every p-injective S-system is completely p-injective.*
(5) *Every injective S-system is completely p-injective.*

PROOF. As a corollary of Propositions 2.1 and 2.2 we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) since S is projective. Se we need only to prove (2) $\Leftrightarrow$ (3).

Necessity follows from Proposition 2.2 as a corollary. For sufficiency, let aS be a principal right ideal of S ($a \in S$). From Lemma 2.5 it follows that aS is S-projective and from this it follows that aS is a retract of S. Hence aS is projective, since S is always projective. Therefore, S is a *pp*-monoid. This completes the proof.

THEOREM 3.
For a monoid S, the following assertions are equivalent:
(1) *S is von Neumann regular.*
(2) *S is regular and p-injective.*
(3) *S is completely p-injective.*
(4) *Every S-system is p-injective.*
(5) *Every cyclic S-system is p-injective.*

PROOF. As a corollary of Proposition 2.3, we have (1) $\Leftrightarrow$ (2). The proof of (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) is found in [12]. So we need only to prove (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3): This follows as a corollary of Proposition 2.2

(3) $\Rightarrow$ (2): Suppose S is completely p-injective. For every $a \in S$, consider a surjection $\pi : S \to aS$ defined by $\pi(s) = as$ for all $s \in S$. Since S is completely p-injective, aS is p-injective. Consider the inclusion map $k : aS \to S$ and the identity map $1_{aS} : aS \to aS$. Since aS is p-injective, we have an S-homomorphism $g : S \to aS$ such that $a = g(a) = g(1)a$. Since π is surjective, there exists $x \in S$ such that $g(1) = ax$. Thus we have $a = axa$ with $x \in S$. This shows S is von Neumann regular. Since (1) $\Leftrightarrow$ (2), this completes the proof.

PROPOSITION 2.6.
If a right S-system M is left M-cancellative, then M is regular.

PROOF.

Let $a \in M$ be any element. Then we may define an S-homomorphism $g : aS \to S$ by $g(as) = s$ for all $s \in S$ by the assumption. Since $g(a) = 1$, $a = a \cdot 1 = ag(a)$. Hence M is regular.

COROLLARY.
If S is left cancellative, then S is a pp-monoid.

PROPOSITION 2.7.
For a monoid S the following assertions are equivalent:
(1) *S is a pp-monoid with a unique idempotent,*
(2) *S is left cancellative.*

PROOF. From the above corollary we have (2) $\Rightarrow$ (1).

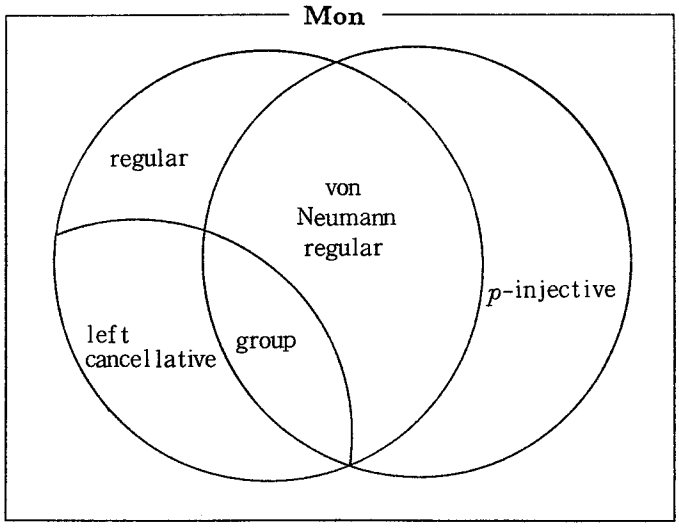
(1) $\Rightarrow$ (2): Suppose we have $as = as'$ for a, s, s' in S. Since S is regular, there exists an S-homomorphism $f : aS \to S$ such that $a = af(a)$. Then $f(a) = f(a)f(a)$. By the uniqueness of the idempotent element, it follows that $f(a) = 1$. Then the equation $as = as'$ implies that $s = 1 \cdot s = f(a)s = f(as) = f(as') = f(a)s' = 1 \cdot s' = s'$, that is, $s = s'$. Therefore S is left cancellative.

PROPOSITION 2.8.
For a monoid S the following assertions are equivalent:
(1) *S is left cancellative and S is p-injective,*
(2) *S is a group.*

PROOF. (2) $\Rightarrow$ (1): Suppose S is a group. Then S is left cancellative and S is S-divisible. It follows that S is p-injective by Proposition 1.1.

(1) $\Rightarrow$ (2): Let $a \in S$. Since S is left cancellative, S is S-divisible by Proposition 1.3. It follows that S is a group.

To summarize, we may now describe the following figure of the category **Mon** (cf. [16]):



3. Weakly regular monoids and normal S-systems

In [2], Brown and McCoy considered the notion of weakly regular rings. These rings were later studied by Ramamurthy [14] and others. Adopting this notion we have the following definition.

DEFINITION 3.1.
A semigroup S is right weakly regular if for all $x \in S$, $x \in (xS)^2$.

Thus if S is commutative and weakly regular, then S is von Neumann regular.

In this section we shall define normal homomorphisms and normal S-systems, and characterize weakly regular monoids by the property that each S-system is normal.

First we introduce the following notations.

Let S be a monoid, I a two-sided ideal of S and A a right S-system. Then

$$AI = \{ax \mid a \in A \quad \text{and} \quad x \in I\}$$

is a right S-subsystem of A. By A_I we shall denote the Rees factor of A by AI, that is,

$$A/AI = (A \backslash AI) \cup \{AI\}.$$

Let $\alpha : A \to B$ be an S-homomorphism. Then we define the map $\alpha_I : A_I \to B_I$ by

$$\alpha_I(a) = \begin{cases} \alpha(a) & \text{if } a \in A \backslash AI \\ \{BI\} & \text{if } a = \{AI\}. \end{cases}$$

Clearly, α_I is an S-homomorphism. Moreover, if α is an epimorphism then so is α_I. We also note that if $\alpha : A \to B$ is an S-homomorphism and I is a two-sided ideal of S, then we always have the inclusion $\alpha(AI) \subseteq BI \cap \alpha(A)$. When equality holds, α has special interest.

DEFINITION 3.2.

An S-homomorphism $\alpha : A \to B$ is called normal if $\alpha(AI) = BI \cap \alpha(A)$ for every two-sided ideal I of S.

PROPOSITION 3.1. *Let $\alpha : A \to B$ be an S-monomorphism and I a two-sided ideal of S. Then the following assertions are equivalent:*

(1) *$\alpha_I : a_I \to B_I$ is an S-monomorphism,*

(2) *$\alpha(AI) = BI \cap \alpha(A)$.*

PROOF. (1) $\Rightarrow$ (2): Suppose α_I is an S-monomorphism. We verify that $\alpha(AI) = BI \cap \alpha(A)$. Since we always have the inclusion $\alpha(AI) \subseteq BI \cap \alpha(A)$, we only show that $BI \cap \alpha(A) \subseteq \alpha(AI)$. Let $\alpha(x) \in BI$ for some $x \in A$. Then $\alpha_I(x) = BI = \alpha_I(AI)$. Since α_I is an S-monomorphism, the coset x and AI are same, that is, $x \in AI$. Hence $\alpha(x) \in \alpha(AI)$. Thus it follows that $\alpha(AI) = BI \cap \alpha(A)$.

(2) $\Rightarrow$ (1): Assume that $\alpha(AI) = BI \cap \alpha(A)$. Suppose we have $\alpha_I(a) = \alpha_I(a')$ for $a, a' \in A_I$. Assume $a, a' \notin AI$. Then $\alpha_I(a) = \alpha_I(a')$ implies that $\alpha(a) = \alpha(a')$. Since α is an S-monomorphism, it follows that $a = a'$.

If $a \in AI$, then a' also belongs to AI. For otherwise $\alpha_I(a) = BI = \alpha_I(a') = \alpha(a')$. This means that $\alpha(a') \in BI$, that is, $\alpha(a') \in BI \cap \alpha(A) = \alpha(AI)$. Since α is a monomorphism, it follows that $a' \in AI$ which is absurd. Also, since a and a' are both in AI, they represent the same coset in A_I, that is, $a = a'$.

COROLLARY. *Let $\alpha : A \to B$ be an S-monomorphism. Then α is normal if and only if α_I is an S-monomorphism for every two-sided ideal I of S.*

DEFINITION 3.3.

An S-subsystem N of an S-system M is normal in M if the inclusion $k : N \to M$ is normal. M is called normal if every S-subsystem N of M is normal in M.

From the above definition and Proposition 3.1, it follows that M is normal if and only if $NI = MI \cap N$ for every S-subsystem N of M and every two-sided ideal I of S.

DEFINITION 3.4.

A two-sided ideal I of a semigroup S is called right (left) pure if for each $x \in I$, there exists $y \in I$ such that $x = xy$ ($x = yx$) : in other words, I is right pure if and only if for every $a \in I$ the equation $a = ax$ has a solution in I.

LEMMA 3.2. *Each two-sided ideal of a right weakly regular semigroup is right weakly regular as a semigroup.*

PROOF. Let I be a two-sided ideal of a right weakly regular semigroup S and let $x \in I$. Then $x \in (xS)(xS)$. Hence $x \in xSxSxSxS \subseteq x(SxS)x(SxS) \subseteq (xI)(xI)$. This means that I is right weakly regular.

LEMMA 3.3 *Let S be a right weakly regular semigroup. Then each S-monomorphism $\alpha : A \to B$ is normal.*

PROOF. By the corollary of Proposition 3.1, in order to prove that α is normal, we must show that $\alpha_I : A_I \to B_I$ is an S-monomorphism for every two-sided ideal I of S. By Proposition 3.1, it is sufficient to prove that for every two-sided ideal I of S, we have $BI \cap \alpha(A) \subseteq \alpha(AI)$. Let $x \in BI \cap \alpha(A)$. Since $x \in BI$, we can write $x = bt$, where $b \in B$ and $t \in I$. By Lemma 3.2, I (considered as a semigroup) is right weakly regular. Hence $t \in I$ implies that there exists an element $s \in I$ such that $t = ts$. Hence $x = bt = bts = xs$. Since $x \in \alpha(A)$, it follows that $x = xs \in \alpha(A)s \subseteq \alpha(As) \subseteq \alpha(AI)$. Hence $BI \cap \alpha(A) \subseteq \alpha(AI)$. This proves the lemma.

PROPOSITION 3.4.
For a monoid S the following assertions are equivalent:
(1) *S is right weakly regular,*
(2) *$B^2 = B$ for all right ideals B of S,*
(3) *$BA = B \cap A$ for all right ideals B and all two-sided ideals A of S,*
(4) *Every two-sided ideal of S is right pure.*

PROOF. (1) $\Rightarrow$ (2): Let B be a right ideal of S. Clearly, $B^2 \subseteq B$. Let $x \in B$. Then $s \in (xS)(xS) \subseteq B \cdot B = B^2$. This proves that $B = B^2$.

(2) $\Rightarrow$ (3): Let B be a right ideal and A a two-sided ideal of S. Clearly, $BA \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$. Since $x \in xS = (xS)(xS) = x(SxS) \subseteq xA \subseteq BA$, we have $B \cap A \subseteq BA$ and so $B \cap A = BA$.

(3) $\Rightarrow$ (1): Let $x \in S$ and let $A = SxS$ be the two-sided ideal generated by x. If B is the right ideal xS generated by x, then $x \in B \cap A = BA = (xS)(SxS) \subseteq xS^2xS \subseteq (xS)(xS)$. This implies that S is right weakly regular.

(1) $\Leftrightarrow$ (4): Suppose S is right weakly regular. Let A be a two-sided ideal of S and $a \in A$. Since S is right weakly regular, $a \in (aS)(aS) = a(SaS) \subseteq aA$.

Hence there exists an element $x \in A$ so that $a = ax$. Thus A is right pure.

Conversely, assume that each two-sided ideal of S is right pure. In order to show that S is right weakly regular, let $x \in S$ and $A = SxS$ be the two-sided ideal of S generated by x. By the hypothesis, $x \in xA = x(SxS) = (xS)(xS)$. Hence S is right weakly regular. This completes the proof.

We are now ready to prove

THEOREM 4.
For a monoid S the following assertions are equivalent:
(1) *S is right weakly regular.*
(2) *S is normal.*
(3) *Every S-monomorphism is normal.*
(4) *Every S-system is normal.*

PROOF. (1) $\Leftrightarrow$ (2): Suppose S is right weakly regular. If B is a right ideal of S then the inclusion map is normal by Lemma 3.3. Hence B is normal in S, and so S is normal.

Conversely, assume that S is normal. If B is any right ideal of S, then B, considered as a right S-system, is normal in S. This means that the inclusion map $\alpha : B \to S$ is a normal monomorphism. Hence it follows from Definition 3.2 that for any two-sided ideal A of S, $BA = B \cap A$. Thus, by Proposition 3.4, S is right weakly regular.

(1) $\Rightarrow$ (3): This follows from Lemma 3.3.

(3) $\Rightarrow$ (4): Immediate.

(4) $\Rightarrow$ (1): Since S is normal by the hypothesis, S is right weakly regular by the argument (1) $\Leftrightarrow$ (2) proved as above.

(1) $\Rightarrow$ (4): This follows from Lemma 3.3. This completes the proof.

COROLLARY.
For a commutative monoid S the following assertions are equivalent:
(1) *S is von Neumann regular.*
(2) *Every ideal of S is pure.*
(3) *S is normal.*
(4) *Every S-system is normal.*

PROOF. For a commutative monoid S, S is von Neumann regular if and only if S is weakly regular by definitions. Thus this follows from Propositin 3.4 and Theorem 3 as corollary.

PROPOSITION 3.5. *Let S be a pp-monoid (commutative or not).*

If A is a two-sided ideal of S such that the Rees factor S/A is p-injective as an S-system, then A is right pure in S.

PROOF. Let $a \in A$. Then aS is a principal right ideal of S. Consider the Rees factor aS/aA of the right S-system aS. Let $g : aS \to aS/aA$ be the natural map defined by

$$g(as) = \begin{cases} as & \text{if } as \notin aA \\ aA & \text{if } as \in aA. \end{cases}$$

Also define $f : S/A \to aS/aA$ by

$$f(s) = \begin{cases} as & \text{if } s \in S \backslash A \\ aA & \text{if } s = \{A\}. \end{cases}$$

Clearly, f is an S-epimorphism. Since S is a right pp-monoid, the principal ideal aS is projective as an S-system. It follows that there exists an S-homomorphism $h : aS \to S/A$ such that $f \circ h = g$. Hence $f \circ h = g$.

Let us now consider the inclusion map $k : aS \to S$. Since S/A is p-injective, for the S-homomorphism h there exists an S-homomorphism $\overline{h} : S \to S/A$ such that $\overline{h} \circ k = h$. Note that $\overline{h}$ extends $h : \overline{h} \circ k = h$. Let $\overline{h}(1) = s$ for $s \in S/A$. Let $\overline{g} = f \circ \overline{h}$. Then $\overline{g} : S \to aS/aA$ is an S-homomorphism.

We now verify that $\overline{g}$ is an extension of g. Let $x \in aS$. Then $\overline{g}(x) = f \circ \overline{h}(x) = f \circ \overline{h} \circ k(x) = f \circ h(x) = g(x)$.

Now if $a \in aA$ then we are done. If $a \notin aA$ then $g(a) = a$. Since $\overline{g}$ is an extension of g, so

$$a = g(a) = \overline{g}(a) = \overline{g}(1 \cdot a) = \overline{g}(1)a = (f \circ \overline{h}(1))a =$$

$$(f(s))a = \begin{cases} asa & \text{if } s \notin A \\ aAa & \text{if } s \in A. \end{cases}$$

This implies that $a \in aA$, in any case. Hence A is right pure.

COROLLARY 1. *Let S be a right pp-monoid.*

If for each two-sided ideal A of S, the Rees factor S/A is p-injective as an S-system, then S is right weakly regular.

PROOF. By the above proposition, each two-sided ideal of S is right pure. Hence by Proposition 3.4(4), S is right weakly regular.

COROLLARY 2.

A commutative monoid S is von Neumann regular if and only if S is a pp-monoid such that for each two-sided ideal A of S, the Rees factor S/A is p-injective.

Recall that a right S-system M $(\neq \theta)$ is called *simple* if M has no proper non zero S-subsystem.

PROPOSITION 3.6

If S is a monoid for which every simple S-system is p-injective, then S is right weakly regular.

PROOF. We prove that each right ideal A of S is idempotent, i.e. $A^2 = A$. Now suppose $A \neq A^2$. Let $a \in A$ such that $a \notin A^2$. Then $aS \neq (aS)^2$. By Zorn's Lemma, the set of right ideals I such that $(aS)^2 \subseteq I \subset aS$ has a maximal element B (say). Then aS/B is simple and so p-injective by hypothesis. Let $f : aS \to aS/B$ be the natural S-homomorphism defined by

$$f(as) = \begin{cases} as & \text{if } as \in aS \backslash B \\ \{B\} & \text{if } as \in B. \end{cases}$$

By the p-injectivity of aS/B, there exists an S-homomorphism $g : S \to aS/B$ which extends f. Let $g(1) = at$ $(t \in S)$. Then $f(as) = g(as) = g(1)as = (at)(as)$. Since $f(a) = a$, we have $a = f(a) = g(a) = (at)a = (at)(a \cdot 1) \in (aS)^2 \subseteq B$. Hence $a \in B$ and so $B = aS$. This is a contradiction. Hence $A = A^2$. From Proposition 3.4(2), it follows that S is weakly regular.

COROLLARY. ([11])

A commutative monoid S is von Neumann regular if and only if each simple S-system is p-injective.

Acknowledgments. An initial version of this paper was read by Professor Peeter Normak. We take this opportunity to thank him for suggesting improvements. We are also grateful to Professor Mati Kilp for his interest in this work.

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