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(Citation)
Kobe Journal of Mathematics,9(1):39-42

(Issue Date)
1992-06

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0003768

(URL)
https://hdl.handle.net/20.500.14094/E0003768

MOTEGI, K. and SHIBUYA, T.
KOBE J. MATH.,
9 (1992) 39—42

ARE KNOTS OBTAINED FROM A PLAIN PATTERN ALWAYS PRIME?

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(Received April 15, 1991)

For the terminology used in this paper, the reader is referred to [1], [5].

Let V be a standardly embedded solid torus in an oriented three sphere S^3, and K a trivial knot contained in V. Following Litherland [3], we adopt the name *plain pattern* for such a pair (V, K). In what follows we assume that the *wrapping number* of K in V (i.e. the geometric intersection number of K and a meridian disk of V) is greater than one.

Using an orientation preserving embedding f from V into S^3, we can construct a new knot $f(K)$ in S^3. If $f(V)$ is an unknotted solid torus in S^3, then this procedure essentially corresponds with twisting along the solid torus V. If $f(V)$ is a knotted solid torus in S^3, then this procedure corresponds with the construction of satellite knots. In the first case if f maps *longitude* to $\varepsilon((longitude) + n(meridian))$ for $\varepsilon = \pm 1$, we use the symbol K_n to denote the knot $f(K)$.

The purpose in this short note is to answer the following question given by Y. Mathieu [4] and the authors [2] and study primeness of knots constructed in the above way.

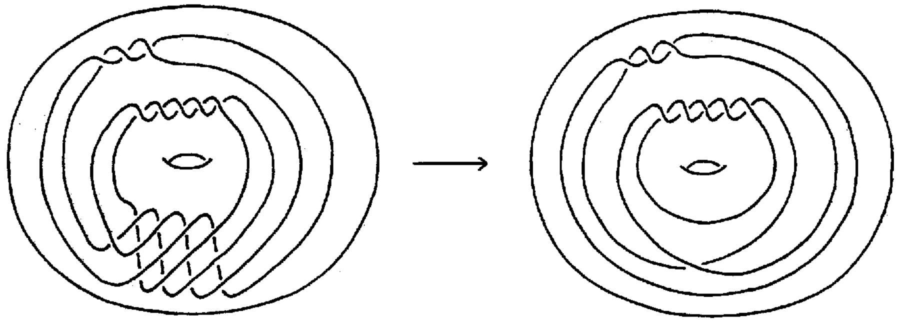
QUESTION 1. Can we obtain nonprime knots from a trivial knot by twisting?

In the case where the wrapping number of K in V equals two, Scharlemann-Thompson [7], Eudave-Munoz and Gordon have shown that any knot obtained from K by twisting along V is prime. This result is a generalization of "unknotting number one knots are prime"[6].

But in general, we have the following.

EXAMPLE. Let (V, K) be a plain pattern depicted in Figure 1, then K_1 is a nonprime knot whose prime factors are torus knots of types (2, 3) and (2, 5).

* Research partially supported by Grant-in-Aid for Encouragement of Young Scientists 03740047, The Ministry of Education, Science and Culture.



(V, K) $\qquad\qquad$ K_1

Fig. 1

This example gives an answer to the above question.

In [2], we proved that K_n's ($n \in \mathbf{Z}$) are stably prime, namely;

THEOREM 1. *Let (V, K) be a plain pattern. For all but at most finitely many integers n, K_n is prime.*

From this we can ask the following question.

QUESTION 2. (1) Is there a universal bound for such integers n?

(2) Except for only one integer n, is K_n prime?

In the case where $f(V)$ is a knotted solid torus in S^3, we can show the following result.

THEOREM 2. *Let (V, K) be a plain pattern. Then $f(K)$ is prime for any embedding f from V into S^3 such that $f(V)$ is knotted in S^3.*

Actually we can prove the stronger result below in which K is not necessarily unknotted in S^3, and which gives a necessary and sufficient conditions for pattern (V, K) so that it yields only prime satellite knots.

THEOREM 3. *Let V be a standardly embedded solid torus in S^3 and K a knot contained in V such that the wrapping number of K in V is greater than one. Then $f(K)$ is prime for any embedding f from V into S^3 such that $f(V)$ is knotted, if and only if K does not have a locally knotted arc in V.*

PROOF. The "only if" part is obvious. Now assume that $f(K)$ is a nonprime knot for some embedding f from V into S^3 such that $f(V)$ is knotted in S^3. Then by the definition, we have a splitting 2-sphere S which meets $f(K)$ in only two points, and $f(K)$ meets each 3-ball bounded by S in knotted arc. Consider the intersection of S and $\partial f(V)$. We may assume $S \cap \partial f(V)$ consists of only finitely many disjoint circles. Let C be any innermost circle of $S \cap \partial f(V)$, then it is easy to check that C is a meridian or inessential in $\partial f(V)$ since $f(V)$ is knotted. In addition if C is a meridian of $\partial f(V)$, then the disk in S bounded by C is a meridian disk of $f(V)$. To prove the "if" part, it suffices to find a 2-sphere S' in $f(V)$ satisfying the above property of S. To do this, we modify the given S so that the result becomes S'. If $S \cap \partial f(V)$ is empty, then S is the required 2-sphere.

Step 1. $S \cap \partial f(V)$ consists of only one component.

In this case, this circle C must be inessential in $\partial f(V)$, otherwise C is a meridian of $\partial f(V)$ and it is also inessential in $S^3 - int f(V)$, and this contradicts the fact that C is a generator of $H_1(S^3 - int f(V))$ isomorphic to the infinite cyclic group $\mathbf{Z}$. Let D be the disk in S bounded by C contained in $f(V)$ and D' the disk in $\partial f(V)$ bounded by C. Then $D \cup D'$ becomes a 2-sphere intersecting $f(K)$ in two points. Moreover the 3-ball bounded by this 2-sphere $D \cup D'$ meets $f(K)$ in knotted arc. Thus $D \cup D'$ is a required splitting 2-sphere S'.

Step 2. $S \cap \partial f(V)$ has at least two components.

Then there are at least two innermost circles of $S \cap \partial f(V)$ in S and one of them must be inessential in $\partial f(V)$. Otherwise we have two innermost circles C_1 and C_2 which are meridians in $\partial f(V)$ and two disks D_1, D_2 bounded by C_1, C_2 respectively. Recall that D_1 and D_2 are meridian disks of $f(V)$. Since the wrapping number of $f(K)$ in $f(V)$ is greater than one, the geometric intersection number of D_i and $f(K)$ is greater than or equal two for $i = 1, 2$. This implies the intersection number of S and $f(K)$ is greater than or equal four. This is a contradiction. Let C be an inessential innermost circle of $S \cap \partial f(V)$ in S, and D the disk in S bounded by C and D' the disk in $\partial f(V)$ bounded by C. If $D \cap f(K)$ is not empty, then $D \cup D'$ is a required splitting 2-sphere by the same reason in Step 1. If $D \cap f(K)$ is empty, then by an isotopic deformation which leaves $f(K)$ fixed we can eliminate the intersection C. By the inductive argument we can find the required 2-sphere S'. This completes the proof. ■

Since a trivial knot does not have a locally knotted arc, Theorem 2 is a direct consequence of Theorem 3.

REMARK 1. From Theorem 3, we see that "$f(K)$ is prime for some embedding f" if and only if "$f(K)$ is prime for any embedding f".

REMARK 2. Even when K does not have a locally knotted arc in V, K itself may be nonprime (see Figure 2).

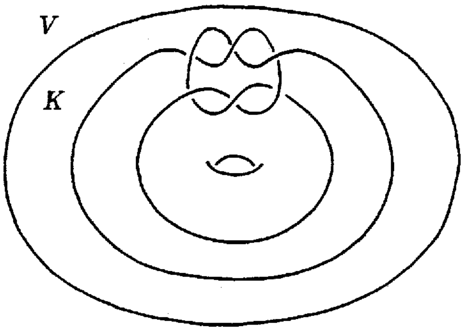


Fig. 2.

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