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DUALITIES BETWEEN Hom_R AND $\otimes_R$ AND THEIR APPLICATIONS

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Introduction

The results contained here concern modules over commutative valuation rings, mostly valuation domains.

In section 1, we study the categories $\mathcal{C}(I)$ and $\mathcal{D}(I)$ (I is a rank one module) of modules M and N respectively satisfying the natural isomorphisms:

$$I \otimes_R \text{Hom}_R(I, M) \cong M \quad \text{and} \quad N \cong \text{Hom}_R(I, I \otimes_R N).$$

Theorem 1.8 characterizes $\mathcal{C}(I)$ as the class of modules M for which M equals to the trace $M(I) = \bigcup_{f: I \rightarrow M} \text{im } f$ of I in M and $\mathcal{D}(I)$ as the class of modules N satisfying the condition

$$\forall a \in M \quad \bigcap_{i \in I} \chi_{IM}(ia) = \chi(a).$$

The objects of the categories $\mathcal{C}(I)$ and $\mathcal{D}(I)$ are in a bijective correspondence, via the functors $\text{Hom}_R(I, -)$ and $I \otimes_R -$. However the closure properties of these categories differ, because these functors are not full. Both $\mathcal{C}(I)$ and $\mathcal{D}(I)$ are subcategories of the category of $R_{I^\dagger}$ -modules, the former contains $\mathcal{H}(I)$ – the category of I -homogeneous modules and the latter contains $\mathcal{H}(R_{I^\dagger})$. Consequently, there are bijections between classes of objects of $\mathcal{H}(I)$ and $\mathcal{H}(R_{I^\dagger})$.

An ideal is called endomorphic, if it satisfies $I \cong R_{I^\dagger}$; several characterizations of these ideals are given. One is that the class $\mathcal{C}(I)$ is closed with respect to the direct products. Theorem 1.14 states that products of I -homogeneous modules are likewise I -homogeneous if and only if I is endomorphic and CGCD. An immediate spin-off is a result on products of separable and homogeneous R -modules for a discrete rank one valuation domain R .

Section 2 is devoted to balanced exact sequences i.e. pure exact sequences $0 \rightarrow A \xrightarrow{*} B \xrightarrow{q} C \rightarrow 0$ with the property that for every $c \in C$, there is a $b \in B$ with $qb = c$ and $\chi_C(c) = \chi_B(b)$. We give several equivalent definitions, establish some properties and prompted by the natural isomorphism

$$R_{I^\dagger} R_{J^\dagger} \cong R_{(I \otimes_R J)^\dagger},$$

we indulge in finding out when the natural monomorphism

$$\text{Hom}_R(I, M) \otimes_R \text{Hom}_R(J, N) \cong \text{Hom}_R(I \otimes_R J, M \otimes_R N),$$

is an isomorphism (I and J are of rank one). Sufficient conditions are given in Theorem 2.4: If M is I -homogeneous and N is tensor balancing (i.e. balanced exact sequences remain balanced exact after tensoring by N), then the monomorphism is an isomorphism in any of the cases $N \in \mathcal{H}(J)$ or $N \in \mathcal{D}(I)$.

Finally, in section 3, the endomorphism rings of separable homogeneous modules are discussed. In Theorem 3.5, we give necessary and sufficient conditions for a ring S_R to be an endomorphism ring of a torsion free, separable, homogeneous R -module. The characterization is in terms of minimal idempotents of S and completeness of S in the annihilator topology.

For most of special notation and terminology, we refer to [D2].* Throughout the paper we will assume that R is a commutative valuation ring (in fact a domain most of the time) with 1, not a field, and all R -modules will be unitary and torsion free, although, upon close inspection of definitions and proofs, one can see that some of the results remain valid for wider class of rings as well as modules. Since rank one torsion free modules are isomorphic either to Q – the quotient field of R or to an ideal of R , we will identify such modules with submodules of Q . Moreover, our ideals I or R would often include the cases $I = R$ or $I = Q$. The maximal ideal of R will be denoted by P and the monoid of units by U . $N \leq_* M$ will mean that N is a pure submodule of M . $\mathcal{M}(\rho)$ will denote the category of ρ -modules over a commutative ring ρ . For unexplained terminology and theoretical background we refer to the standard reference on modules over valuation domains [FS].

For an R -module M define

$$M^b = \{r \in R \mid rM = M\} \quad (\text{read: “}M \text{ flat”}).$$

This is apparently a multiplicatively closed subset of R which contains units and it measures how near is a module to a divisible module. Closely related to this set is

$$M^\sharp = \{r \in R \mid rM < M\} = R \setminus M^b \quad (\text{read: “}M \text{ sharp”})$$

which is consequently a prime ideal of R (enabling localizations of R).

For an R -module M and $a \in M$, the set

$$\chi_M(a) = \{r \in R : a = ra_1 \text{ for some } a_1 \in M\}$$

* The expanded version of the present manuscript dates back to June 1989.

is called the characteristic of a in M . Characteristics $\chi_M(a)$ and $\chi_M(b)$ are equivalent if there is an $r \in R$ with either $\chi(r)\chi_M(b) = \chi_M(a)$ or $\chi(r)\chi_M(a) = \chi_M(b)$. The equivalence class of this relation is the type $t_M(a)$ of a . The Baer-like theorem holds: Two rank one modules are isomorphic if and only if they are of the same type and every type is the type of a rank one module.

An R -module M is homogeneous of type t if, for every $a \in M$, $t_M(a) = t$. By Theorem 2.5 [D2], we can interchangeably use the following equivalent definition: If $I \leq Q$, then an R -module M is called I -homogeneous or homogeneous of type I if, for every $a \in M$, the purification $\langle a \rangle_* \cong I$. Obviously $tI = t$. $\mathcal{H}(I)$ (or $\mathcal{H}(t)$, if $tI = t$) will denote the category of I -homogeneous (or t -homogeneous) R -modules.

For a given cardinal κ , an ideal I of R is a κ -greatest common divisor ideal (or abbreviated: κ -GCD), if for every set $\{i_\alpha\}$ of elements of I , of cardinality κ , there is an $i_0 \in I$ such that

$$\bigcap_{\alpha \in \kappa} \chi_I(i_\alpha) = \chi_I(i_0)$$

If I is κ -GCD, for finite κ , we call it a GCD -ideal and if the property is satisfied for all κ , I will be called a *complete greatest common divisor ideal* (or a $CGCD$ -ideal).

If the ring itself satisfies the properties, it will be given the appropriate names. Notice that every Bezout domain is a GCD ring while every factorial domain is a $CGCD$ ring. GCD as well as $CGCD$ rings are completely integrally closed.

1. The Categories $\mathcal{C}(I)$ and $\mathcal{D}(I)$

A special, easy-to-establish but useful property is that all non-zero homomorphisms from rank one modules into torsion free modules are injective.

LEMMA 1.1. *If I and M are torsion free modules where I is of rank one, then every non-zero homomorphism $f : I \rightarrow M$ is injective.*

PROOF. Let $i_0 \in I$ be such that $fi_0 \neq 0$. For every non-zero $i \in I$ either $i = ri_0$, in which case $fi = rfi_0 \neq 0$ or $i_0 = ri$, in which case $fi_0 = rfi \neq 0$, hence $fi \neq 0$.

The “flats” and “sharps” of product are calculated in

PROPOSITION 1.2. *For an ideal I of R and a torsion free R -module M ,*

- (a) $(I \otimes_R M)^\flat = I^\flat \cdot M^\flat = I^\flat \cup M^\flat$;
- (b) $(I \otimes_R M)^\sharp = I^\sharp \cap M^\sharp$;

PROOF. (a) One inclusion follows from Lemma 1.2(b), [D2]. Let, for the reverse inclusion, $r \in (IM)^b$. Notice that in general $J < I$ and $N < M$ imply $JN < IM$. If $r \in I^b$, we are done. If $rI < I$, and also $rM < M$, then $(rI)(rM) < IM$, i.e. $r(rIM) = rIM = IM < IM$, a contradiction. Hence $r \in M^b$. The remainder of the proof is a straightforward checking.

(b) This is simply the R -complement of (a).

The last proposition along with Proposition 1.3, [D2] (see also Lemma 1.4 below) suffice to prove the following

COROLLARY 1.3. *If I and J are rank one modules, then*

$$R_{I^b} \otimes_R R_{J^b} \cong R_{(IJ)^b}.$$

We would like to expand this isomorphism however and begin by considering the mapping

$$\xi : \text{Hom}_R(I, M) \otimes_R \text{Hom}_R(J, N) \longrightarrow \text{Hom}_R(I \otimes_R J, M \otimes_R N).$$

defined by

$$(\xi \sum_{\alpha, \beta} \alpha \otimes \beta)(i \otimes j) = (\sum_{\alpha, \beta} \alpha \otimes \beta)(i \otimes j) = \sum_{\alpha, \beta} \alpha(i) \otimes \beta(j),$$

where $\alpha \in \text{Hom}_R(I, M)$ and $\beta \in \text{Hom}_R(J, N)$, $i \in I$, $j \in J$. Here M and N are torsion free R -modules and I and J are rank one torsion free R -modules.

ξ is a well defined injective homomorphism since every homomorphism on the right-hand side is uniquely determined by the image of one of the elements $i \otimes j \in I \otimes J$, by [D1, Lemma 6]. Moreover, there is no difficulty in showing that ξ is natural both in M and N . By the appropriate substitutions for modules in the homomorphism ξ we arrive at the following injective homomorphisms:

$$\phi : I \otimes_R \text{Hom}_R(I, M) \longrightarrow M,$$

defined by $\phi(i \otimes_R f) = f(i)$ and

$$\psi : N \longrightarrow \text{Hom}_R(I, I \otimes_R N),$$

defined by $(\psi n)(i) = i \otimes n$. Again both ϕ and ψ are natural in M and N respectively.

If R is a valuation domain whose ideals are principal, all the monomorphisms ξ, ϕ, ψ are isomorphisms but this is not true in general, and it can be seen from Example 1.7 below.

First we need some auxiliary tools such as the following lemma having one of its consequences the fact that an R -module M may be regarded as an $R_{I^\#}$ -module as well, as long as the inequality $I^\# \geq M^\#$ is satisfied.

LEMMA 1.4. *If I and M are torsion free modules, where I is of rank one, then the following are equivalent:*

- (a) $I^\# \geq M^\#$;
- (b) $R_{I^\#} \otimes_R M \cong M$ (natural isomorphism), i.e. M is an $R_{I^\#}$ -module.

PROOF. If (a), then $R_{I^\#} \leq R_{M^\#}$, thus $M \cong M \otimes_R R_{M^\#} \geq M \otimes_R R_{I^\#}$, but because of $R \leq R_{I^\#}$, $M \cong R \otimes_R M \leq R_{I^\#} \otimes_R M$, so we have (b). Conversely, if (b), then every $s \in I^\#$ divides M i.e. $s \in M^\#$.

Let us call an ideal I of R *endomorphich* if $I \cong R_{I^\#}$. I is *idempotent* if $I^2 = I$ and finally I is an *iso-idempotent* ideal if $I \cong I^2$.

We first record a lemma (Lemma 4.8, [FS], p.16) on endomorphich ideals and we will give several more characterizations of these ideals in the sequel:

LEMMA 1.5. *An ideal I of R is endomorphich if and only if $I^\# I < I$.*

An immediate consequence is that the maximal ideal P is idempotent if and only if it is not principal, and it is endomorphich, (but not idempotent) if and only if it is principal.

Proper idempotent ideals must be at least countably generated. Both endomorphich and idempotent ideals are iso-idempotent and the relation between idempotent and endomorphich ideals is described by

PROPOSITION 1.6. *If I is a proper ideal of R , then I is idempotent if and only if I is prime and is not endomorphich.*

PROOF. If I is idempotent, then, for all natural n , $I^n = I$ and by [FS, Proposition 1.6, (d)], $\bigcap_{n \in \mathbb{N}} I^n = I$ must be prime. Hence $I^\# I = I^2 = I$ and Lemma 1.5, implies that I is not endomorphich. Conversely, if I is prime and not endomorphich, then $I^\# I = I^2$, hence I is idempotent.

EXAMPLE 1.7. The isomorphisms $\text{Hom}_R(P, R) \cong \text{Hom}_R(P, P) \cong R$ are self evident and if P is not principal, the following non-isomorphisms give examples for which the monomorphisms ξ, ϕ and ψ respectively are not epimorphisms:

$$\text{Hom}_R(P, R) \otimes_R \text{Hom}_R(R, P) \not\cong \text{Hom}_R(P \otimes_R R, R \otimes_R P).$$

$$P \otimes_R \text{Hom}_R(P, R) \not\cong R,$$

$$P \not\cong \text{Hom}_R(P, P \otimes_R P)$$

It is easier to characterize the classes (categories) $\mathcal{C}(I)$ and $\mathcal{D}(I)$ of modules for which ϕ and ψ are canonical isomorphisms respectively, then finding conditions for the two-dimensional homomorphism ξ to be an isomorphism.

Define the *trace* of an ideal I in an R -module M to be

$$M(I) = \bigcup_{f:I \rightarrow M} \text{im} f = \{a \in M : \exists f \in \text{Hom}_R(I, M) \text{ with } a \in \text{im} f\}$$

By property (10), [D2] for characteristics,

$$M(I) = \{a \in M : \exists i \in I \chi_I(i) \leq \chi_M(a)\}.$$

Some properties of the trace are as follows:

- (1) $M(I)$ is a fully invariant submodule of M ;
- (2) $M(I) \cong N(I)$, for all ideals I of R , is equivalent to $M \cong N$;
- (3) If $f : A \rightarrow B$ is a homomorphism, then $fA(I) \subseteq B(I)$, i.e. $- (I)$ is a subfunctor of the identity;
- (4) $I \cong J$ implies $M(I) \cong M(J)$;
- (5) $N \leq M$ implies $N(I) \leq M(I)$;
- (6) If $N \leq_* M$, then $N(I) = M(I) \cap N$, thus $N(I) \leq_* M(I)$;
- (7) $IM \leq M(I)$;
- (8) $(\oplus M_k)(I) = \oplus M_k(I)$.

THEOREM 1.8. (a) $\mathcal{C}(I)$ is the class of all the modules M with the property that $\forall a \in M, \exists i \in I$ such that $\chi_I(i) \leq \chi_M(a)$ i.e. modules M for which $M = M(I)$.

(b) $\mathcal{D}(I)$ is the class of modules M with the property that

$$\forall a \in M \bigcap_{i \in I} \chi_{IM}(ia) = \chi(a).$$

PROOF. (a) The isomorphism $I \otimes_R \text{Hom}_R(I, M) \cong M(I)$ is easy to check.

(b) Suppose that the homomorphism ψ is surjective. $\chi_M(a) \leq \bigcap_{i \in I} \chi_{IM}(ia)$ is always true, for if $a = ra'$, then $ia = ria'$ for all $i \in I$. In order to prove the reverse inequality, let $r \in \bigcap_{i \in I} \chi_{IM}(ia)$. From here, $ia/r \in IM$, for all $i \in I$ hence we may define a homomorphism $f : I \rightarrow IM$ by $fi = ia/r$. Since ψ is surjective, there is an $a_0 \in M$ with $fi = ia_0 = ia/r$, for all $i \in I$, therefore $a = ra_0$ and $r \in \chi(a)$.

For the converse, we point out that ψ is surjective if and only if $\text{im} \psi \leq_* \text{Hom}_R(I, IM)$. In fact $\text{im} \psi$ is an essential submodule of $\text{Hom}_R(I, IM)$: If $f : I \rightarrow IM$ is a homomorphism, then $\forall i \in I$ either $fi = rim' = i(rm')$, for some

$r \in R$, $m' \in M$, and then $f \in \text{im } \psi$ or $fi = \frac{i}{r}m'$, in which case $rf \in \text{im } \psi$. Pure essential submodule must coincide with the ambient module. Suppose now the equality satisfied and let $f \in \text{Hom}_R(I, IM)$. If $rf \in \text{im } \psi$, for some $r \in R$, then $(rf)i = (\psi m_0)i = im_0$, for some $m_0 \in M$. Thus $r(fi) = im_0$, for all $i \in I$ and then $r \in \chi_{IM}(im_0)$, for all $i \in I$. This implies $r \in \chi(m_0)$ and $fi = i\frac{m_0}{r} \in IM$ i.e. $f \in \text{im } \psi$. This means that $\text{im } \psi$ is pure in $\text{Hom}_R(I, IM)$.

We would like to examine the categories $\mathcal{C}(I)$ and $\mathcal{D}(I)$ in more detail, to find out the closure properties as well as relate them to other categories. As for some closure properties (and more in the following section), we have the following

PROPOSITION 1.9. (a) $\mathcal{C}(I)$ is closed with respect to pure submodules, pure homomorphic images and direct sums.

(b) $\mathcal{D}(I)$ is closed under pure submodules, pure extensions, direct products and direct sums.

PROOF. (a) Let $M \in \mathcal{C}(I)$ and $N \leq_* M$. For $n \in N$, there is an $i \in I$ with $\chi(i) \leq \chi_M(n)$. But $\chi_M(n) = \chi_N(n)$, hence $N \in \mathcal{C}(I)$.

For the exact sequence $0 \rightarrow N \xrightarrow{*} M \xrightarrow{q} L \rightarrow 0$ take $a \in L$ and let $m \in M$ be such that $qm = a$. For this m there is a homomorphism $f : I \rightarrow M$ and an $i \in I$ such that $fi = m$. The homomorphism $qf : I \rightarrow L$ is such that $qfi = a$ proving that $L \in \mathcal{C}(I)$; thus, $\mathcal{C}(I)$ is closed under pure homomorphic images.

Let $M_k \in \mathcal{C}(I)$ and $m = (m_k)_k \in \oplus M_k$ (almost all coordinates equal to 0). There are $i_k \in I$, $k \in \text{supp } m$ such that $\chi(i_k) \leq \chi_{M_k}(m_k)$. Because of finiteness of the support and property (3) for characteristics, there is an $i \in I$ such that $\chi(i) = \bigcap_k \chi(i_k) \leq \bigcap_k \chi(m_k) = \chi(m)$. This proves that $\oplus M_k \in \mathcal{C}(I)$.

(b) Let again $E : 0 \rightarrow A \xrightarrow{*} B \xrightarrow{q} C \rightarrow 0$ be a pure exact sequence. If $B \in \mathcal{D}(I)$, let $f \in \text{Hom}_R(I, IA)$. There is a $b \in B$ such that $fi = ib$, for all $i \in I$. But $fi \in A$ and this implies $b \in A$, as $A \leq_* B$. Hence $\mathcal{D}(I)$ is closed under submodules.

If in the exact sequence E , A and C are in $\mathcal{D}(I)$, let $f \in \text{Hom}_R(I, IB)$. There is a $c \in C$ such that $qfi = ci$, for all $i \in I$. Choose $b \in B$ with $qb = c$. For all $i \in I$, $gi = fi - bi \in \text{Hom}_R(I, IA)$, therefore there is an $a \in A$ with $gi = fi - bi = ai$. Now $\forall i \in I$, $fi = (a + b)i$ and this shows that $B \in \mathcal{D}(I)$ and that $\mathcal{D}(I)$ is closed under pure extensions.

Assume that all $M_k \in \mathcal{D}(I)$ and let $f \in \text{Hom}_R(I, I \prod M_k)$. Then $f_k = (id \otimes \pi_k)f \in \text{Hom}_R(I, IM_k)$, thus there exist $m_k \in M_k$ such that $f_k(i) = im_k$, for all $i \in I$. If $fi = i_1(x_k)_k$, then $f_k(i) = i_1 x_k = im_k$ for all k . We then have $fi = i(m_k)_k$ i.e. $\mathcal{D}(I)$ is closed with respect to direct products. Closeness under direct sums then immediately follows.

PROPOSITION 1.10. *The following are functors:*

$$\text{Hom}_R(I, -) : \mathcal{C}(I) \longrightarrow \mathcal{D}(I) \text{ and } I \otimes_R - : \mathcal{D}(I) \longrightarrow \mathcal{C}(I).$$

They are bijections on objects but, in general, they are not equivalences of categories, as they are not full.

PROOF. Let $M \in \mathcal{C}(I)$. Then $\text{Hom}_R(I, I \otimes_R \text{Hom}_R(I, M)) \cong \text{Hom}_R(I, M)$ i.e. $\text{Hom}_R(I, M) \in \mathcal{D}(I)$. If $N \in \mathcal{D}(I)$, then $I \otimes_R \text{Hom}_R(I, I \otimes_R N) \cong I \otimes_R N$, and $I \otimes_R N \in \mathcal{C}(I)$. $R \in \mathcal{D}(P)$, hence $\oplus R \in \mathcal{D}(P)$, by Proposition 1.9. Not all pure homomorphic images of modules in $\mathcal{D}(P)$ are in $\mathcal{D}(P)$ as, by Example 1.7 $P \notin \mathcal{D}(P)$, if P is not principal. Thus $\mathcal{D}(P)$ is not closed with respect to pure homomorphic images and hence cannot be equivalent to $\mathcal{C}(P)$.

PROPOSITION 1.11. *The following inclusions (which are proper if P is not principal) hold:*

- (a) $\mathcal{M}(R_{I^\sharp}) \supset \mathcal{C}(I) \supset \mathcal{H}(I)$.
- (b) $\mathcal{M}(R_{I^\sharp}) \supset \mathcal{D}(I) \supset \mathcal{H}(R_{I^\sharp})$.

PROOF. (a) The isomorphism $I \otimes_R \text{Hom}_R(I, M) \cong M$ and Lemmas 1.1 and 1.2 in [D2] imply $M^\sharp \leq M^\sharp \cap I^\sharp$, hence $M^\sharp \leq I^\sharp$ which, by Lemma 1.4 is equivalent to $M \in \mathcal{M}(R_{I^\sharp})$. This establishes the first inclusion. The strict inequality holds in case of non-principal P , which is evident from Example 1.7.

For an I -homogeneous module M , and an $a \in M$, there is always an isomorphism $f : I \longrightarrow \langle a \rangle_*$ and this proves the second inequality. It is strict, as $\mathcal{C}(R) = \mathcal{M}(R)$ and P is not R -homogeneous, if it is not principal.

(b) The reasoning for the first inclusion is same as above: use the isomorphism $\text{Hom}_R(I, I \otimes_R M) \cong M$. Example 1.7 ensures non-equality again.

Assume that M is $R_{I^\sharp}$ -homogeneous and let $f \in \text{Hom}_R(I, IM)$ with $fi_0 = i'm'$. Now $f : I \longrightarrow \langle i'm' \rangle_* = I\langle m' \rangle_*$ (use Lemma 4.5, [D2]). There is an isomorphism $iso : \langle m' \rangle_* \longrightarrow R_{I^\sharp}$ and the equality $IR_{I^\sharp} = I$, so that $(id_I \otimes iso) \circ f \in \text{Hom}_R(I, I) \cong R_{I^\sharp}$. There is then an $\frac{r}{s} \in R_{I^\sharp}$ such that $((id_I \otimes iso) \circ f)i = \frac{r}{s}i$, for all $i \in I$ and we have now $fi = (iso^{-1}(\frac{r}{s}))i = im_0$, for $m_0 \in M$. Thus $M \in \mathcal{D}(I)$. For the strict inclusion, set $I = R$. Then $\mathcal{D}(I) = \mathcal{M}(R)$, but not all R -modules are R -homogeneous.

We note that Warfield had considered in [W] the class $\mathcal{D}(I)$ for the category of Abelian groups. In this case it coincides with $\mathcal{M}(R_{I^\sharp})$. If R is a Noetherian valuation domain, then the classes $\mathcal{C}(I)$, $\mathcal{D}(I)$ and $\mathcal{M}(R_{I^\sharp})$ coincide with the class of all R -modules.

Since $\mathcal{C}(I)$ is not closed under the products the following theorem settles this question, giving another characterization of endomorphic ideals at the same time.

For this, we need a helping device.

LEMMA 1.12. *The following are equivalent:*

- (a) $I^\flat I < I$;
- (b) I is endomorphic;
- (c) $\exists i_0 \in I$ such that $I = R_{I^\flat} i_0$;
- (d) $\exists i_0 \in I$ such that $i_0 \in \bigcap_{i \in I} \chi_{R_{I^\flat}}(i)$;
- (e) $\exists i_0 \in I$ such that $\chi_I(i_0) = I^\flat$, or equivalently $tI = I^\flat$.

PROOF. Lemma 1.5 takes care of the equivalence of (a) and (b). That (c) implies (b) is trivial. Assume (b) and let $\phi : R_{I^\flat} \rightarrow I$ be an isomorphism. Then, there is a $q \in Q$, such that, for every $\frac{r}{s} \in R_{I^\flat}$, $q \frac{r}{s} = \phi(\frac{r}{s})$ and $q \frac{r}{s} \in I$. If $q \in R$, then, for all $s \in I^\flat$, $\frac{q}{s} \in I$. This implies $q \in I$ and we may use $i_0 = q$ for the claim in (c). If $q = \frac{1}{t} \notin R$, then $\frac{1}{t} \frac{1}{s} \in I$, hence t is a unit, a contradiction. (c) is easily verifiable to be equivalent to (d). That (c) implies (e) follows from the fact that $I^\flat \leq \chi_I(i_0)$ and for $ri_1 = i_0$, $R_{I^\flat} i_1 r \subseteq I$ i.e., for $i \in I \exists i' \in I$ with $ri' = i$ and thus $r \in I^\flat$. (e) implies (b) since $R_{I^\flat} = \frac{R}{I^\flat} \cong \frac{Ri_0}{\chi_I(i_0)} = \langle i_0 \rangle_* = I$.

THEOREM 1.13. *The following are equivalent:*

- (a) I is endomorphic;
- (b) $\prod I \in \mathcal{C}(I)$;
- (c) $\mathcal{C}(I)$ is closed under direct products.

PROOF. If I is endomorphic, then we show that the homomorphism $F : I \prod_k \text{Hom}_R(I, M_k) \rightarrow \prod_k I \text{Hom}_R(I, M_k)$ defined by $F((i(f_k))_k) = (if_k)_k$ is an isomorphism. Without loss of generality, we may assume that k 's are in the index set identified with I . We only need surjectivity, so let $(i_k g_k)_k \in \prod I \text{Hom}_R(I, M_k)$. By Lemma 1.12(d), there is an $i_0 \in I$ with $i_0 \in \chi_{R_{I^\flat}}(i_k)$, for all k , i.e. there are $\frac{r_k}{s_k} \in R_{I^\flat}$, such that $i_k = i_0 \frac{r_k}{s_k}$. Define $f_k = \frac{r_k}{s_k} g_k$. By Lemma 1.2, [D2], $I^\flat \cup M^\flat \subset (\text{Hom}_R(I, M))^\flat$ and since s_k are in $I^\flat$, $s_k | g_k$. Thus every $f_k \in \text{Hom}_R(I, M_k)$. Clearly $F(i_0(f_k)_k) = (i_k g_k)_k$. If every M_k is in $\mathcal{C}(I)$, then $I \otimes_R \text{Hom}_R(I, \prod M_k) \cong I \otimes_R \prod \text{Hom}_R(I, M_k) \cong \prod I \otimes_R \text{Hom}_R(I, M_k) \cong \prod M_k$, hence $\prod M_k \in \mathcal{C}(I)$ and we have (c).

(c) implies (b), as the latter is a special case of the former.

Assume (b) and index elements of I by a well-ordered set $\mathbb{N}$. For $(i_k)_{k < \mathbb{N}} \in \prod_{k < \mathbb{N}} I$, there is an $f \in \text{Hom}_R(I, \prod_k I)$ and $i_0 \in I$ such that $fi_0 = (i_k)_{k < \mathbb{N}}$. Composing with the projections $\pi_k f \in \text{Hom}_R(I, I)$, there are $\frac{r_k}{s_k} \in R_{I^\flat}$, such that $\frac{r_k}{s_k} i_0 = i_k$, for all k and then $i_0 \in \bigcap_{i \in I} \chi_{R_{I^\flat}}(i)$. By Lemma 1.12 this is sufficient to ensure that I is endomorphic.

We are now able to settle an important question on products of homogeneous

modules:

THEOREM 1.14. *The following are equivalent for an ideal I of R :*

- (a) *Products of I -homogeneous modules are I -homogeneous;*
- (b) *$\prod I$ is I -homogeneous;*
- (c) *I is endomorphic and $\forall r_\alpha \in R \exists r \in R \cap \chi(r_\alpha)I^b = \chi(r)I^b$;*
- (d) *I is endomorphic and CGCD.*

PROOF. It is self evident that (a) implies (b).

If (b) is satisfied, then by Proposition 1.11 and Theorem 1.13, I is endomorphic. By Lemma 1.12, there exists an $i_0 \in I$ such that $\chi_I(i_0) = I^b$. Given a family $\{r_\alpha\} \subseteq R$, set $a = (r_\alpha i_0)_\alpha \in \prod_\alpha I$. By homogeneity, $ta = tI$, hence $\chi(a)$ is equivalent to I^b . But $\chi(a) = \bigcap_\alpha \chi_I(r_\alpha i_0) = \bigcap_\alpha \chi(r_\alpha) \chi_I(i_0) = \bigcap_\alpha \chi(r_\alpha) I^b$. The two possibilities are as follows: for some $r \in R$, either $\bigcap_\alpha \chi(r_\alpha) I^b = \chi(r) I^b$ or $\chi(r)(\bigcap_\alpha \chi(r_\alpha) I^b) = I^b$. With the first case we get the desired and with the latter we get $\bigcap_\alpha \chi(r_\alpha) I^b = I^b = \chi(1) I^b$, hence we again have (c).

If we assume (c), then $\chi_I(i_0) = I^b$, for some $i_0 \in I$, since I is endomorphic. For a collection i_α in I , it can happen that $i_\alpha = r_\alpha i_0$, for some $r_\alpha \in R$ and $\alpha \in K$, where K is a subset of the index set. In this case, $\chi_I(i_\alpha) = \chi(r_\alpha) \chi_I(i_0)$, for all $\alpha \in K$ and consequently, $\bigcap_{\alpha \in K} \chi_I(i_\alpha) = \chi(r) I^b = \chi_I(r i_0) = \chi_I(i')$. It might happen however that there are indices $\alpha \in T$ (T is the complement of K in the index set), such that $i_0 = s_\alpha i_\alpha$, for some $s_\alpha \in R$ and consequently $I^b = \chi_I(i_0) = \chi(s_\alpha) \chi_I(i_\alpha)$. Then, for all $\alpha \in T$, $\chi_I(i_\alpha) = I^b$. Finally, $\bigcap_\alpha \chi_I(i_\alpha) = I^b \cap \chi_I(i')$, which equals to either $\chi_I(i_0)$ or $\chi_I(i')$ and this is the CGCD condition for R .

Assuming (d), let M_α be I -homogeneous modules and let $a = (m_\alpha)_\alpha \in \prod_\alpha M_\alpha$. Since every $\langle m_\alpha \rangle_* \cong I$, $\chi_{M_\alpha}(m_\alpha)$ is equivalent to $I^b = \chi_I(i_0)$, for some $i_0 \in I$. The two possibilities are $\chi_{M_k}(m_k) = \chi(r_k) \chi_I(i_0)$, for some sub-index set $K \ni k$ or $\chi(s_k) \chi_{M_k}(m_k) = \chi_I(i_0) = I^b$, for a sub-index set $T \ni k$. By Lemma 4.4 [D2], $\langle m_\alpha \rangle_*^b = I^b$, for all α , thus $\chi_{M_k}(m_k) = I^b = \chi_I(i_0)$, for all $k \in T$ and $\bigcap_{k \in K} \chi_{M_k}(m_k) = \chi_I(i')$, for some $i' \in I$. This finally establishes the fact that $\chi(a) = \chi_I(i)$, for an $i \in I$, which is the I -homogeneity condition.

Combining Propositions 4.6 in [D2], 1.10 and 1.11, we get the following special case of Proposition 1.10, which is in turn an improvement of a result in [FV] (also [FS], Proposition 1.7 p.272).

PROPOSITION 1.15. *The following are functors:*

$$\text{Hom}_R(I, -) : \mathcal{H}(I) \longrightarrow \mathcal{H}(R_I) \text{ and } I \otimes_R - : \mathcal{H}(R_I) \longrightarrow \mathcal{H}(I).$$

They are bijections on objects but, in general, they are not equivalences of categories, as they are not full.

PROOF. Let the value group of R be the reals. Then R will be a CGCD ring with non-principal maximal ideal P . Applying Theorem 1.14 to this case and $I = P$, we conclude that $\mathcal{H}(R_{I^\dagger}) = \mathcal{H}(R)$ is closed with respect to direct products but $\mathcal{H}(I)$ is not as P is not endomorphic. The categories $\mathcal{H}(R_{I^\dagger})$ and $\mathcal{H}(I)$ are not equivalent and the functors in Proposition 1.15 are not full.

As for the products of separable homogeneous modules, we give an answer only in a special case:

COROLLARY 1.16. *If R is a Noetherian valuation domain then products of modules are separable and homogeneous if and only if the components are separable and homogeneous of the same type.*

PROOF. If the products are homogeneous and separable, then the components are likewise homogeneous and separable, as they are direct summands.

Conversely, assume that all M_k are homogeneous and separable. In the case of Noetherian valuation domains all ideals are endomorphic and these ideals are CGCD thus, by Theorem 1.14, the product $\prod_k M_k$ is homogeneous.

If $a = (a_k)_k \in \prod_k M_k$, then $a \in \prod_k \langle a_k \rangle_* \leq_* \prod_k M_k$, thus all we need to prove is that $\prod_k R$ is separable. We may assume $a \in \prod_k R$. Since R is Noetherian, there is an index α , such that $\chi(a) = \bigcap_k \chi(a_k) = \chi(a_\alpha)$ and $a = a_\alpha (a'_k)_k = a_\alpha a'$ (here $a'_\alpha = 1$). Clearly $\langle a' \rangle_* \oplus \prod_{k \neq \alpha} R = \prod_k R$ and $\langle a \rangle_* = \langle a' \rangle_*$; so that separability follows.

We conclude this section by giving conditions for the monomorphism ξ to be an isomorphism in a special case $M = I$ – a rank one module.

PROPOSITION 1.17. *If either (1) $N \in \mathcal{H}(J)$ or (2) $N \in \mathcal{D}(I)$, then the monomorphism*

$$\xi : \text{Hom}_R(I, I) \otimes_R \text{Hom}_R(J, N) \longrightarrow \text{Hom}_R(I \otimes_R J, I \otimes_R N).$$

is an isomorphism.

PROOF. (a) Given $F \in \text{Hom}_R(I \otimes_R J, I \otimes_R N)$, let $F(ij) = i_1 n_1$. Then, since $\langle i_1 n_1 \rangle_* = I \langle n_1 \rangle_*$, F may be regarded as a homomorphism $IJ \longrightarrow I \langle n_1 \rangle_*$. If $\text{iso} : \langle n_1 \rangle_* \longrightarrow J$ is a fixed isomorphism, then $f = (\text{id}_I \otimes \text{iso})F \in \text{Hom}_R(IJ, IJ)$, thus, by Corollary 1.3, there are $\alpha \in \text{Hom}_R(I, I)$ and $\beta \in \text{Hom}_R(J, J)$ such that $f(ij) = \alpha(i)\beta(j)$, therefore $F(ij) = \alpha(i)(\text{iso}^{-1}\beta)j$. This proves the claim.

(b) $\text{Hom}_R(I \otimes_R J, I \otimes_R N) \cong \text{Hom}_R(J, \text{Hom}_R(I, I \otimes_R N)) \cong \text{Hom}_R(J, N)$. On the other hand, by Proposition 1.11, $N \in \mathcal{M}(R_{I^\dagger})$ and because of $(\text{Hom}_R(J, N))^\# \leq J^\# \cap N^\# \leq I^\#$, $\text{Hom}_R(J, N)$ is in $\mathcal{M}(R_{I^\dagger})$. Lemma 1.4 then gives

$\text{Hom}_R(I, I) \otimes_R \text{Hom}_R(J, N) \cong \text{Hom}_R(J, N)$, thus the isomorphism again holds. In this case, J need not be of rank one, nor related to N .

We will extend this result to a wider setting, only after we introduce additional notions in the following section.

2. Balanced exact sequences

A pure exact sequence $E : 0 \longrightarrow A \xrightarrow{*} B \xrightarrow{q} C \longrightarrow 0$, of torsion free R -modules is called *balanced* if for every $c \in C$, there is a $b \in B$ such that $qb = c$ and $\chi_C(c) = \chi_B(b)$. We will also say that A is a *balanced submodule* of B .

It is always useful to have in hand several characterizations of a notion and this is why we start with a

LEMMA 2.1. *The following are equivalent:*

- (a) $E : 0 \longrightarrow A \xrightarrow{*} B \xrightarrow{q} C \longrightarrow 0$ is balanced exact;
- (b) For every torsion free rank one module I , the sequence $E \otimes_R I : 0 \longrightarrow A \otimes_R I \xrightarrow{*} B \otimes_R I \xrightarrow{q \otimes id} C \otimes_R I \longrightarrow 0$ is balanced exact;
- (c) For every torsion free rank one module I the sequence $0 \longrightarrow \text{Hom}_R(I, A) \longrightarrow \text{Hom}_R(I, B) \xrightarrow{\bar{q}} \text{Hom}_R(I, C) \longrightarrow 0$ is balanced exact;
- (d) For every torsion free rank one module I the sequence $0 \longrightarrow A(I) \longrightarrow B(I) \xrightarrow{q} C(I) \longrightarrow 0$ is balanced exact.

PROOF. Setting $I = R$ in sequences (b), (c), (d), each of them reduces to the sequence in (a) so that each of the conditions implies (a).

We prove the converses, assuming (a):

To get (b) let $ci \in CI$. There is a $b \in B$ with $qb = c$ and $\chi_B(b) = \chi_C(c)$. Since $(q \otimes id)(bi) = ci$, we have $\chi_{BI}(bi) \leq \chi_{CI}(ci)$. On the other hand, let $r \mid ci$. If $r \mid i$, then $r \mid bi$; if $r = ir'$, then $r' \mid c$, $r' \mid b$, thus $r \mid bi$ and $\chi_{BI}(bi) = \chi_{CI}(ci)$.

For (c), Proposition 5.1, [D2] justifies the pure exactness of the sequence. If $f \in \text{Hom}_R(I, C)$ and $fi = c$, for an $i \in I$, then there is a $b \in B$ with $qb = c$ and $\chi_B(b) = \chi_C(c)$. f is the image of $g \in \text{Hom}_R(I, B)$, defined by $gi = b$, hence $\chi(f) \geq \chi(g)$. We need the reverse inequality: if $f = rf'$, $f'i = c'$, then $\chi(i) \leq \chi(c')$ and there is a $b' \in B$ with $qb' = c'$ and $\chi_B(b') = \chi_C(c')$. For $g'i = b'$, we have $g = rg'$. This establishes balanceness of the sequence $\text{Hom}_R(I, E)$.

For (d), let $c \in C(I)$; then there is a $b \in B$ with $\chi_B(b) = \chi_C(c)$ and consequently $b \in B(I)$.

An R -module N is called *tensor balancing*, if for every balanced exact sequence $E : 0 \longrightarrow A \xrightarrow{*} B \xrightarrow{q} C \longrightarrow 0$, the sequence $E \otimes_R N : 0 \longrightarrow A \otimes_R N \xrightarrow{*} B \otimes_R N \xrightarrow{q \otimes id} C \otimes_R N \longrightarrow 0$ is balanced exact.

$B \otimes_R N \xrightarrow{q \otimes id} C \otimes_R N \rightarrow 0$ is again balanced exact.

We have the previous lemma to ensure that the rank one modules are tensor balancing. Moreover, completely decomposable modules are tensor balancing. In fact, direct sums of tensor balancing modules are tensor balancing: If M_k are tensor balancing, then the sequences $E \otimes_R M_k$ are balanced exact, hence elements of Bext_R (see [D2]); their direct sum is again in Bext_R , thus $\oplus M_k$ is tensor balancing. Or else, use the definition of a balanced exact sequence and property (3), [D2], for characteristics. Pure submodules of tensor balancing modules are tensor balancing.

PROPOSITION 2.2. *If $E : 0 \rightarrow A \xrightarrow{*} B \xrightarrow{q} C \rightarrow 0$ is a balanced exact sequence, then*

- (a) $B \in \mathcal{C}(I)$ if and only if A and C are in $\mathcal{C}(I)$;
- (b) $B \in \mathcal{D}(I)$ if and only if A and C are in $\mathcal{D}(I)$;
- (c) $B \in \mathcal{H}(I)$ if and only if A and C are in $\mathcal{H}(I)$.

PROOF. (a) Proposition 1.9 settles one half of the claim. Assume now $A, C \in \mathcal{C}(I)$. If $b \in B$, $qb = c$, let $b' \in B$ be such that $qb' = c$ and $\chi(b') = \chi(c)$. $b - b' = a \in A \in \mathcal{C}(I)$, hence there is an $i \in I$ with $\chi(i) \leq \chi(a)$ and, since $C \in \mathcal{C}(I)$, there is an $i' \in I$ with $\chi(i') \leq \chi(b') = \chi(c)$. Let i_0 be the one of i, i' such that $\chi(i_0) = \chi(i) \cap \chi(i')$. Then $\chi(i_0) \leq \chi(a + b') = \chi(b)$, therefore $B \in \mathcal{C}(I)$.

(b) If A and C are in $\mathcal{D}(I)$, then Proposition 1.9(b) implies $B \in \mathcal{D}(I)$. If $B \in \mathcal{D}(I)$, then $A \in \mathcal{D}(I)$, by the same proposition. By Lemma 2.1, the sequence $E : 0 \rightarrow A \otimes_R I \xrightarrow{*} B \otimes_R I \xrightarrow{q \otimes id} C \otimes_R I \rightarrow 0$ is balanced, hence, by Proposition 5.1, [D2], the sequence $0 \rightarrow \text{Hom}_R(I, A \otimes_R I) \rightarrow \text{Hom}_R(I, B \otimes_R I) \rightarrow \text{Hom}_R(I, C \otimes_R I) \rightarrow 0$ is pure exact. The natural isomorphisms $\text{Hom}_R(I, A \otimes_R I) \cong A$ and $\text{Hom}_R(I, B \otimes_R I) \cong B$ give $\text{Hom}_R(I, C \otimes_R I) \cong C$, which proves $C \in \mathcal{D}(I)$.

(c) Let B be I -homogeneous. A is I -homogeneous as $A \leq_* B$. If $c \in C$, then there is a $b \in B$ such that $qb = c$ and $\chi_B(b) = \chi_C(c)$. This implies $tb = tc$ and, by Theorem 2.3, [D2], $\langle c \rangle_* \cong \langle b \rangle_* \cong I$. Thus C is I -homogeneous. If now A and C are I -homogeneous, let $b \in B$ and $qb = c$. There is then a $b' \in B$ with $qb' = c$ and $\chi_B(b') = \chi_C(c)$ and consequently, $b - b' = a \in A$. Let us fix isomorphisms $f : I \rightarrow \langle a \rangle_*$ and $g : I \rightarrow \langle c \rangle_*$ and the induced isomorphism $g' : I \rightarrow \langle b' \rangle_*$. Set $fi = a$ and $g'i' = b'$. If i_0 denotes the one of the i, i' such that $\chi(i_0) = \chi(i) \cap \chi(i')$, then there is a homomorphism $h : I \rightarrow \langle a + b' \rangle_* = \langle b \rangle_*$ with $h(i_0) = b$. We only need to prove that it is an isomorphism. For if $sb_1 = b = a + b'$, then $sq(b_1) = q(b) = q(b') = c$, hence $s \in \chi_C(c) = \chi_B(b')$. Then also $s \in \chi_A(a)$. As f, g' are isomorphisms, $\frac{a}{s} \in \text{im } f$, $\frac{b'}{s} \in \text{im } g'$, and thus $\frac{b}{s} = b_1 \in \text{im } h$.

At the end of this section, we give some conditions for the homomorphism ξ to be an isomorphism, but for this we need a preliminary result:

LEMMA 2.3. *An R -module M is I -homogeneous if and only if there is a balanced exact sequence*

$$0 \longrightarrow L \xrightarrow{*} \oplus I \xrightarrow{q} M \longrightarrow 0$$

(and, in this case, L will be separable and I -homogeneous).

PROOF. If such a homogeneous resolution exists, then M will be I -homogeneous by Proposition 2.2. Conversely, if M is I -homogeneous, define $C = \oplus_{a \in M} \langle a \rangle_* \cong \oplus I$ and $q : C \longrightarrow M$ by $q(\oplus_{a \in M} a) = \sum_{a \in A} a$. The rest of the proof is same mutatis mutandis as that of Proposition 5.2, [D2]. The claim about L is supported by Examples 4.1 and 4.2, [D2]

THEOREM 2.4. *Let M and N be R -modules such that M is I -homogeneous and N tensor balancing. Then the isomorphism*

$$\text{Hom}_R(I, M) \otimes_R \text{Hom}_R(J, N) \cong \text{Hom}_R(I \otimes_R J, M \otimes_R N)$$

holds in either of the following two cases:

- (1) $N \in \mathcal{H}(J)$; (2) $N \in \mathcal{D}(I)$.

PROOF. We first prove the claim for the case $M = \oplus I_k$, where all $I_k \cong I$, so let $F' \in \text{Hom}_R(I \otimes_R J, \oplus_k (I_k \otimes_R N))$ and $F'(i'j') = i_1 n_1 \oplus \cdots \oplus i_t n_t$. Then $\chi_{IJ}(i'j') \leq \chi_{\oplus_k (I_k N)}((i_i n_i)_i) = \bigcap_{i=1}^t \chi_{I_i N}(i_i n_i)$. This implies $\chi_{IJ}(i'j') \leq \chi_{I_i N}(i_i n_i)$, for all $i = 1, \dots, t$ and then there are $F_i \in \text{Hom}_R(IJ, I_i N)$ such that $F_i(i'j') = i_i n_i$, for all i . By Proposition 1.17, there are $f_i \in \text{Hom}_R(I, I_i) \hookrightarrow \text{Hom}_R(I, \oplus_k I_k)$ and $g_i \in \text{Hom}_R(J, N)$ such that $F_i(i'j') = (f_i \otimes g_i)(i'j')$ and then $F'(i'j') = \sum_{i=1}^t (f_i \otimes g_i)(i'j')$. Hence $F' = \sum_{i=1}^t (f_i \otimes g_i)$ and the isomorphism follows.

In the general case, for any I -homogeneous M , Lemma 2.3 ensures the existence of a balanced exact sequence

$$0 \longrightarrow L \xrightarrow{*} \oplus I_k \xrightarrow{q} M \longrightarrow 0,$$

where each $I_k \cong I$. By tensoring, we obtain another balanced exact sequence

$$0 \longrightarrow L \otimes_R N \xrightarrow{*} (\oplus I_k) \otimes_R N \xrightarrow{q \otimes id} M \otimes_R N \longrightarrow 0.$$

If $F \in \text{Hom}_R(I \otimes_R J, M \otimes_R N)$, then because of the projective property, there is an $F' \in \text{Hom}_R(I \otimes_R J, (\oplus I_k) \otimes_R N)$, such that $(q \otimes id) \circ F' = F$. By the previous

part, $F' = \sum_{i=1}^t f_i \otimes g_i$, for some $f_i \in \text{Hom}_R(I, \oplus I_k)$, $g_i \in \text{Hom}_R(J, N)$. Now $F = (q \otimes id) \circ (\sum_{i=1}^t f_i \otimes g_i) = \sum_{i=1}^t (q \otimes id) \circ (f_i \otimes g_i) = \sum_{i=1}^t (qf_i) \otimes g_i$ and $qf_i \in \text{Hom}_R(I, M)$, which finishes the proof.

The possibilities for the isomorphism in the last theorem are not exhausted by the conditions listed. For instance if I and J are principal ideals, then the isomorphism holds for all M and N .

3. Endomorphism Algebras of Separable Modules

In this section we will study the endomorphism algebras of torsion free, separable, homogeneous R -modules and give criteria for a ring (an R -algebra) to be realized as an endomorphism algebra of a torsion free, separable, homogeneous R -module.

Using properties (4) and (10) for characteristics (see [D2]), we easily check the validity of

LEMMA 3.1. $\forall i \in I \text{ Hom}_R(I, M) \cong \{a \in M : \chi_I(i) \leq \chi_M(a)\}$.

Recall that Theorem 7 in [D1] states that for I -homogeneous separable R -modules M and N , every isomorphism $\text{End}_R M \cong \text{End}_R N$ is induced by an isomorphism $M \cong N$ in a natural way.

PROPOSITION 3.2. *If M and I are torsion free R -modules where I is of rank one, and M is separable I -homogeneous, then $\text{Hom}_R(I, M) \cong M_0 = \{(f\pi)x_0 \mid f \in \text{End}_R M\}$, for some $x_0 \in M$ and the natural projection $\pi : M \longrightarrow \langle x_0 \rangle_*$.*

PROOF. Observe that if $M \in \mathcal{C}(I)$, then $\text{End}_R(\text{Hom}_R(I, M)) \cong \text{End}_R(I \otimes_R \text{Hom}_R(I, M), M) \cong \text{End}_R M$. Proposition 9 in [D1], secures $\text{End}_R M_0 \cong \text{End}_R M$ and since both M_0 and $\text{Hom}_R(I, M)$ are separable and homogeneous of type R_I , then $M_0 \cong \text{Hom}_R(I, M)$.

The following is a result obtained in [D1, Theorem 10, Corollary 11], in a slightly different way.

THEOREM 3.3. *If M, N are torsion free separable, homogeneous R -modules of types I and J respectively and $\text{End}_R M \cong \text{End}_R N$, then*

- (1) $\text{Hom}_R(I, M) \cong \text{Hom}_R(J, N) (\cong H)$;
- (2) $I \otimes_R H \cong M$, $J \otimes_R H \cong N$;
- (3) $J \otimes_R M \cong I \otimes_R N$ canonically.

PROOF. (1) We have already established that, $\text{End}_R(\text{Hom}_R(I, M)) \cong$

End_RM and $End_R(Hom_R(I, N)) \cong End_RN$. $Hom_R(I, M)$ and $Hom_R(J, N)$ are separable (Proposition 3.6(b), [D2]) and R_{I^1} - and R_{J^1} -homogeneous respectively (Proposition 3.6(a), [D2]). However, because of separability of M and N , as well as the isomorphism of the endomorphism rings, we obtain $R_{I^1} = R_{J^1}$, therefore we may apply [D1, Theorem 7] to prove the claim. Or else, use the fact that; $M_0 = \{(f\pi)x_0 \mid f \in End_RM\} \cong N_0 = \{(g\rho)y_0 \mid g \in End_RN\}$, as $End_RM \cong End_RN$

- (2) Use the fact that $M \in \mathcal{C}(I)$ and $N \in \mathcal{C}(J)$;
- (3) Trivially, from (2).

The conditions of separability and homogeneity are both needed for Theorem 3.3 (and Theorem 7 in [D1]) to function: For a torsion free rank 2 indecomposable module M must be homogeneous (say of type I), and there are such modules M with the property that $End_RM \cong R$ (and $End_RR \cong R$), but $M \otimes_R R \not\cong I \otimes_R R$. Hence we cannot dispose of separability condition.

On the other hand, homogeneity cannot be replaced by a weaker condition, say membership in $\mathcal{C}(I)$, for let $I = J = R$; in this case $\mathcal{C}(I) = \mathcal{M}(R)$, $End_RP \cong End_RR$, both $P, R \in \mathcal{C}(I)$ and they are separable, but $P \not\cong R$ in most cases.

Let us point out at this stage that the bijections in Proposition 1.14 enable us to reduce our considerations of I -homogeneous modules to that of R_{I^1} -homogeneous modules, in a number of cases. The latter have some advantages over the general case, one of them being that R_{I^1} is an endomorphic ideal. Once established in $\mathcal{H}(R_{I^1})$, the results can be transferred, via the bijections into $\mathcal{H}(I)$. Indeed, this is what has been done in Theorem 3.3...

We single out several properties inherent in the endomorphism algebra of a torsion free separable homogeneous R -module:

PROPOSITION 3.4. *Let M be a torsion free R -module, which is separable and homogeneous of type R_{I^1} ($I \leq Q$); denote by $E = End_RM$ (as an R -algebra), by $\mathcal{E}$ the set of all minimal idempotents of E , by $\mathcal{E}_0$ the left ideal of E generated by $\mathcal{E}$ and by $\mathcal{F}$ the topology on E having the annihilator set $\{Ann_{Ee}\}_{e \in \mathcal{E}}$ of left ideals as a subbase of its neighbourhoods of 0. Then:*

- (a) $\mathcal{E} \neq \emptyset$;
- (b) All left ideals Ee ($e \in \mathcal{E}$) are isomorphic (as R - and E -modules) and they are torsion free, separable homogeneous of type R_{I^1} ;
- (c) For every $e \in \mathcal{E}$, $eEe \cong R_{I^1}$;
- (d) E is Hausdorff and complete in the annihilator topology and $\mathcal{E}_0$ is dense in it.

PROOF. (a) All minimal idempotents are of the form $\pi_x : M \longrightarrow \langle x \rangle_*$, $x \in M$, because M is separable;

(b) If $M_x = \{(f\pi)x \mid f \in \text{End}_R M\}$, then define $\alpha : E\pi \longrightarrow M_x$ by $\alpha(f\pi) = (f\pi)x = fx$. This α is an isomorphism: it is easily checked that α is an R - and an E -homomorphism. It is injective, for if $\alpha(f\pi) = 0 = fx$, then $f\langle x \rangle_* = 0$, hence $f\pi = 0$. α is trivially surjective. Moreover, $M_x \cong M$, because M_x is of type $R_{I^\sharp}$ by [D1, Proposition 9(b)] (by Proposition 1.3(d) in [D2] $(R_{I^\sharp})^\sharp \cong I^\sharp$). Proposition 9(d) (ibid.) give $\text{End}_R M_x \cong \text{End}_R M$ and Theorem 7 (ibid.) ensures that $M \cong M_x \cong E\pi$ (as R -modules). α is clearly an E -isomorphism.

(c) If $\alpha : E\pi \longrightarrow M$ is an E -isomorphism, then $\beta : \pi E\pi \longrightarrow \pi M$ is also an isomorphism and the claim follows from the fact that $\pi M = \langle x \rangle_* \cong R_{I^\sharp}$, since M is $R_{I^\sharp}$ -homogeneous;

(d) Recall that the finite topology on E (cf. for example [F, §107]) is defined by the subbase $U_x = \{\eta \in U \mid \eta x = 0\}$, $x \in M$. This is obviously a Hausdorff topology since $\bigcap_{x \in M} U_x = \{\eta \mid \eta M = 0\} = 0$. There is no difficulty in checking that E is complete in this topology and that $\mathcal{E}_0$ is dense (follow Propositions 107.2 and 107.3 ibid., mutatis mutandis; see also the comment after its proof).

On the other hand, the annihilator topology and the finite topology are equivalent: 1) Given $\text{Ann}_E(\pi)$, $\pi \in \mathcal{E}$, there is an $x \in M$ such that $\pi : M \longrightarrow \langle x \rangle_*$, since M is separable. If $f \in U_x$, then $f\pi M = f\langle x \rangle_* = 0$, thus $f \in \text{Ann}_E(\pi)$ and therefore $U_x \subseteq \text{Ann}_E(\pi)$; 2) Given U_x in the finite topology, then $f \in \text{Ann}_E(\pi)$ ($\pi : M \longrightarrow \langle x \rangle_*$) implies $f\pi = 0$ i.e. $f\pi x = 0 = fx$ hence $f \in U_x$.

The main result of this section states that the converse of Proposition 3.4 is also true. We apply the proof of an analogous theorem given for abelian groups in [MS].

THEOREM 3.5. *Let S be a (not necessarily commutative) ring with identity and R a commutative valuation domain with the identity, such that S is an R -module. Let $\mathcal{I}$ be the set of minimal idempotents of S , and let $\mathcal{I}_0$ be the left ideal of S generated by $\mathcal{I}$ and F the annihilator topology $\{\text{Ann}_S i\}_{i \in \mathcal{I}}$ on S . Assume also that:*

- (a) $\mathcal{I} \neq \emptyset$;
- (b) *All left ideals Si ($i \in \mathcal{I}$) are isomorphic as S -modules and they are torsion free, separable homogeneous R -modules of an endomorphic type J , for some $J \leq Q$;*
- (c) *iSi are isomorphic (to $R_{J^\sharp}$), for every $i \in \mathcal{I}$;*
- (d) *S is Hausdorff and complete in F and $\mathcal{I}_0$ is dense in it.*

Then, there is a torsion free, separable, homogeneous R -module M (of type $R_{J^\sharp}$) such that $\text{End}_R M \cong S$. Up to isomorphism, $M = Si$, for any $i \in \mathcal{I}$.

PROOF. By (a), there is an $i_0 \in \mathcal{I}$. We prove that $E = \text{End}_R(Si_0) \cong S$, therefore $M = Si_0$ would suffice as a torsion free, separable, homogeneous R -

module. Uniqueness would then follow from [D1, Theorem 7]. Define $\phi : S \rightarrow E$ by $\phi(s)(ti_0) = s(ti_0)$ (multiplication by $s \in S$; $t \in S$); abbreviate $\phi(s) = s^*$. It is a straightforward check to see that ϕ is an R -algebra homomorphism.

ϕ is injective:

If $\phi(s) = 0$, then, for every $ti_0 \in Si_0$, $s(ti_0) = 0$. Consider any $i \in \mathcal{I}$. By (b), there is an isomorphism of S -modules $\alpha : Si \rightarrow Si_0$. It satisfies $\alpha(si) = s(\alpha i) = s(ti_0) = 0$ hence $s \in \text{Ann}_S i$, for all $i \in \mathcal{I}$ thus, by (d), $s = 0$. Thus we may identify S with a submodule of E .

The rest of the proof is devoted to showing that ϕ is surjective, in fact that S equals to E . Before proving this, observe first that $\phi(Si_0)$ is a left ideal of E i.e. $E\phi(Si_0) \leq \phi(Si_0)$ (and consequently $Ei_0 \leq Si_0$), for the following reason: Let $e \in E$; if $t \in S$, let $e(ti_0) = t_1i_0$. Notice that every element of E is also an $R_{J_1} \cong Si_0$ -homomorphism and that $E = \text{End}_R(Si_0) = \text{End}_{R_{J_1}}(Si_0)$. For $si_0 \in Si_0$, $(e \circ \phi(ti_0))(si_0) = e((ti_0)(si_0)) = (e(ti_0))(si_0) = (t_1i_0)(si_0) = (\phi(t_1i_0))(si_0)$; this shows that $e \circ \phi(ti_0) = \phi(t_1i_0)$. Moreover, $Ei_0 = Si_0$, since $S \leq E$. The same is true more generally, for every $i \in \mathcal{I}$, $Ei = Si$ (*): Let us denote by $\psi : Si \rightarrow Si_0$, $ti \mapsto tsi_0$ and $\psi_1 : Si_0 \rightarrow Si$, $t_1i_0 \mapsto ts_1i$, for some $s, s_1 \in S$ the mutually inverse isomorphisms ensured by (b). Define $\xi : Ei \rightarrow Ei_0$ by $fi \mapsto fsi_0$ and $\xi_1 : Ei_0 = Si_0 \rightarrow Ei$ by $f_1i_0 \mapsto fs_1i$, for any $f \in E$. $Si \cong Si_0$ is torsion free thus, by definitions of ψ and ψ_1 , $ss_1i = i$, $s_1si_0 = i_0$ and $isi_0 = si_0$, $i_0s_1i = s_1i$ which proves that ξ and ξ_1 are well-defined E -isomorphisms such that $\xi\xi_1 = \xi_1\xi$.

Now, when the preparatory work has been done, we resume the proof that $S = E$. Denote by $\mathcal{F}$ the finite topology on E . As before, it is identical with the annihilator topology $\{\text{Ann}_E e\}_{e \in \mathcal{E}}$, where $\mathcal{E}$ is the set of minimal idempotents of E . $(E, \mathcal{F})$ is complete, as we already mentioned earlier on. The induced topology on a subspace S satisfies $\mathcal{F} \cap E = F$ (**), (which is the fact to be justified later on), consequently S must be closed in E , because of completeness. We prove that S is dense in E by proving $\mathcal{I}_0$ dense in E and using (d).

To this end, suppose that $\mathcal{U}$ is a member of the base of $\mathcal{F}$ and let $f \in E$. Then, there is a base element $U = \bigcap_{k=1}^n \text{Ann}_S(i_k) \subset \mathcal{U} \cap S$. $\mathcal{I}_0$ is dense in S , thus there is a $t \in \mathcal{I}_0$ such that $t \in 1_S + U$ therefore, for $k = 1, \dots, n$, $(t-1)i_k = 0$ and also $f(t-1)i_k = 0$. This implies $f(t-1) \in U$ and $f(t-1) \in \mathcal{U}$ i.e. $ft \in f + U$. $ft \in \mathcal{I}_0$ finishes the proof that $\mathcal{I}_0$ is dense in $\mathcal{F}$.

It remains to prove only that $\mathcal{F} \cap E = F$ (**). If $i \in \mathcal{I}$, then $\phi i \in \mathcal{E}$ because $\phi i \circ \phi i(si_0) = \phi i(isi_0) = i^2si_0 = isi_0 = \phi i(si_0)$, hence $\phi i \circ \phi i = \phi i$. ϕi is minimal, for if $\phi i = e_1 + e_2$, where e_1, e_2 are R -independent in E , then $\phi i S \phi i = \phi i S(e_1 + e_2) = \phi i S e_1 + \phi i S e_2$, which is, by condition (c) isomorphic to the rank one torsion free R -module and thus either $e_1 = 0$ or $e_2 = 0$. We conclude therefore that $i \in \mathcal{E}$, hence $F \subset \mathcal{F}$.

For $\mathcal{F} \subset F$ it suffices to prove that, for every $e \in \mathcal{E}$, there are $i_1, \dots, i_n \in \mathcal{I}$ such that $\text{Ann}_S(e) \supset \bigcap_{k=1}^n \text{Ann}_S(i_k)$ ($\text{Ann}_E(e) \supset \text{Ann}_S(e)$ because $fe = 0$ in S implies $fe = 0$ in E , since $S \leq E$). For an $y \in Si$, $eSi \cong iSi \cong R_{J_1}y$ and $Si = iSi \oplus (1-i)Si$. It suffices to prove that $Si = \sum_{i_k \in \mathcal{I}} i_k Si$, as in this case $y = i_1(t_1) + \dots + i_n(t_n)$, ($t_k \in Si$) and thus $\text{Ann}_{Se} = \text{Ann}_{Sy} \supset \bigcap_{k=1}^n \text{Ann}_S(i_k)$. ($\text{Ann}_{Se} = \text{Ann}_{Sy}$ since $eSi = R_{J_1}y$, $1 \in S$, $ei = r_0y$, for some $r_0 \in R_{J_1}$, $fei = fr_0y$ therefore $fe = 0$ is equivalent to $fy = 0$.)

Let $z \in Si$; by condition (c), $iSi = R_{J_1}x$, for some $x \in Si$. $Si = R_{J_1}x \oplus N$, where $z = rx + n$, $r \in R_{J_1}$, $n \in N$. By (b), $\langle x + n \rangle_* = \langle x \rangle_{*Si} \cong R_{J_1}$ as R -modules. Since $Si = R_{J_1}(x + n) \oplus N$, there is an $e_1 \in \mathcal{E}$ such that $e_1 Si = R_{J_1}(x + n)$ and $(1 - e_1)Si = N$ (hence $e_1 N = 0$). If $i_1 = e_1 i$, then $i_1 \in Si$, since $Ei = Si$, by (*). One deduces that $i_1 N = e_1(iN) = e_1 i(1 - i_1)Si = e_1 0 = 0$ and $i_1(x + n) = e_1 i(x + n) = e_1(x) = e_1(x + n - n) = x + n$. This ensures $i_1 \in \mathcal{I}$ and then $z = rx + n = (r - 1)x + (x + n) = i((r - 1)x) + i_1(x + n) \in \sum_{i_k \in \mathcal{I}} i_k Si$.

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