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UNIQUENESS OF POSITIVE SOLUTIONS OF $\Delta u + f(|x|, u) = 0$ IN $\Omega(\theta_1, \theta_2)$

By Fu-Hsiang WONG and Shiueh-Ling YU

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Abstract

Consider the following exterior boundary value problem

$$\text{(BVP)} \qquad \begin{cases} u''(r) + \frac{m}{r}u'(r) + f(r, u(r)) = 0 & \text{in } (\theta_1, \theta_2), \\ \lim_{r \to \theta_2} u(r) = 0 \quad \text{and} \quad u(r) > 0 & \text{in } (\theta_1, \theta_2), \\ \text{and, in addition : } u'(\theta_2) = 0 & \text{if } \theta_2 \text{ is finite,} \end{cases}$$

where $0 \leq \theta_1 < \theta_2 \leq \infty$ and $m \geq 0$ is a constant. We show that (BVP) has at most one positive solution in (θ_1, θ_2) under suitable conditions on $f(r, u)$.

1. Introduction

Given $0 \leq \theta_1 < \theta_2 \leq \infty$, and an integer $n \geq 2$. Consider the annular domain

$$\Omega(\theta_1, \theta_2) := \begin{cases} \{x \in R^n \,|\, \theta_2 < |x| < \theta_2\} & \text{if } \theta_1 > 0, \\ \{x \in R^n \,|\, |x| < \theta_2\} & \text{if } \theta_1 = 0, \end{cases}$$

where R^n is the usual n-dimensional Euclidean space. There has recently been an increase in studying the uniqueness of non-trivial solutions of the interesting semilinear elliptic differential equation

$$\Delta u + f(u) = 0 \quad \text{in } \Omega(\theta_1, \theta_2)$$

satisfying various boundary conditions. See, for example, Atkinson and Peletier [1], Coffman [3], Gidas, Ni and Nirenberg [4], Ni and Nussbaum [14], Kaper and Kwong [5, 6, 7], Kwong [8, 9], Kwong and Zhang [10], Wong and Yeh [17, 18], Mcleod and Serrin [12], Peletier and Serrin [15, 16].

Recently, Wong and Yeh [17] established the following extension of Coffman [3] and Kwong [8].

THEOREM A. *If $\beta < 0 < \alpha$ and $p > q \geq 1$, then*

$$\text{(II)} \qquad \begin{cases} \Delta u + \alpha u^p + \beta u^q = 0 & \text{in } \Omega(0,\infty), \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{if } |x| = 0, \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

has at most one positive radially symmetric solution in $\Omega(0,\infty)$.

Furthermore, Kaper and Kwong [7] showed the following interesting result:

THEOREM B. *Let*

1°. f *be continuous on* $[0,\infty)$ *and locally Lipschitz in* $(0,\infty)$,

2°. *there exist* $u_0 > 0$ *such that* $F(u) < 0$ *for* $0 < u < u_0$, $F(u_0) = 0$,

$$f(u) > 0 \quad \text{for} \quad u \geq u_0, \quad \text{where} \quad F(u) := \int_0^u f(s)ds,$$

3°. $u \to \frac{f(u)}{u - u_0}$ *be decreasing for* $u > u_0$.

Then, the free boundary problem

$$\text{(III)} \qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(0,\theta_2), \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{if } |x| = 0, \\ u(x) = \dfrac{\partial u}{\partial n}(x) = 0 & \text{if } |x| = \theta_2, \end{cases}$$

has at most one positive radially symmetric solution in $\Omega(0,\theta_2)$ *if* θ_2 *is finite.*

Taking $f(u) = u^2 - u^{\frac{1}{2}}$, we see easily that $u_0 = 2^{\frac{2}{3}} > 0$ and $u \to \frac{f(u)}{u-u_0}$ is increasing for $u > 2^{\frac{5}{3}} > u_0$. Thus, Theorem B can not be applied to this function. Since $\frac{1}{2} < 1$, Theorem A can not be applied to this function, too. Moreover, if $f(u) = u^p - u^q - \varepsilon$ or f is "non-autonomous", for example, $f = \frac{1}{r^\sigma}(u^p - u^q - \varepsilon)$, for $p > q > 0$ and $\varepsilon, \sigma \geq 0$, then Theorems A and B can not applied. Inspired by the above-mentioned results, we will establish a uniqueness theorem of positive radially symmetric solutions of the exterior boundary value problem

$$\text{(IV)} \qquad \begin{cases} \Delta u + f(|x|,u) = 0 & \text{in } \Omega(\theta_1,\theta_2), \\ \lim_{|x|\to\theta_2} u(x) = 0, \\ \text{and, in addition : } \dfrac{\partial u}{\partial n}(x) = 0 & \text{if } |x| = \theta_2 \text{ and } \theta_2 \text{ is finite,} \end{cases}$$

which generalizes the results in [3, 5, 7, 8, 10, 11, 12, 15, 16, 17]. It is well-known that problem (IV) is equivalent to the following ordinary differential equation

$$\text{(E}_1\text{)} \qquad u''(r) + \frac{m}{r}u'(r) + f(r,u(r)) = 0 \quad \text{in } (\theta_1,\theta_2)$$

with exterior boundary conditions

$$\text{(BCS)} \qquad \begin{cases} \lim\limits_{r\to\theta_2} u(r) = 0, \\ \text{and, in addition : } u'(\theta_2) = 0 \quad \text{if } \theta_2 \text{ is finite,} \end{cases}$$

where r is the radial variable, if u is a radially symmetric solution of (IV). Hence, we may restrict our attention to the ordinary differential equation (E_1). For the other related results, we refer to [1–18].

2. Main Results

Let $m \geq 0$ and $0 \leq \theta_1 < \theta_2 \leq \infty$. Throughout this paper, we assume that the following assumptions hold:

- (A_1) $f : (\theta_1, \theta_2) \times (0, \infty) \to (-\infty, \infty)$ is locally Lipschitz continuous in $u \in (0, \infty)$ for all fixed r,
- (A_2) There exist $\alpha \in (\theta_1, \theta_2)$ near θ_2 enough and $\beta > 0$ small enough such that $f(r, u) < 0$ for all $(r, u) \in (\alpha, \theta_2) \times (0, \beta)$, $f(r, u)$ is increasing with respect to r in (α, θ_2) for each fixed $u \in (0, \beta)$ and $f(r, u)$ is strictly decreasing with respect to u in $(0, \beta)$ for each fixed $r \in (\alpha, \theta_2)$.

Clearly, if $f(r, u) = \frac{1}{1+r^\sigma}(u^p - u^q - \varepsilon)$, $e^{-r}(u^p - u^q - \varepsilon)$, $-r^{-\sigma}e^u$, then (A_1) and (A_2) hold, where $p > q > 0$ and $\varepsilon, \sigma \geq 0$. In particular, if $\theta_2 < \infty$, then $f(r, u) = e^r u^p - u^q - \varepsilon$, $r^\sigma u^p - u^q - \varepsilon$ satisfy (A_1) and (A_2), where $p > q > 0$ and $\varepsilon, \sigma \geq 0$.

We now consider the uniqueness of the following exterior boundary value problem:

$$\text{(BVP)} \qquad \begin{cases} u''(r) + \frac{m}{r} u'(r) + f(r, u(r)) = 0 & \text{in } (\theta_1, \theta_2), \\ \lim\limits_{r\to\theta_2} u(r) = 0 \quad \text{and} \quad u(r) > 0 & \text{in } (\theta_1, \theta_2), \\ \text{and, in addition : } u'(\theta_2) = 0 & \text{if } \theta_2 \text{ is finite.} \end{cases}$$

The significance of uniqueness of (BVP) lies in the impossible co-existence of the positive solution of (E_1) with different interior boundary conditions at θ_1.

LEMMA 1. *If u is a solution of* (BVP), *then there exists $a \in (\theta_1, \theta_2)$ such that $u'(r) < 0$ in (a, θ_2). In particular, if $\theta_2 = \infty$, then $\lim_{r\to\theta_2} u'(r) = 0$.*

PROOF. Assume, on the contrary, that there exist two increasing sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \theta_2$,

$$\theta_1 < a_n < b_n < a_{n+1} < b_{n+1} < \theta_2, \ u'(a_n) = u'(b_n) = 0, \ u'(r) > 0$$

in (a_n, b_n) and $u'(r) < 0$ in (b_n, a_{n+1}) for all $n \in N$, where N denotes the set of positive integer. Since $\lim_{r\to\theta_2} u(r) = 0$, there exists $n_0 \in N$ such that $0 < u(b_{n_0}) < \beta$ and $\alpha < b_{n_0} < \theta_2$. It follows from (E_1) and (A_2) that

$$0 = u''(b_{n_0}) + \frac{m}{b_{n_0}} u'(b_{n_0}) + f(b_{n_0}, u(b_{n_0})) < 0$$

which gives a contradiction. Thus, there exists $a \in (\theta_1, \theta_2)$ such that $u'(r) < 0$ in (a, θ_2). Furthermore, if $\theta_2 = \infty$, then

$$u''(r) = -\frac{m}{r} u'(r) - f(r, u(r)) > 0$$

in (r_0, ∞), where $r_0 \geq \max\{a, \alpha\}$ and satisfies $0 < u(r) < \beta$ in (r_0, ∞). Hence, $u'(r)$ is increasing and bounded above by 0 in (r_0, ∞) which imply $\lim_{r\to\infty} u'(r) := \xi \leq 0$. We claim that $\xi = 0$. In fact, if $\xi < 0$, then $\lim_{r\to\infty} u(r) = -\infty$ which contradicts to $\lim_{r\to\infty} u(r) = 0$. Hence, $\lim_{r\to\theta_2} u'(r) = \lim_{r\to\infty} u'(r) = 0$.

LEMMA 2. *If u and v are two distinct solutions of* (BVP), *then there exists $b \in (\theta_1, \theta_2)$ such that $u(r) \neq v(r)$ in (b, θ_2).*

PROOF. Assume, on the contrary, that there exists an increasing sequence $\{a_n\}_{n=1}^{\infty} \subset (\theta_1, \theta_2)$ such that $\lim_{n\to\infty} a_n = \theta_2$ and $u(a_n) = v(a_n)$ for each $n \in N$. Since $\lim_{r\to\theta_2} u(r) = \lim_{r\to\theta_2} v(r) = 0$, there exists $n_0 \in N$ and $d \in (a_{n_0}, a_{n_0+1}) \subset (\alpha, \theta_2)$ such that $0 < v(r) < u(r) < \beta$ in (a_{n_0}, a_{n_0+1}), $u'(d) = v'(d)$ and $u''(d) \leq v''(d)$. It follows from (E_1) and (A_2) that

$$0 \geq u''(d) - v''(d) = -\frac{m}{d}\{u'(d) - v'(d)\} - \{f(d, u(d)) - f(d, v(d))\} > 0$$

which gives a contradiction. Hence, there exists $b \in (\theta_1, \theta_2)$ such that $u(r) \neq v(r)$ in (b, θ_2).

LEMMA 3. *If u and v are two distinct solutions of* (BVP), *then there exists $c \in (\theta_1, \theta_2)$ satisfying $u(r) \neq v(r)$ in (c, θ_2) and if $v(r) < u(r)$ in (c, θ_2), then $u'(r) \leq v'(r)$ in (c, θ_2).*

PROOF. It follows from Lemma 2 that there exists $b \in (\theta_1, \theta_2)$ such that $u(r) \neq v(r)$ in (b, θ_2). Without loss of generality, we may assume that $u(r) > v(r)$ in (b, θ_2). Let $w(r) := u(r) - v(r)$ in (b, θ_2). We claim that there exists $c \in (b, \theta_2)$ such that $u'(r) \leq v'(r)$ in (c, θ_2). Assume, on the contrary, that there exists an increasing sequence $\{a_n\}_{n=1}^{\infty} \subset (b, \theta_2)$ such that $\lim_{n\to\infty} a_n = \theta_2$, $w'(a_n) = 0$ and $w''(a_n) \leq 0$ for each $n \in N$. Since $\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = 0$, there

exists $n_0 \in N$ such that $0 < v(a_{n_0}) < u(a_{n_0}) < \beta$ and $a_{n_0} \in (\alpha, \theta_2)$. It follows from (E_1), (A_2) and the definition of w that

$$0 > w''(a_{n_0}) + \frac{m}{a_{n_0}} w'(a_{n_0}) + f(a_{n_0}, u(a_{n_0})) - f(a_{n_0}, v(a_{n_0})) = 0$$

which gives a contradiction. Hence, there exists $c \in (b, \theta_2)$ such that $u'(r) \leq v'(r)$ in (c, θ_2). Hence, we obtain the desired result.

THEOREM 4. (*Uniqueness theorem*) (BVP) *has at most one solution in* (θ_1, θ_2).

PROOF. Assume, on the contrary, that u and v are two distinct solutions of (BVP). It follows from Lemmas 1 and 3 that there exists $c \in (\theta_1, \theta_2)$ such that $0 < v(y) < u(y)$ and $u'(y) \leq v'(y) < 0$ in (c, θ_2). Since u and v are strictly decreasing in (c, θ_2), they have the inverse functions, say r and s, respectively. Clearly, r and s are defined on $[0, u(c)]$ and $[0, v(c)]$ respectively. Moreover,

$$u(r(t)) = t \quad \text{for} \quad t \in [0, u(c)]; \; v(s(t)) = t \quad \text{for} \quad t \in [0, v(c)].$$

Define $R : [0, u(c)] \to [0, \infty)$ and $S : [0, v(c)] \to [0, \infty)$ by $R(t) := u'^2(r(t))$ and $S(t) := v'^2(s(t))$, respectively. It follows from Lemma 1 and the definitions of r, s, R, S that R and S satisfy the initial value problems:

$$(E_2) \qquad R'(t) = 2\{\frac{m}{r(t)} \sqrt{R(t)} - f(r(t), t)\} \quad \text{on} \quad (0, u(c)]; \; R(0) = 0.$$

and

$$(E_3) \qquad S'(t) = 2\{\frac{m}{s(t)} \sqrt{S(t)} - f(s(t), t)\} \quad \text{on} \quad (0, v(c)]; \; S(0) = 0.$$

respectively. It follows from Theorem of Kaper and Kwong [6] (see also Murakami [13]) that these initial value problems (E_2) and (E_3) have the unique positive non-trivial solutions R and S, respectively. Since $u(y) > v(y)$ in (c, θ_2), $r(t) > s(t)$ in $(0, v(c))$. Hence, by (A_2) we see that

$$R'(t) \leq 2\{\frac{m}{s(t)} \sqrt{R(t)} - f(s(t), t)\} \quad \text{on} \quad (0, t_0],$$

where $t_0 \in (0, v(c))$ such that $s(t_0) > \max\{c, \alpha\}$ and $v(s(t_0)) = t_0 \in (0, \beta)$. A simple comparison argument shows that

$$R(t) \leq S(t) \quad \text{on} \quad [0, t_0].$$

Hence, $u'(r(t)) \geq v'(s(t))$ on $[0, t_0]$, which implies

$$r'(t) = \frac{1}{u'(r(t))} \leq \frac{1}{v'(s(t))} = s'(t) \leq 0 \quad \text{on} \quad (0, t_0].$$

Hence,

$$-\frac{r'(t)}{u'(r(t))} = -r'^2(t) \leq -s'^2(t) = -\frac{s'(t)}{v'(s(t))} \quad \text{on} \quad (0, t_0].$$

Integrating the above inequality from $t \in (0, t_0]$ to t_0, we see that

$$\int_{s(t_0)}^{r(t)} \frac{1}{u'(y)}\, dy \leq \int_{r(t_0)}^{r(t)} \frac{1}{u'(y)}\, dy \leq \int_{s(t_0)}^{s(t)} \frac{1}{v'(y)}\, dy.$$

Let

$$H(t) := \int_{s(t_0)}^{r(t)} \frac{1}{u'(y)}\, dy - \int_{s(t_0)}^{s(t)} \frac{1}{v'(y)}\, dy \quad \text{on} \quad (0, t_0].$$

It is clear that

$$H'(t) := \frac{r'(t)}{u'(r(t))} - \frac{s'(t)}{v'(s(t))} \geq 0 \quad \text{and} \quad H(t) \leq 0 \quad \text{on} \quad (0, t_0].$$

Hence,

$$\int_{s(t_0)}^{\theta_2} \left\{ \frac{1}{u'(y)} - \frac{1}{v'(y)} \right\} dy \leq H(0+) \leq 0.$$

It follows from $s(t_0) > c$ and $u'(y) \leq v'(y) < 0$ in (c, θ_2) that

$$0 \leq \int_{s(t_0)}^{\theta_2} \left\{ \frac{1}{u'(y)} - \frac{1}{v'(y)} \right\} dy \leq 0,$$

which implies $u'(y) = v'(y)$ on $[s(t_0), \theta_2)$. Since $\lim_{y \to \theta_2} u(y) = \lim_{y \to \theta_2} v(y) = 0$, we see that $u(y) = v(y)$ on $[s(t_0), \theta_2)$ which contradicts to $0 < v(y) < u(y)$ in (c, θ_2). Hence, (BVP) has at most one solution in (θ_1, θ_2).

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