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DENOMINATOR IDEALS AND THE EXTENSIONS OF NOETHERIAN DOMAINS

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Introduction

In this article, we study relationships between denominator ideals and extensions of Noetherian domains. Let $A \supset R$ be an extensions of Noetherian domains, and let K and L are quotient fields of R and A respectively. Assume that $A = R[\alpha_1, \ldots, \alpha_n]$ for some elements $\alpha_i (1 \leq i \leq n)$ of A and that L/K is an algebraic extension.

For a non-zero element $\alpha \in A$, let $\phi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the minimal polynomial of α over K.

Then we denote by I_α the ideal $\bigcap_{i=1}^{d}(R :_R \eta_i)$ where $(R :_R \eta_i) = \{r \in R \,|\, r\eta_i \in R\}$. Then one can see that for a prime ideal p of R, I_α is not contained in p if and only if $\eta_i \in R_p$ for all i, namely $\phi_\alpha(X) \in R_p[X]$. Note that if $\phi_\alpha(X) \in R_p[X]$, then A_p is a free R_p-module of rank d. If R and A are birational, and $A = R[\alpha]$, then $\phi_\alpha(X) = X - \alpha$, and hence we see that $I_\alpha = \{r \in R \,|\, r\alpha \in R\}$, which is usually called the denominator ideal of α. Thus our definition is a generalization of the birational case. We are interested in characterization of the flatness and the integrality of the extensions of Noetherian domains in terms of the set $\{I_{\alpha_i} \,|\, 1 \leq i \leq n\}$.

First, we treat the birational case. Indeed we obtain a result that A is flat over R if and only if $I_{\alpha_i} A = A$ for all i. Further, under certain condition, we can show that A is integral over R if and only if $\alpha I_\alpha \subseteq I_\alpha$.

Then, we consider the non-birational case. In fact, under certain condition, similar results as birational case are proved, and we determine the non-flat locus for a simple extension. Throughout this article, R denotes a commutative integral domain with quotient field K, unless otherwise specified. As for our notation, we refer to [3] and [4].

§1. Birational Case

Now recall the definition of a "denominator ideal". For a non-zero element $\alpha \in K$, the denominator ideal I_α of α is $\{r \in R \,|\, r\alpha \in R\}$.

First, we have the following proposition.

PROPOSITION 1. *Let $\{\alpha_i \mid 1 \leq i \leq n\}$ be a finite subset of K. Then $A = R[\alpha_1, \ldots, \alpha_n]$ is flat over R if and only if $I_{\alpha_1} \cdot I_{\alpha_2} \cdots I_{\alpha_n} A = A$ or, equivalently $I_{\alpha_i} A = A$ for all i.*

PROOF. ($\Rightarrow$): [6, Theorem 1] shows that A is flat over R if and only if $I_\alpha A = A$ for every $\alpha \in A$. This is applied to $\alpha = \alpha_i$ for each i.
($\Leftarrow$): Consider each $p \in \mathrm{Spec}(R)$. If $I_{\alpha_1} \cdot I_{\alpha_2} \cdots I_{\alpha_n} \subseteq p$, then $pA = A$ by assumption. If $I_{\alpha_1} \cdot I_{\alpha_2} \cdots I_{\alpha_n}$ is not contained in p, then I_{α_i} is not contained in p for all i, and hence $\alpha_i \in R_p$ for all i. Therefore it follows $A_p = R_p$ in this case. Thus, by [6, Theorem 1], we see that A is flat over R. Q.E.D.

For an ideal I of R, we write $V(I) = \{p \in \mathrm{Spec}(R) \mid p \supset I\}$. Then we have;

COROLLARY 2. *Let $\{\alpha_i \mid 1 \leq i \leq n\}$ be a finite subset of K, and assume that $A = R[\alpha_1, \ldots, \alpha_n]$ is flat over R. Then we have* $\mathrm{Spec}(A) \cong \mathrm{Spec}(R) \setminus \bigcup_{i=1}^n V(I_{\alpha_i})$.

COROLLARY 3. *For $\alpha \in K$, $A = R[\alpha, \alpha^{-1}]$ is flat over R if and only if $I_\alpha A = A$.*

PROOF. If A is flat over R, then $I_\alpha \cdot I_{\alpha^{-1}} A = A$ by Proposition 1. Note that $I_{\alpha^{-1}} = \alpha I_\alpha$. Hence we have $\alpha (I_\alpha)^2 A = A$. Since α is a unit in A, it holds $(I_\alpha)^2 A = A$, and $I_\alpha A = A$.
Conversely, we assume that $I_\alpha A = A$. Then, since α is a unit in A, we have $I_{\alpha^{-1}} A = \alpha I_\alpha A = A$. Thus A is flat over R by Proposition 1. Q.E.D.

Now, we introduce the following definition.

DEFINITION 4. *We say that $\alpha \in K$ is a flat element over R if $R[\alpha]$ is flat over R.*

Flat elements have the following property.

PROPOSITION 5. *Let $\{\alpha_i \mid 1 \leq i \leq n\}$ be a finite subset of K. If $\alpha_i (1 \leq i \leq n)$ are flat elements over R, then $R[\alpha_1, \ldots, \alpha_n]$ is flat over R.*

PROOF. Since $\alpha_i (1 \leq i \leq n)$ are flat elements over R, by definition, we have $R[\alpha_i]$ are flat over R, and hence $I_{\alpha_i} R[\alpha_i] = R[\alpha_i]$ by Proosition 1. Thus $I_{\alpha_i} R[\alpha_1, \ldots, \alpha_n] = R[\alpha_i, \ldots, \alpha_n]$ for all $i (1 \leq i \leq n)$. Therefore $I_{\alpha_1} \cdot I_{\alpha_2} \cdots I_{\alpha_n} R[\alpha_1, \ldots, \alpha_n] = R[\alpha_1, \ldots, \alpha_n]$. This shows that $R[\alpha_1, \ldots, \alpha_n]$ is flat over R by Proposition 1. Q.E.D.

Next we consider integral extensions. Let R be a Noetherian domain with quotient field K. Assume that the integral cosure $\overline{R}$ of R in K is a finite R-module.

We recall the following definition introduced by J. Lipman ([2]).

DEFINITION 6. (cf. [2]) *Let A be an overring of R and ${}^*_A R$ be the subring of A given by*

$${}^*_A R = \{\alpha \in A \mid \alpha \otimes 1 = 1 \otimes \alpha \in A \otimes_R A\}.$$

We say that R is strictly closed in A if ${}^*_A R = R$.

An intermidiate ring A between R and $\overline{R}$ will be called a birational-integral extension of R. For the strictly closedness of a birational-integral extension of R, the following proposition is important.

PROPOSITION 7. (cf. [1, Proposition 2]) *Let A be a birational-integral extension of R. If R is strictly closed in A, then we have $I_\alpha \bigcap I_\beta \subseteq I_{\alpha\beta}$ for any elements $\alpha, \beta \in A$.*

Now, we have the following proposition, which is a characterization of birational-integral extensions in terms of denominator ideals.

PROPOSITION 8. *Consider $A = R[\alpha]$ with $\alpha \in K$. Suppose that R is strictly closed in A. Then α is integral if and only if $I_{\alpha^{-1}} = \alpha I_\alpha \subseteq I_\alpha$.*

PROOF. ($\Rightarrow$): By the determinant trick, this is clear.
($\Leftarrow$): Let c be the conductor ideal of A over R. Then it is easy to see that $c = \bigcap_{i=1}^{n} I_{\alpha^i}$, for some n. Since, by assumption, R is strictly closed in A, $I_\alpha \bigcap I_\beta \subseteq I_{\alpha\beta}$ for any elements by Proposition 7. Hence it holds that $I_\alpha \subseteq I_{\alpha^i}$ for all i, and $I_\alpha \subseteq c$. Thus $c = I_\alpha$. Therefore I_α is an ideal in A, and $\alpha I_\alpha \subseteq I_\alpha$. Q.E.D.

§2. General Case

Hereafter we treat the non-birational case. Let R be a Noetherian domain with quotient field K, and let L be an algebraic extension of finite degree over K. Consider the polynomial ring $R[X]$ of one variable over R. For a subset I of $R[X]$, we denote by $c(I)$ the ideal of R generated by the coefficients of the elements of I. For any $\beta \in L$, let $\pi : R[X] \to R[\beta]$ be the natural R-algebra homomorphism sending X to β and let $\phi_\beta(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ be the minimal polynomial of β over K. Recall that $I_\beta := \bigcap_{i=1}^{d}(R :_R \eta_i)$ where $(R :_R \eta) = \{r \in R \mid r\eta_i \in R\}$. For any $\beta \in L$, we denote $c(I_\beta \phi_\beta(X))$ by J_β.

DEFINITION 9. *Under the circumstances, we say that $\alpha \in L$ is an anti-integral element of degree d over R if $\mathrm{Ker}(\pi) = I_\alpha \phi_\alpha(X) R[X]$. When $\alpha \in L$ is an anti-integral element, we say that $R[\alpha]$ is an anti-integral extension of R.*

For the flatness of anti-integral extension, denominator ideals play an important role. In fact we have the following proposition.

PROPOSITION 10. *Let $\alpha \in L$ be an anti-integral element of degree d over R. Consider $A = R[\alpha]$. If $I_\alpha A = A$, then A is flat over R.*

PROOF. Since $I_\alpha A = A$, there exists elements $a_i \in I_\alpha (0 \leq i \leq n)$ such that $1 = \sum_{i=0}^n a_i \alpha^i$. Take $p \in \mathrm{Spec}(R)$. Replacing R and A by R_p and A_p, we may assume that R is a local ring with maximal ideal m. If I_α is not contained in m, then $I_\alpha = R$, and hence $\phi_\alpha(X) \in R[X]$. Thus A is a free R-module in this case. If $I_\alpha \subseteq m$, then we put $f(X) = 1 - \sum_{i=0}^n a_i X^i$. Then $f(\alpha) = 0$. Since α is anti-integral element over R, there exist elements $g_i(X), h_i(X) \in R[X]$ $(0 \leq i \leq n)$ such that $f(X) = \sum_{i=0}^n g_i(X)h_i(X)$ and $g_i(X) \in I_\alpha \phi_\alpha(X)$ for all i. If $c(I_\alpha \phi_\alpha(X)) \subseteq m$, then it follows that $g_i(X) \in m[X]$ for all i. Hence we have $f(X) \in m[X]$. However, considering a constant term of $f(X)$, we see that $1 - a_0 \in m$. Since $a_0 \in I_\alpha \subseteq m$, this is a contradiction. Therefore we see $c(I_\alpha \phi_\alpha(X)) = R$. Hence, by [5, 2.6. Proposition], A is flat over R. Q.E.D.

Next we determine the non-flat locus for an integral extension. For this purpose, we prove the following lemma.

LEMMA 11. *Let $R \subset A$ be an extension of Noetherian rings. Assume that $A = R[\alpha]$ for some element $\alpha \in A$ and that A is a free R-module of rank d. Then we have $A = R + R\alpha + \cdots + R\alpha^{d-1}$.*

PROOF. Let $B = R + R\alpha + \cdots + R\alpha^{d-1} \subseteq A$. Then we have only to show that $A_p = B_p$ for all $p \in \mathrm{Spec}(R)$. Hence we may assume that R is a local ring with maximal ideal m. Since A is a free R-module of rank d, A/mA is a free R/m-module of rank d, and hence A/mA is a d-dimensional vector space over R/m. Note that $A/mA = (R/m)[\overline{\alpha}]$, $\overline{\alpha} \in A/mA$. Now suppose that $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha^s}$ $(s < d)$ are linearly dependent over R/m and $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha^{s-1}}$ are linearly independent over R/m. Then we have $A/mA = R/m + R/m\overline{\alpha} + \cdots + R/m\overline{\alpha^{s-1}}$. This contradicts the fact that $\dim(A/mA) = d$. Hence $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha^{d-1}}$ are linearly independent over R/m and we see that $\overline{1}, \overline{\alpha}, \ldots, \overline{\alpha^d}$ are linearly dependent over R/m. Therefore we see that $A = R + R\alpha + \cdots + R\alpha^{d-1}$ by Nakayama's lemma, as desired. Q.E.D.

Hence we have the following;

PROPOSITION 12. *Let L be an algebraic extension of finite degree over K. Consider $A = R[\alpha]$ for an integral element L over R. Then $V(I_\alpha) = \{p \in \mathrm{Spec}(R) \mid A_p$ is not flat over $R\}$.*

PROOF. Let $[K[\alpha] : K] = d$ and consider $p \in \mathrm{Spec}(R)$ such that I_α is not contained in p. Then $\phi_\alpha(X)$ is monic polynomial of degree d over R_p. Since $[K[\alpha] : K] = d$, we see that $A_p = R_p[\alpha]$ is a free R_p-module of rank d, and A_p is a flat R_p-module, then $A_p = R_p + R_p\alpha + \cdots + R_p\alpha^{d-1}$ by Lemma 11. Thus there exists a monic relation of α of degree d over R_p. Therefore $\phi_\alpha(X) \in R_p[X]$, and I_α is not contained in p. This complete the proof. Q.E.D.

PROPOSITION 13. *Let L be an algebraic extension of a finite degree over K. Consider $A = R[\alpha]$ for an element $\alpha \in L$. Assume that $I_\alpha A = A$. Then $V(I_\alpha + I_{\alpha^{-1}}) \supseteq \{p \in \mathrm{Spec}(R) \mid A_p$ is not flat over $R_p\}$.*

PROOF. Let $[K[\alpha] : K] = d$ and assume that p does not contain $I_\alpha + I_{\alpha^{-1}}$ for $p \in \mathrm{Spec}(R)$. Then p does not contain I_α or $I_{\alpha^{-1}}$. First assume that p does not contain I_α. Then $\phi_\alpha(X) \in R_p[X]$. Hence A_p is a free R_p-module. Next, assume that p does not contain $I_{\alpha^{-1}}$ and $p \supseteq I_\alpha$. Since p does not contain $I_{\alpha-1}$, by the definition of $I_{\alpha^{-1}}$, we have that $R_p[\alpha^{-1}]$ is a free R_p-module of rank d. Since $I_\alpha A = A$ by the assumption, it holds that α^{-1} is integral over R_p. Thus we have the integral relation $(\alpha^{-1})^n + (\alpha^{-1})^{n-1} \cdots + a_n = 0,\ a_i \in R_p\ (1 \leq i \leq n)$, and hence $\alpha^{-1} = -(a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1}) \in R_p[\alpha]$. Therefore $A_p = R_p[\alpha] = R_p[\alpha, \alpha^{-1}]$. Since A_p is flat over $R_p[\alpha^{-1}]$, we conclude that A_p is flat over R_p.

Q.E.D.

REMARK 14. Under the circumstances, we can see easily that if $I_\alpha + I_{\alpha^{-1}} = R$ and $I_\alpha A = A$, then A is flat over R.

As another application of denominator ideals, we see the following proposition.

PROPOSITION 15. *Let L be an algebraic extension of a finite degree over K. Assume that $\alpha, \beta_1, \ldots, \beta_n \in L$, $[K[\alpha] : K] = d$, $L = K[\alpha]$ and that $R[\alpha]$ is flat over R. Consider $A = R[\alpha, \beta_1, \ldots, \beta_n]$. Then we can write $\beta_i = \eta_{i,0} + \eta_{i,1}\alpha + \cdots + \eta_{i,d-1}\alpha^{d-1}$ with $\eta_{i,j} \in K(1 \leq i \leq n,\ 1 \leq j \leq d-1)$. Now, we define $J_{\beta_i} = \bigcap_{j=0}^{d-1} I_{\eta_{i,j}}$ for all i. If $J_{\beta_1} \cdots J_{\beta_n} A = A$, then A is flat over R.*

PROOF. Since $R[\alpha]$ is flat over R, we have only to show that A is flat over $R[\alpha]$. Let $P \in \mathrm{Spec}(R[\alpha])$ and let $P \cap R = p$. If $p \supseteq J_{\beta_1} \cdots J_{\beta_n}$, then $pA = A$. This contradicts the fact that $P \supseteq pA$. Hence p does not contain $J_{\beta_1} \cdots J_{\beta_n}$, so that p does not contain J_{β_i} for all i. Thus, by the definition of J_{β_i}, it holds p does not contain $I_{\eta_{i,j}}$ for all i, j. Hence we see that $\beta_i \in R_p[\alpha]$. Therefore it follows that $A_p = R_p[\alpha]$. This implies that A_p is flat over R_p, and hence A is flat over R. Q.E.D.

COROLLARY 16. *Under the circumstances, if $J_{\beta_i}R[\alpha,\beta_i] = R[\alpha,\beta_i]$ for all i, then we have $R[\alpha,\beta_1,\dots,\beta_n]$ is flat over R.*

PROOF. Since $J_{\beta_1}\cdots J_{\beta_n}R[\alpha,\beta_1,\dots,\beta_n] = R[\alpha,\beta_1,\dots,\beta_n]$, the assertion is easily seen by Proposition 15. Q.E.D.

Finally the following example shows that Propositions 5 does not holds in general for non-birational extensions.

EXAMPLE 17. Let $R \subset A$ be a non-birational extension of Noetherian domains, and let K and L be the quotient fields of R and A, respectively. Assume that there are elements $\alpha \in L\backslash K$ and $\eta \in K$ such that $L = K[\alpha]$, $\alpha^2 = a \in R$ both η and $\eta^{-1} \notin R_p$ for some $p \in \mathrm{Spec}(R)$ and that $a\eta^2 \in R$. Then we have $L = K[\beta] \supset K$ with $\beta = \eta\alpha$ and $\beta^2 = \eta^2\alpha^2 = a\eta^2 \in R$. Hence we see that $R[\alpha]$ and $R[\beta]$ are free R-modules. Furthermore we can show that $A = R[\alpha,\beta]$ is not a flat R-module. Indeed, let us assume that $R[\alpha,\beta]$ is a flat R-module. By localizing at p, we may assume that R is a local ring with maximal ideal m. Note that $A = R + R\alpha + R\beta + R\alpha\beta$. Since A is a free R-module of rank 2, $A/mA = R/m + (R/m)\overline{\alpha} + (R/m)\overline{\beta} + (R/m)\overline{\alpha\beta}$, $\overline{\alpha},\overline{\beta} \in A/mA$ is a 2-dimensional vector space over R/m. Since $\overline{1} \in A/mA$ is linearly independent over R/m, we have the following possibilities:

(1) $A/mA = R/m + (R/m)\overline{\alpha}$,
(2) $A/mA = R/m + (R/m)\overline{\beta}$,
(3) $A/mA = R/m + (R/m)\overline{\alpha\beta}$.

Therefore, by Nakayama's lemma, we have $A = R + R\alpha$ or $A = R + R\beta$ or $A = R + R\alpha\beta$. If $\alpha \in R + R\alpha\beta$, then we have $\alpha = x + y\alpha\beta$ for some elements $x, y \in R$. Since η and $\eta^{-1} \notin R$ by assumption, we notice that x, y are non-zero elements of R. Now, it holds that $R \ni y^2\alpha^2\beta^2 = (\alpha - x)^2 = \alpha^2 - 2x\alpha + x^2$, and $\alpha = (a + x^2 - y^2\alpha^2\beta^2)/2x \in K$. This is a contradiction. Therefore $\alpha \notin R + R\alpha\beta$. In the same way, we see that $\beta \notin R + R\alpha\beta$. Thus it follows that $\alpha \in R + R\alpha\beta$ or $\beta \in R + R\alpha$. Suppose that $\beta \in R + R\alpha$. Then $\beta = x + y\alpha$ for some elements $x, y \in R$. Note that x, y are non-zero elements of R. Since $\beta^2 = x^2 + 2xy\alpha + \alpha^2 + a$, we have $2xy\alpha = \beta^2 - a - x^2 \in R$. Since $xy \neq 0$, it holds that $\alpha \in K$, a contradiction. Therefore A is not a flat R-module. Finally, we construct the example satisfying these conditions. Let k be a field and let X, Y, Z be the indeterminates over k. Let $R = k[X,Y]/(Y^2 - X^2Y + X^2) = k[x,y]$, where x, y denote the residue classes of X, Y in R. Let $A = R[Z]/(Z^2 - (y-1)) = R[z]$, where z denotes the residue class of Z in A. Now, put $\alpha = z$, $\eta = y/x$ and $p = (x,y) \in \mathrm{Spec}(R)$. Then we see that $\alpha \notin K$. Now assume that $\eta \in R_p$. Then we have $y \in pR_p$, and hence $pR_p = xR_p$. Thus R_p is a discrete valuation ring. This is a contraciction. Therefore we have $\eta \notin R_p$. Similarly, we see $\eta^{-1} \notin R_p$. Further, we have $\eta^2 = y^2/x^2 = y - 1 = a \in R$

and $a\eta^2 = (y-1)^2 \in R$. Therefore it follows that $R[z, yz/x]$ is not a flat R-module but $R[z]$ and $R[yz/x]$ are flat R-modules by the preceeding argument.

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