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LOCAL WEIGHTED INEQUALITIES FOR THE FRACTIONAL INTEGRAL OPERATOR

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Abstract

Necessary conditions and sufficient conditions are found here for a local version of the fractional integral operator to be bounded on weighted Lebesgue spaces. Actually a characterization of this boundedness is obtained for certain classes of weights.

1. Introduction

Throughout this paper $B(x_0, R_0) = \{z \in \mathbb{R}^n; |x_0 - z| < R_0\}$ denotes the ball centered at $x_0 \in \mathbb{R}^n$, $n \geq 1$, and with the radius $R_0 > 0$. Define the *local* fractional integral operator of order α , $0 < \alpha < n$, by

$$\left(I_{\alpha, B(x_0, R_0)} f\right)(x) = \int_{y \in B(x_0, R_0)} |x - y|^{\alpha-n} f(y) dy.$$

Our purpose in this paper is to derive conditions on the weights $u(\cdot)$ and $v(\cdot)$ for which $I_{\alpha, B(x_0, R_0)}$ is bounded from the weighted Lebesgue space $L^p(B(x_0, R_0), v(y)dy)$ to $L^q(B(x_0, R_0), u(x)dx)$ when $1 \leq p \leq q < \infty$. It means that for some constant $C > 0$

$$(1.1) \quad \left(\int_{B(x_0, R_0)} \left(I_{\alpha, B(x_0, R_0)} f \right)^q(x) u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_{B(x_0, R_0)} f^p(y) v(y) dy \right)^{\frac{1}{p}}$$

for all functions $f(\cdot) \geq 0$ whose supports are $B(x_0, R_0)$.

Local weighted inequalities like (1.1) arise naturally in studies of Sobolev and Poincaré inequalities [Fr-Pe-Wh] which are involved in some partial elliptic differential equations in divergence form. These inequalities (1.1) play also crucial role in solving multidimensional Riccati's equations [Ve-Wh] and some superlinear integral equations [Ve].

The literature concerning (1.1) when $B(x_0, R_0) = \mathbb{R}^n$ (e.g. for $R_0 = \infty$) is vast and references should be found over [Ge-Go-Ko], [Ge-Go-Ko-Kr], [Pe],

[Ra2], [Sa-Wh], [Sa-Wh-Zh] and [Ve-Wh]. A. Cianchi and D. Edmunds [Ci-Ed] were able to characterize inequality (1.1) even $B(x_0, R)$ is replaced by an open subset of $\mathbb{R}^n$. Their characterizing conditions are unfortunately given in terms of nonincreasing rearrangements of $u(\cdot)$ and $v(\cdot)$, and consequently it is not so clear in general to decide whether (1.1) holds for given weights. This situation leads us to raise the problem of deriving new criteria for (1.1) and which can be computed at least for some classes of usual weights. To answer this question is among of our motivations in this work.

E. Sawyer and R. Wheeden [Sa-Wh] have pointed out and explained in the proof of their Theorem 5 (p. 839) that, for $1 < p = q < \infty$ and $p' = \frac{p}{p-1}$, inequality (1.1) holds for some constant C which is a product of

$$(1.2) \quad C_0(t) = \sup_{B(x,r) \subset B(x_0, 8R_0)} \left\{ r^\alpha \left(r^{-n} \int_{B(x,r)} u^t(z) dz \right)^{\frac{1}{tp}} \left(r^{-n} \int_{B(x,r)} v^{(1-p')t}(z) dz \right)^{\frac{1}{tp}} \right\}$$

by another constant $c > 0$ only depending on α, p, q and n . Here t is some real strictly bigger than 1. This result seems to be the only one available concerning (1.1) and not expressed in terms of nonincreasing rearrangements of weights. However (1.2) leads to make three observations. Indeed first it is implicitly assumed that a knowledge of values of the functions $u(\cdot)$ and $v(\cdot)$ outside the ball $B(x_0, R_0)$ is required to apply the result. Next the constant $C_0(t)$ could become infinite (for all $t > 1$) for some weights $u(\cdot)$ and $v(\cdot)$ for which the inequality (1.1) holds. Such a fact, for $R_0 = \infty$, was pointed out by the author in [Ra1] and had motivated him in [Ra2] for an investigation on new sufficient conditions for (1.1), with $p = q$ and $R_0 = \infty$. This possible infiniteness of $C_0(t)$ is due to the high integrability required on the weights $u(\cdot)$ and $v^{1-p'}(\cdot)$ rather than to the supremum over balls. The third difficult with (1.2) is that an explicit computation of the constant $C_0(t)$ is not so easy to handle in general though the expressions in question do not involve the fractional integral operator itself. All of these observations will be taken into consideration in writing this paper.

Therefore we will provide here some results for the local weighted inequality (1.1) which would be an alternative to those found in [Ci-Ed]. Our conditions related to (1.1) require only the knowledge of values of $u(\cdot)$ and $v(\cdot)$ inside the ball $B(x_0, R_0)$. An interesting point here is that, results for the global boundedness (e.g. for (1.1) with $R_0 = \infty$ and $x_0 = 0$) can be deduced directly just by taking $x_0 = 0$ and $R_0 = \infty$ in each of our conditions. For $1 < p = q < \infty$ the conditions, which will be introduced here, allow more weights $u(\cdot)$ and $v(\cdot)$ than those permitted by (1.2). So even for $R_0 = \infty$ our results are new and would complete the criteria described in (1.2) and [Ra2]. The computability aspect of our conditions related to (1.1) will be illustrated over examples 2.5 to 2.12. The

case $p = 1$ or $q = 1$ will also deserve our attention since these range limits can be meaningful in applications mainly when $R_0 < \infty$.

Our main results will be stated in §2. They will be deduced from basic results displayed in §3. Their proofs will be performed in §4. And these basic results will be proved in §4.

2. Main Results

Throughout this paper it is assumed that

$$1 \leq p \leq q < \infty \quad \text{with} \quad \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n}.$$

This last inequality can be seen as a necessary condition for (1.1) to hold after using the Lebesgue differentiation theorem. As usual let us set

$$p' = \frac{p}{p-1} \quad \text{if } p > 1, \quad \text{and} \quad \frac{1}{p'} = 0 \quad \text{if } p = 1.$$

Hereafter $u(\cdot)$ and $v(\cdot)$ are weights on $B(x_0, R_0)$ e.g. nonnegative and locally integrable functions on $B(x_0, R_0)$ and such that $0 < \int_{B(x_0, r)} v^{1-p'}(y) dy < \infty$ for all nonnegative $r < R_0$ if $p > 1$ and $0 < \sup_{y \in B(x_0, r)} \frac{1}{v(y)} < \infty$ if $p = 1$.

For convenience the boundedness in (1.1) is denoted by

$$I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$$

where $B_0 = B(x_0, R_0)$. We will also write that " $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$ holds with the bound C " to qualify (1.1).

Here are some conditions which will be needed in our results when $p = 1$:

$$(2.1) \quad \left(\int_{2R < |x-x_0| < R_0} |x-x_0|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(x_0, R)} \frac{1}{v(y)} \right] \leq A$$

for all $R > 0$ with $2R < R_0$;

$$(2.1^*) \quad \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{2R < |y-x_0| < R_0} |y-x_0|^{(\alpha-n)} \frac{1}{v(y)} \right] \leq A$$

for all $R > 0$ with $2R < R_0$

$$(2.2) \quad R^{\alpha-n} \left(\int_{2^{-1}R < |x-x_0| < R} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{2^{-1}R < |y-x_0| < R} \frac{1}{v(y)} \right] \leq A$$

for all R with $0 < R < R_0$.

For $p > 1$ we will make use of the following conditions:

$$(2.3) \quad \left(\int_{2R < |x-x_0| < R_0} |x-x_0|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R)} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all $R > 0$ with $2R < R_0$;

$$(2.3^*) \quad \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left(\int_{2R < |y-x_0| < R_0} |y-x_0|^{(\alpha-n)p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all $R > 0$ with $2R < R_0$

$$(2.4) \quad R^{\alpha-n} \left(\int_{2^{-1}R < |x-x_0| < R} u(x) dx \right)^{\frac{1}{q}} \left(\int_{2^{-1}R < |y-x_0| < R} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all R with $0 < R < R_0$.

REMARK 1. Since (2.2) can be written as

$$(2.5) \quad |z-x_0|^{\alpha-n} \left(\int_{2^{-1}|z-x_0| < |x-x_0| < |z-x_0|} u(x) dx \right)^{\frac{1}{q}} \\ \times \left[\operatorname{ess\,sup}_{2^{-1}|z-x_0| < |y-x_0| < |z-x_0|} \frac{1}{v(y)} \right] \leq A,$$

for all z with $|z-x_0| < R_0$, then it appears that (2.2) is implied by the (stronger) condition

$$(2.6) \quad \left(\int_{B(x_0, R_0) \cap C(x_0, |z-x_0|)} |z-x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \\ \times \left[\operatorname{ess\,sup}_{y \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} \frac{1}{v(y)} \right] \leq A$$

for all z with $|z-x_0| < R_0$. Here $C(x_0, R)$, $R > 0$, denotes the annuli $\{y; 2^{-1}R < |y-x_0| < 2R\}$. An analogue form of (2.5) for $p > 1$ leads to see that condition (2.4) can be strengthened as

$$(2.7) \quad \int_{x \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} |z-x|^{(\alpha-n)t} \\ \times \left[\int_{y \in B(x_0, R_0) \cap C(x_0, |x-x_0|)} |x-y|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx \leq A^q$$

for all z with $|z - x_0| < R_0$ and $\frac{1}{t} = \frac{1}{q} + \frac{1}{p'}$. Proofs of implications (2.6) $\implies$ (2.2) and (2.7) $\implies$ (2.4) are given in §5. Note that $B(x_0, R_0) \cap C(x_0, |z - x_0|)$ is just an annuli but not any arbitrary subset of the ball $B(x_0, R_0)$. This particular geometry is of great importance in concrete computations.

We are now in position to state our main results.

THEOREM 2.1.

Part A. *If the boundedness $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$ holds with the bound C then both the Hardy conditions (2.1), (2.1*) and the Muckenhoupt (2.2) are satisfied with the constant $A = 2^{n-\alpha}C$. Conversely if conditions (2.1), (2.1*) and (2.6) are satisfied with some constant $A > 0$ then $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$ holds with the bound $C = cA$ where $c > 0$ depends only on α, q and n .*

Part B. *For $q < \frac{n}{n-\alpha}$, the reinforced Muckenhoupt condition (2.6) in Part A is implied by*

$$(2.8) \quad R^{\alpha + \frac{n}{q} - n} \left(\sup_{x \in B(x_0, R_0) \cap C(x_0, R)} u(x) \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(x_0, R_0) \cap C(x_0, R)} \frac{1}{v(y)} \right] \leq A$$

for all $R > 0$ with $R < R_0$.

THEOREM 2.2. *Let $p > 1$.*

Part A. *If the boundedness $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$ holds with the bound C then both the Hardy conditions (2.3), (2.3*) and the Muckenhoupt (2.4) are satisfied with the constant $A = 2^{n-\alpha}C$. Conversely if conditions (2.3), (2.3*) and (2.7) are satisfied for some constant $A > 0$ then $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$ holds with the bound $C = cA$ where $c > 0$ depends only on α, q and n .*

Part B. *If $\frac{n}{\alpha} \leq p \leq q$ or $p < \frac{n}{\alpha}$ with $p \leq q < p^*$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{n}{\alpha}$ then the reinforced Muckenhoupt condition (2.7) is implied by*

$$(2.9) \quad R^{\alpha + \frac{n}{q} - \frac{n}{p}} \left(\sup_{x \in B(x_0, R_0) \cap C(x_0, 2R)} u(x) \right)^{\frac{1}{q}} \left(\sup_{y \in B(x_0, R_0) \cap C(x_0, 2R)} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A$$

for all $R > 0$ with $R < R_0$.

Part C. *Better if $p < \frac{n}{\alpha}$ and $p \leq q \leq p^*$ then condition (2.7) in Part A can be replaced by*

$$(2.10) \quad |x - x_0|^{\alpha + \frac{n}{q} - \frac{n}{p}} \left(u(x) \right)^{\frac{1}{q}} \left(\sup_{y \in B(x_0, R_0) \cap C(x_0, 2|x-x_0|)} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A$$

for all x with $|x - x_0| < R_0$.

or

$$(2.10') \quad |y - x_0|^{\alpha + \frac{n}{q} - \frac{n}{p}} \left(\sup_{x \in B(x_0, R_0) \cap C(x_0, 2|y - x_0|)} u(x) \right)^{\frac{1}{q}} \left(v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A$$

for all y with $|y - x_0| < R_0$.

REMARKS 2. 1) Results for the global boundedness $I_\alpha : L^p(v) \rightarrow L^q(u)$, e.g.

$$(2.11) \quad \left(\int_{\mathbb{R}^n} (I_\alpha f)^q(x) u(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} f^p(y) v(y) dy \right)^{\frac{1}{p}} \quad \text{for all } f(\cdot) \geq 0$$

with $(I_\alpha f)(x) = \int_{y \in \mathbb{R}^n} |x - y|^{\alpha-n} f(y) dy$, can be deduced from the above Theorems by taking $x_0 = 0$ and $R_0 = \infty$ in all of our conditions since the bound involved in $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$ is of the form $C = cA$, where $c > 0$ depends just on α, n, p and q but not on R_0 . However for $R_0 < \infty$ and certain weights $u(\cdot)$ and $v(\cdot)$, the constant $A > 0$ may depend on the radius R_0 , as will seen in the below examples. The boundedness $I_\alpha : L^p(v) \rightarrow L^p(u)$, with $p > 1$, was treated in [Ra2] and the present work takes its inspiration from this previous paper.

2) The introduction of conditions (2.9), (2.10) and (2.10') in replacement of (2.7) deserves some explanations. To simplify let us consider the particular case of $p > 1$, $x_0 = 0$, $R_0 = \infty$ and for which (2.7) becomes

$$(2.12) \quad \int_{2^{-1}|z| < |x| < 2|z|} |z - x|^{(\alpha-n)t} \times \left[\int_{2^{-1}|x| < |y| < 2|x|} |x - y|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx \leq A^q$$

for all $z \neq 0$ and with $\frac{1}{t} = \frac{1}{q} + \frac{1}{p'}$. A direct computation of a term as

$$W_\alpha(z) = \int_{2^{-1}|z| < |x| < 2|z|} |x - z|^{(\alpha-n)t} w(x) dx,$$

for a general weight $w(\cdot)$, is not so easy to do. However if $w(\cdot) = 1$ then it is clear, by using polar coordinates, that

$$\int_{2^{-1}|z| < |x| < 2|z|} |x - z|^{(\alpha-n)t} dx \approx \int_0^{c|z|} r^{(\alpha-n)t+n-1} dr$$

for some $c > 0$ independent of z . From now an expression as $\mathcal{Q}_1(r) \approx \mathcal{Q}_2(r)$ means that for some (fixed) $c_1, c_2 > 0$ then $c_1 \mathcal{Q}_1(r) \leq \mathcal{Q}_2(r) \leq c_2 \mathcal{Q}_1(r)$ for all

variables r . Thus for $w(\cdot) = 1$ then $W_\alpha(z) \approx |z|^{(\alpha-n)t+n}$ whenever $(\alpha-n)t+n > 0$ or $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{n}$. This last inequality leads to understand why (2.12) can be useless if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. The case of general weights $w(\cdot)$ remains in somewhat to the above model since

$$W_\alpha(z) \leq \left(\sup_{2^{-1}|z| < |y| < 2|z|} w(y) \right) \int_{2^{-1}|z| < |x| < 2|z|} |x-z|^{(\alpha-n)t} dx.$$

Therefore (2.9) is introduced for computational reasons. On the other hand (2.10) is useful to include the limit case $q = p^*$ which cannot be reached by (2.12). Finally (2.10') is a sort of dual form of (2.10). By the way, remind that boundedness (1.1) is meaningful only whenever $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n}$.

3) The pointwise conditions (2.8), (2.9), (2.10) and (2.10') are very often easy to handle and manage in explicit computations. Some of them are already introduced in [Ra2]. At present, our emphasize on the use of conditions involving integral expressions like (2.6) and (2.7) lies on the fact that there are weights $u(\cdot)$ and $v(\cdot)$ for which (1.1) holds but (2.9) or (2.10) is not satisfied since one of the supremums becomes infinite. An illustration will be given in example 2.12 below.

4) For $q = p^*$ and $p < \frac{n}{\alpha}$, the question of finding a condition not involving supremum of the weights on annuli, as in (2.9) or (2.10) and which replaces (2.7), remains unsolved.

5) The philosophy of our above results can be roughly summarized as follows:

Necessary conditions for $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$ are the fulfillement of two Hardy conditions and a Muckenhoupt condition type. And conversely the boundedness holds if the former Muckenhoupt condition is strengthened.

Consequently a characterizing result should be available if the reinforced condition coincides with the necessary Muckenhoupt condition type. This situation will happen if the weights satisfy an additional suitable property. To be clear let us introduce

$$\text{the class } RI_{\alpha, p, q}(B(x_0, R_0))$$

which consists of weights $w(\cdot)$ such that for some constants $c_1, c_2 > 0$

(2.13)

$$\int_{y \in B(x_0, R_0) \cap C(x_0, R)} |z-y|^{(\alpha-n)t} w(y) dy \leq c_1 R^{(\alpha-n)t} \int_{B(x_0, R_0) \cap B(x_0, c_2 R)} w(y) dy$$

for all nonnegative $R = |z - x_0| \leq R_0$ and where as above $\frac{1}{t} = \frac{1}{q} + \frac{1}{p'}$.

Of course the set $B(x_0, R_0) \cap B(x_0, c_2 R)$ is also a ball centered at x_0 and contained in $B(x_0, R_0)$. This fact is useful for explicit computations as we will illustrate

in the below examples. Assumption (2.13) seems to be a very natural condition since it is a sort of converse of the inequality

$$R^{(\alpha-n)t} \int_{B(x_0, R_0) \cap B(x_0, c_2 R)} w(y) dy \leq c_3 \int_{y \in B(x_0, R_0) \cap B(x_0, c_2 R)} |z - y|^{(\alpha-n)t} w(y) dy$$

(where $R = |z - x_0|$) which is always true for all weights $w(\cdot)$. Readers who are well-familiarized with weights theory may interpret (2.13) as a consequence of reverse doubling condition of order θ , $\theta > \frac{n-\alpha}{n}t$, for the weight $w(\cdot)$ in the sense that

$$\int_{B(z, 2^{-k}|z-x_0|)} w(y) dy \leq c_1 2^{-kn\theta} \int_{B(x_0, c_2|z-x_0|) \cap B(x_0, R_0)} w(y) dy$$

for all $z \in B(x_0, R_0)$ and all integers $k \geq 1$. Observe that for $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{n}$ then (2.13) is implied by the

$$\text{assumption } w(\cdot) \in \mathcal{H}(B(x_0, R_0))$$

which means

$$(2.14) \quad \left(\sup_{y \in B(x_0, R_0) \cap C(x_0, 2R)} w(y) \right) \leq c_1 R^{-n} \int_{B(x_0, R_0) \cap B(x_0, c_2 R)} w(y) dy$$

Here (2.14) is a local version of the weight class $\mathcal{H} = \mathcal{H}(\mathbb{R}^n)$ already introduced in [Ra2]. It is seen, for example, in this former paper that $w(\cdot) \in \mathcal{H}$ whenever $w(\cdot)$ is given by a radial and monotone weight function. But the assumption $\mathcal{H}$ can be also satisfied though $w(\cdot)$ is not necessarily monotone.

Proof of first part of Theorem 2.1 enables us to see that a necessary condition for $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$, stronger than (2.2), is that for some constant $A > 0$

$$(2.15) \quad R^{\alpha-n} \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(x_0, R)} \frac{1}{v(y)} \right] \leq A$$

for all R with $0 < R < R_0$.

Similarly for $p > 1$ then $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$ implies

$$(2.16) \quad R^{\alpha-n} \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R)} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all R with $0 < R < R_0$.

The expected characterization for boundedness (1.1), announced in the abstract, can be now stated.

PROPOSITION 2.3. *The boundedness $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$ is characterized by both the Hardy condition (2.1) and the Muckenhoupt condition (2.15) whenever one of the following is satisfied:*

- i) $u(\cdot) \in RI_{\alpha, 1, q}(B(x_0, R_0))$;
- ii) $q < \frac{n}{n-\alpha}$ and $u(\cdot) \in \mathcal{H}(B(x_0, R_0))$.

PROPOSITION 2.4. *Let $p > 1$. The boundedness $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$ is characterized by both the Hardy conditions (2.3), (2.3*) and the Muckenhoupt condition (2.16) whenever one of the following is satisfied:*

- i) $v^{1-p'}(\cdot) \in RI_{\alpha, p, q}(B(x_0, R_0))$ and $u(\cdot) \in RI_{\alpha, p, q}(B(x_0, R_0))$;
- ii) $p \leq q \leq p^*$ with $p < \frac{n}{\alpha}$ and $v^{1-p'}(\cdot) \in \mathcal{H}(B(x_0, R_0))$ and $u(\cdot) \in \mathcal{H}(B(x_0, R_0))$.

REMARKS 3. 1) We have no result concerning $I_{\alpha, B_0} : L^1(v) \rightarrow L^{1^*}(u)$ where $\frac{1}{1^*} = 1 - \frac{\alpha}{n}$. In this direction it can be noted that the boundedness $I_\alpha : L^1(1) \rightarrow L^{1^*}(1)$ is well-known to be false. However examples of weights $u(\cdot)$ and $v(\cdot)$ for which $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$, with $1 \leq q < 1^*$, do exist as is seen in below examples 2.5 to 2.7.

2) For $p = 1$ the dual Hardy condition (2.1*) is not involved in Proposition 2.3. This fact appears since (2.15) is strong enough in order that it implies (2.1*). But for $p > 1$ the implication (2.16) $\implies$ (2.3*) is false in general. Indeed taking $x_0 = 0$, $R_0 = \infty$, $u(x) = (1 + |x|)^{-(\beta+n)}$ and $v(y) = |y|^{\alpha p - n}$ with $\beta > 0$ then the Muckenhoupt condition (2.16) is satisfied since $\int_{|x| < R} u(x) dx \leq c$, for some fixed constant $c > 0$, and $\int_{|y| < R} v^{1-p'}(y) dy \approx R^{p'(n-\alpha)}$. And the dual Hardy condition (2.3*) cannot be held because $\int_{2R < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) dy = \infty$.

3) Observe that (2.3), (2.3*) and (2.16) together can be summarized by

$$(2.17) \quad \left(\int_{B(x_0, R_0)} (|x - x_0| + R)^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R)} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

and

$$(2.17^*) \quad \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R_0)} (|y - x_0| + R)^{(\alpha-n)p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all nonnegative $R < R_0$. One part of the result by E. Sawyer and R. Wheeden [Sa-Wh] quoted in the introduction (see at the level of (1.2)) asserts that for $p < q$ a boundedness characterization is given by analogue conditions of (2.17), (2.17*) and where $B(x_0, R_0)$, $B(x_0, R)$, x_0 and R are replaced respectively by $B(x_0, 8R_0)$, $B(x_1, R_1)$, x_1 and R_1 with $B(x_1, R_1) \subset B(x_0, 8R_0)$. These authors left open the question of a similar characterization for $q = p$ and they gave in replacement a sufficient condition given by the expression in (1.2).

4) In [Ra2] we have stated that, for $1 < p = q$, $x_0 = 0$ and $R_0 = \infty$, both (2.17) and (2.17*) characterize $I_\alpha : L^p(v) \rightarrow L^p(u)$ whenever $v^{1-p'}(\cdot) \in \mathcal{H}(\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{H}(\mathbb{R}^n)$. At present, Proposition 2.4 asserts that the characterization holds under $v^{1-p'}(\cdot) \in RI_{\alpha,p,p}(\mathbb{R}^n)$ and $u(\cdot) \in RI_{\alpha,p,p}(\mathbb{R}^n)$. The new point here is that $w(\cdot) \in \mathcal{H}(\mathbb{R}^n) \implies w(\cdot) \in RI_{\alpha,p,p}(\mathbb{R}^n)$ and, as shown in example 2.12 below, weights $u(\cdot)$ and $v(\cdot)$ can be found such that $v^{1-p'}(\cdot) \notin \mathcal{H}(\mathbb{R}^n)$ or $u(\cdot) \notin \mathcal{H}(\mathbb{R}^n)$ but for which $I_\alpha : L^p(v) \rightarrow L^p(u)$.

5) A characterization result as Proposition 2.4 is theoretically important since it tells that the sufficient conditions under consideration are the exact ones required to obtain the boundedness. However for practical computations, when the weights are complicated, it should be better to check directly (2.3), (2.3*) and (2.7) rather than to verify (2.3), (2.3*), (2.16) and that both $v^{1-p'}(\cdot)$ and $u(\cdot)$ satisfy $RI_{\alpha,p,q}(B(x_0, R_0))$. The difficult with these last assumptions is that a knowledge of lower bounds of $\int_{B(x_0, R_0) \cap B(x_0, c_2 R)} u(x) dx$ and $\int_{B(x_0, R_0) \cap B(x_0, c_2 R)} v^{1-p'}(y) dy$ are implicitly involved in computations.

To illustrate the computability of conditions in our results, we will state eight examples.

EXAMPLE 2.5. Let $1 \leq q < \frac{n}{n-\alpha}$ and γ chosen such that $\alpha < \gamma < n$. Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = |x|^{(\gamma-\alpha)q-n} \quad \text{and} \quad v(y) = |y|^{\gamma-n}.$$

Then $I_\alpha : L^1(v) \rightarrow L^q(u)$ holds.

This first example can be considered as describing the good power weights convenient for the boundedness when $R_0 = \infty$. The case $R_0 < \infty$ is contained in the following result.

EXAMPLE 2.6. Let $R_0 < \infty$ and $1 \leq q < \frac{n}{n-\alpha}$. Choose γ and $\beta > 0$ such that $\alpha < \gamma < n$ and $(\gamma - \alpha)q \leq \beta < (n - \alpha)q$. Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = |x - x_0|^{\beta-n} \quad \text{and} \quad v(y) = |y - x_0|^{\gamma-n}.$$

Then $I_{\alpha, B(x_0, R_0)} : L^1(v) \rightarrow L^q(u)$ holds with the bound $C = cR_0^{\frac{\beta}{q} + \alpha - \gamma}$ where the nonnegative constant c depends only on α , n , q and the parameters β and γ .

This second example leads to see, by taking $x_0 = 0$, that there are more suitable weights when $R_0 < \infty$. However the bound C (see (1.1)) depends this time in general on the radius R_0 .

In weighted Sobolev spaces theory, it is custom to use weights $w(x)$ which are powers of the distance of x from the boundary of $B(x_0, R_0)$. Theorem 2.1 above can be also used to treat such a situation, as is illustrated by the following.

EXAMPLE 2.7. Let $R_0 < \infty$ and $1 \leq q < \frac{n}{n-\alpha}$. Choose γ and $\beta > 0$ such that $\gamma \leq n \leq \beta$. Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = [\text{dist}(x, \partial B(x_0, R_0))]^{\beta-n} = (R_0 - |x - x_0|)^{\beta-n}$$

and

$$v(y) = [\text{dist}(y, \partial B(x_0, R_0))]^{\gamma-n} = (R_0 - |y - x_0|)^{\gamma-n}.$$

Then $I_{\alpha, B(x_0, R_0)} : L^1(v) \rightarrow L^q(u)$ holds with the constant $C = cR_0^{\frac{\beta}{q} + \alpha - \gamma}$ where the nonnegative constant c depends only on α, n, q and the parameters β and γ .

Now we will focus on the case of $p > 1$ by starting to recall a more and less known result for power weights.

EXAMPLE 2.8. Let $p > 1$ and $\frac{n}{\alpha} \leq p \leq q$ or $p < \frac{n}{\alpha}$ with $p \leq q \leq p^*$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$. Choose γ such that $\alpha p < \gamma < np$. Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = |x|^{\left(\frac{\gamma}{p} - \alpha\right)q - n} \quad \text{and} \quad v(y) = |y|^{\gamma-n}.$$

Then $I_\alpha : L^p(v) \rightarrow L^q(u)$.

This boundedness does not hold whenever the weight $u(\cdot)$ is replaced by $u(x) = |x|^{\beta-n}$ with $\left(\frac{\gamma}{p} - \alpha\right)q < \beta < (n - \alpha)q$. However more weights $u(\cdot)$ are permitted when $R_0 < \infty$.

EXAMPLE 2.9. Suppose that $R_0 < \infty$ and $p > 1$. Let $\frac{n}{\alpha} \leq p \leq q$ or $p < \frac{n}{\alpha}$ with $p \leq q \leq p^*$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$. Choose γ and β satisfying one of the following assumptions:

$$(i) \quad \alpha p < \gamma < np \quad \text{and} \quad \left(\frac{\gamma}{p} - \alpha\right)q \leq \beta < (n - \alpha)q,$$

$$(ii) \quad \alpha p < \gamma < np \quad \text{and} \quad (n - \alpha)q < \beta,$$

$$(iii) \quad 0 < \gamma < \alpha p \quad \text{and} \quad 0 < \beta < (n - \alpha)q,$$

$$(iv) \quad 0 < \gamma < \alpha p \quad \text{and} \quad (n - \alpha)q < \beta.$$

Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = |x - x_0|^{\beta-n} \quad \text{and} \quad v(y) = |y - x_0|^{\gamma-n}.$$

Then $I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u)$ holds with the constant $C = cR_0^{\frac{p}{q} + \alpha - \gamma}$ where the nonnegative constant c depends only on α, n, p and q .

Example 2.8 is just about power weights. To go beyond this simple and well-known case let us introduce the assumption $\varphi(\cdot) \in \Delta_\sigma$, $\sigma \geq 0$, by which we mean that $\varphi(\cdot)$ is a nonnegative and increasing function on $]0, \infty[$ such that for some constant $c > 0$

$$\varphi(\lambda s) \leq c\lambda^\sigma \varphi(s) \quad \text{for all } s > 0 \text{ and } \lambda > 1.$$

For instance if $\varphi(s) = s^a \ln^b(e+s)$, with $a, b \geq 0$, then $\varphi(\cdot) \in \Delta_\sigma$ where $\sigma = a+b$.

EXAMPLE 2.10. Let $1 < p < \frac{n}{\alpha}$. Choose $\varphi(\cdot) \in \Delta_\sigma$ such that $0 \leq \sigma < (n - \alpha p)\frac{p'}{p}$. Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = |x|^{-\alpha p} \varphi^{-\frac{p}{p'}}(|x|) \quad \text{and} \quad v(y) = \varphi^{1-p}(|y|).$$

Then $I_\alpha : L^p(v) \rightarrow L^p(u)$ holds.

Weights $u(\cdot)$ and $v(\cdot)$ for which $I_\alpha : L^p(v) \rightarrow L^q(u)$ holds, in the same spirit of this last example can be also given. However we have restricted our study to $p = q$ for shortness reason and since (2.11) is renowned to be hard to obtain for the range $p = q$.

We will end with two examples illustrating the strength of criteria introduced in this paper in comparison with past results [Sa-Wh], [Ra2] about weighted inequalities for the fractional integral operator I_α .

EXAMPLE 2.11. Let $p > 1$. Choose $\delta, \beta > 0$ and γ such that

$$\alpha p < \gamma < np, \quad \beta < (n - \alpha)p < \delta \quad \text{and} \quad \alpha p + \beta \leq \gamma.$$

Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = \begin{cases} |x|^{\delta-n} & \text{if } |x| < 2^{-1} \\ |x|^{\beta-n} & \text{if } |x| > 2^{-1}. \end{cases}$$

and

$$v(y) = \begin{cases} |y|^{n(p-1)} \ln^p(|y|^{-1}) & \text{if } |y| < 2^{-1} \\ |y|^{\gamma-n} & \text{if } |y| > 2^{-1}. \end{cases}$$

Then $I_\alpha : L^p(v) \rightarrow L^p(u)$ holds.

First it can be noted that weights $u(\cdot)$ and $v(\cdot)$ as described in this example do exist. For instance it is sufficient to take $n = 2, p = 3, \alpha = 1, \beta = 1, \gamma = 5$ and

$\delta = 4$. Next what is remarkable in example 2.13 is that $\int_{|y| < R} v^{(1-p')^r}(y) dy = \infty$ for all $r > 1$ and $R < 2^{-1}$. Therefore the boundedness $I_\alpha : L^p(v) \rightarrow L^p(u)$ cannot be decided from a criterion like in (1.2). Example 2.11 is not explicated in our previous paper [Ra2] however the spirit is found in [Ra1].

EXAMPLE 2.12. Let $p > 1$. Choose β_1, β_2 and γ such that

$$0 < \beta_1 < n_1, \quad 0 < \beta_2 < n_2, \quad n_1 \text{ and } n_2 \text{ are integers with } n = n_1 + n_2$$

$$\alpha p < \gamma < np, \quad n - \alpha < \beta_1 + \beta_2 = \gamma - \alpha p < (n - \alpha)p.$$

Define the weights $u(\cdot)$ and $v(\cdot)$ by

$$u(x) = |x_1|^{\beta_1 - n_1} |x_2|^{\beta_2 - n_2} \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^n \text{ with } x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$$

and

$$v(y) = |y|^{\gamma - n}.$$

Then $I_\alpha : L^p(v) \rightarrow L^p(u)$ holds.

Weights $u(\cdot)$ and $v(\cdot)$, as described in this example, do exist since it is sufficient to take $n = 2, n_1 = n_2 = 1, p = 3, \alpha = 1, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{4}$ and $\gamma = 3 + \frac{5}{4}$. The relevant point in example 2.14 is that $\sup_{2^{-1}|x| < |z| < 2|x|} u(z) = \infty$. And the boundedness $I_\alpha : L^p(v) \rightarrow L^p(u)$ cannot be decided from a criterion using the pointwise inequality (2.9) or (2.10'). Therefore this last example yields the importance of introduction of integral conditions like (2.12) and (2.7) instead of pointwise conditions as (2.9) and (2.10').

3. Basic Results

We first state the main key for all of our results.

PROPOSITION 3.1. *If $I_{\alpha, B(0, R_0)} : L^p(v) \rightarrow L^q(u)$ holds with the bound C then the following three inequalities are satisfied:*

$$(3.1) \quad \left(\int_{x \in B(0, R_0)} \left[\int_{|y| < 2^{-1}|x|} g(y) dy \right]^q |x|^{(\alpha - n)q} u(x) dx \right)^{\frac{1}{q}} \\ \leq \mathcal{A} \left(\int_{B(0, 2^{-1}R_0)} g^p(y) v(y) dy \right)^{\frac{1}{p}} \\ \text{for all } g(\cdot) \geq 0 \text{ with } \text{supp } g \subset B(0, 2^{-1}R_0)$$

$$\begin{aligned}
(3.1^*) \quad & \left(\int_{x \in B(0, 2^{-1}R_0)} \left[\int_{2|x| < |y| < R_0} |y|^{\alpha-n} f(y) dy \right]^q u(x) dx \right)^{\frac{1}{q}} \\
& \leq \mathcal{A} \left(\int_{B(0, R_0)} f^p(y) v(y) dy \right)^{\frac{1}{p}} \\
& \text{for all } f(\cdot) \geq 0 \text{ with } \text{supp } f \subset B(0, R_0)
\end{aligned}$$

and

$$\begin{aligned}
(3.2) \quad & \left(\int_{B(0, R_0)} \left(\tilde{I}_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \right)^{\frac{1}{q}} \leq \mathcal{A} \left(\int_{B(0, R_0)} f^p(y) v(y) dy \right)^{\frac{1}{p}} \\
& \text{for all } f(\cdot) \geq 0 \text{ with } \text{supp } f \subset B(0, R_0)
\end{aligned}$$

where

$$\left(\tilde{I}_{\alpha, B(0, R_0)} f \right)(x) = \int_{y \in B(0, R_0) \cap C(0, |x|)} |x - y|^{\alpha-n} f(y) dy$$

and $\mathcal{A} = 2^{n-\alpha} C$.

Conversely if inequalities (3.1), (3.1*) and (3.2) are both satisfied with the bound $\mathcal{A}$ then $I_{\alpha, B(0, R_0)} : L^p(v) \rightarrow L^q(u)$ holds with the bound $C = [3 \times 2^{q-1}(1 + 2^{q-1})(1 + 2^{(n-\alpha)q})]^{\frac{1}{q}} \mathcal{A}$.

This Proposition reflects the principle of “three parts proof” we will use to tackle (1.1). In an other word it appears here that, $I_{\alpha, B(0, R_0)} : L^p(v) \rightarrow L^q(u)$ is equivalent both to Hardy inequalities (3.1) and (3.1*) and boundedness (3.2) for the restricted operator $\tilde{I}_{\alpha, B(0, R_0)}$. The notation (3.1*) is used to indicate that it is a dual form of inequality (3.1). Therefore the problem remains to give sufficient conditions (and also necessary, if possible) for each of (3.1), (3.1*) and (3.2).

Characterizations of the Hardy inequalities are available as stated in the following.

PROPOSITION 3.2. *For $p = 1$, if the (resp. dual) Hardy inequality (3.1) (resp. (3.1*)) holds with the bound $\mathcal{A} > 0$ then*

$$\begin{aligned}
(3.3) \quad & \left(\int_{2R < |x| < R_0} |x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left[\text{ess sup}_{y \in B(0, R)} \frac{1}{v(y)} \right] \leq A \\
& \text{for all } R > 0 \text{ with } 2R < R_0.
\end{aligned}$$

(resp.

$$(3.3^*) \quad \left(\int_{B(0,R)} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{2R < |y| < R_0} |y|^{(\alpha-n)} \frac{1}{v(y)} \right] \leq A$$

for all $R > 0$ with $2R < R_0$)

where $A = \mathcal{A}$. Conversely for $1 = p \leq q$, condition (3.3) (resp. (3.3*)) implies inequality (3.1) (resp. (3.1*)) with the bound $\mathcal{A} = 4A$.

Similarly for $p > 1$, if the (resp. dual) Hardy inequality (3.1) (resp. (3.1*)) holds with the bound $\mathcal{A} > 0$ then

$$(3.4) \quad \left(\int_{2R < |x| < R_0} |x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(0,R)} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all $R > 0$ with $2R < R_0$

(resp.

$$(3.4^*) \quad \left(\int_{B(0,R)} u(x) dx \right)^{\frac{1}{q}} \left(\int_{2R < |y| < R_0} |y|^{(\alpha-n)p'} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all $R > 0$ with $2R < R_0$)

where $A = \mathcal{A}$. Conversely for $1 < p \leq q$, then condition (3.4) (resp. (3.4*)) implies inequality (3.1) (resp. (3.1*)) with the bound $\mathcal{A} = 4A$.

Unable to derive a characterization result for the boundedness (3.2), we will be content to state some sufficient conditions close (in certain sense) to natural necessary conditions.

PROPOSITION 3.3. For $p = 1$, if the restricted boundedness (3.2) holds with the bound $\mathcal{A}$ then

$$(3.5) \quad |z|^{\alpha-n} \left(\int_{2^{-1}|z| < |x| < |z|} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{2^{-1}|z| < |y| < |z|} \frac{1}{v(y)} \right] \leq A$$

for all z with $|z| < R_0$

where $A = 2^{n-\alpha} \mathcal{A}$. Similarly for $p > 1$, a necessary condition for (3.2) is

$$(3.6) \quad |z|^{\alpha-n} \left(\int_{2^{-1}|z| < |x| < |z|} u(x) dx \right)^{\frac{1}{q}} \left(\int_{2^{-1}|z| < |y| < |z|} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq A$$

for all z with $|z| < R_0$.

Conversely for $1 = p \leq q$ boundedness (3.2) holds, with the bound $A = A$, whenever

$$(3.7) \quad \left(\int_{x \in B(0, R_0) \cap C(0, |z|)} |z - x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(0, R_0) \cap C(0, |z|)} \frac{1}{v(y)} \right] \leq A$$

for all z with $|z| < R_0$.

Similarly for $1 < p \leq q$ and $\frac{1}{t} = \frac{1}{q} + \frac{1}{p'}$, then a sufficient condition for (3.2) is

$$(3.8) \quad \int_{x \in B(0, R_0) \cap C(0, |z|)} |z - x|^{(\alpha-n)t} \times \left[\int_{y \in B(0, R_0) \cap C(0, |x|)} |x - y|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx \leq A^q$$

for all z with $|z| < R_0$.

Condition (3.7) (resp. (3.8)) is stronger than (3.5) (resp. (3.6)) since (3.7) $\implies$ (3.5) and (3.8) $\implies$ (3.6).

For practical computations, (3.7) and (3.8) seem more difficult to check in comparison with Muckenhoupt conditions (3.5) and (3.6). Therefore for certain situations, it would be preferable to use conditions stronger than (3.7) and (3.8) but easily verifiable. For this purpose let us introduce

$$(3.9) \quad |z|^{\alpha + \frac{n}{q} - n} \left(\sup_{x \in B(0, R_0) \cap C(0, |z|)} u(x) \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(0, R_0) \cap C(0, |z|)} \frac{1}{v(y)} \right] \leq A$$

for all z with $|z| < R_0$

(3.10)

$$|z|^{\alpha + \frac{n}{q} - \frac{n}{p}} \left(\sup_{x \in B(0, R_0) \cap C(0, 2|z|)} u(x) \right)^{\frac{1}{q}} \left(\sup_{y \in B(0, R_0) \cap C(0, 2|z|)} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A$$

for all z with $|z| < R_0$.

$$(3.11) \quad |x|^{\alpha + \frac{n}{q} - \frac{n}{p}} \left(u(x) \right)^{\frac{1}{q}} \left(\sup_{y \in B(0, R_0) \cap C(0, 2|x|)} v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A$$

for all x with $|x| < R_0$,

$$(3.11') \quad |y|^{\alpha + \frac{n}{q} - \frac{n}{p}} \left(\sup_{x \in B(0, R_0) \cap C(0, 2|y|)} u(x) \right)^{\frac{1}{q}} \left(v^{1-p'}(y) \right)^{\frac{1}{p'}} \leq A$$

for all y with $|y| < R_0$.

PROPOSITION 3.4. *If $1 \leq q < \frac{n}{n-\alpha}$ then $(3.9) \implies (3.7)$. And the implication $(3.10) \implies (3.8)$ is true under one of the following: i) $\frac{n}{\alpha} \leq p \leq q$; ii) $p < \frac{n}{\alpha}$ and $p \leq q < p^*$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{\alpha}{n}$.*

For $1 < p < \frac{n}{\alpha}$ and $p \leq q \leq p^$, then the boundedness (3.2) holds if one of (3.11) and (3.11') is satisfied.*

4. Proofs of Main Results

The main key of our main results lies on the fact that

$$I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u) \text{ with the bound } C > 0$$

is equivalent to

$$I_{\alpha, B(0, R_0)} : L^p(v_{x_0}) \rightarrow L^q(u_{x_0}) \text{ for the same bound } C$$

where

$$v_{x_0}(\cdot) = v(\cdot + x_0) \quad \text{and} \quad u_{x_0}(\cdot) = u(\cdot + x_0).$$

The point for this second boundedness is that C does not depend on the individual real x_0 as the weights $v_{x_0}(\cdot)$ and $u_{x_0}(\cdot)$ do.

Also observe that, by translation and change of variables, conditions (2.1), (2.1*), (2.2), (2.3), (2.3*), (2.4), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.10') for the pair of weights (u, v) are respectively the same as (3.3), (3.3*), (3.5), (3.4), (3.4*), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.11') for the pair of weights (u_{x_0}, v_{x_0}) and whose the constants do not depend on x_0 and are the same as in the former conditions.

PROOFS OF THEOREMS 2.1 AND 2.2. We just have to justify Theorem 2.2 since the proof of Theorem 2.1 is quite close.

Part A. Suppose that $I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u)$ with the bound C . Then $I_{\alpha, B(0, R_0)} : L^p(v_{x_0}) \rightarrow L^q(u_{x_0})$ with the same bound C . By Propositions 3.1 to 3.3, conditions (3.4), (3.4*) and (3.6) (related to the pair of weights (u_{x_0}, v_{x_0})) are satisfied with the constant $A = 2^{n-\alpha}C$. It means, following the above observation, that conditions (2.3), (2.3*) and (2.4) (related to the pair of weights (u, v)) are also satisfied with the same constant A .

Conversely to get $I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u)$ it is sufficient to derive $I_{\alpha, B(0, R_0)} : L^p(v_{x_0}) \rightarrow L^q(u_{x_0})$ with some bound $C > 0$ which does not depend on the individual real x_0 . The problem remains, by Proposition 3.1, to check the Hardy inequalities (3.1), (3.1*) and the restricted boundedness (3.2) related to the pair of weights (u_{x_0}, v_{x_0}) .

Inequality (3.1) (with the bound 4A) appears because of Proposition 3.2 and condition (3.4) (for the pair (u_{x_0}, v_{x_0})) which is the same as the Hardy condition (2.3). Similarly inequality (3.1*) (with the bound 4A) follows from the dual Hardy condition (2.3*). The restricted boundedness (3.2) (related to (u_{x_0}, v_{x_0})) and with the bound A) arises from the corresponding condition (3.8) as is seen in Proposition 3.3. And (3.8) (related to (u_{x_0}, v_{x_0})) is nothing else than condition (2.7). Therefore the boundedness $I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u)$ holds with the bound $C = [3 \times 2^{q-1}(1 + 2^{q-1})(1 + 2^{(n-\alpha)q})]^{\frac{1}{q}}(4A)$.

Part B. Under $\frac{n}{\alpha} \leq p \leq q$ or $p < \frac{n}{\alpha}$ and $p \leq q < p^*$, the implication (2.9) $\implies$ (2.7) holds since it is the same as (3.10) $\implies$ (3.8) (related to the pair of weights (u_{x_0}, v_{x_0})). And this last implication is true because of Proposition 3.4.

Part C. The fact that both conditions (2.3), (2.3*) and (2.10) [or (2.10')] lead to the boundedness $I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u)$ is equivalent to say that $I_{\alpha, B(0, R_0)} : L^p(v_{x_0}) \rightarrow L^q(u_{x_0})$ is implied by (3.4), (3.4*) and (3.11) [or (3.11')] (related to the pair of weights (u_{x_0}, v_{x_0})). This last implication is true because of the second part of Propositions 3.4 and Propositions 3.1 to 3.3.

PROOF OF PROPOSITION 2.3.

Necessary Part. Suppose that $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$. Then the Hardy condition (2.1) appears by Theorem 2.1. The fact that (2.15) is a necessary condition can be seen as (2.2).

Sufficient Part. Conversely to derive $I_{\alpha, B_0} : L^1(v) \rightarrow L^q(u)$, by Theorem 2.1, the task remains to check conditions (2.1*) and (2.6) since (2.1) is already assumed by hypothesis.

Observe that the dual Hardy condition (2.1*) is implied by the Muckenhoupt condition (2.15). Indeed for $0 < 2R < R_0$ then

$$\begin{aligned} & |y - x_0|^{\alpha-n} \frac{1}{v(y)} \mathbb{I}_{\{2R < |y-x_0| < R_0\}} \\ &= \sum_{k=1}^{\infty} |y - x_0|^{\alpha-n} \frac{1}{v(y)} \mathbb{I}_{\{2^k R < |y-x_0| < 2^{k+1} R\} \cap B(x_0, R_0)}(y) \\ &\leq c_3 \sum_{k=1}^{\infty} (2^k R)^{\alpha-n} \frac{1}{v(y)} \mathbb{I}_{\{2^k R < |y-x_0| < 2^{k+1} R\} \cap B(x_0, R_0)}(y) \\ &\leq c_4 R^{\alpha-n} \left(\operatorname{ess\,sup}_{z \in B(x_0, R_0)} \frac{1}{v(z)} \right) \sum_{k=1}^{\infty} 2^{k(\alpha-n)} \leq c_5 R^{\alpha-n} \left(\operatorname{ess\,sup}_{z \in B(x_0, R_0)} \frac{1}{v(z)} \right). \end{aligned}$$

Next let us prove that (2.6) is implied by the Muckenhoupt condition (2.15) under the additionnal assumption on $u(\cdot)$. Indeed if $u(\cdot) \in RI_{\alpha, 1, q}(B(x_0, R_0))$

then

$$\begin{aligned}
& \left(\int_{B(x_0, R_0) \cap C(x_0, |z-x_0|)} |z-x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} \frac{1}{v(y)} \right] \\
& \leq c_5 |z-x_0|^{(\alpha-n)} \left(\int_{B(x_0, R_0) \cap C(x_0, c_2|z-x_0|)} u(x) dx \right)^{\frac{1}{q}} \\
& \quad \left[\operatorname{ess\,sup}_{y \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} \frac{1}{v(y)} \right] \\
& \leq c_6 R^{(\alpha-n)} \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left[\operatorname{ess\,sup}_{y \in B(x_0, R)} \frac{1}{v(y)} \right] \leq c_6 A.
\end{aligned}$$

In this last inequality the real R is such that $R = R_0$ if $R_0 \leq 2 \max(1, c_2)|z-x_0|$ else $R = 2 \max(1, c_2)|z-x_0|$. Similarly condition (2.6) is also satisfied whenever $q < \frac{n}{n-\alpha}$ and $u(\cdot) \in \mathcal{H}(B(x_0, R_0))$. Indeed these last assumptions imply $u(\cdot) \in RI_{\alpha, 1, q}(B(x_0, R_0))$ because

$$\begin{aligned}
& \int_{B(x_0, R_0) \cap C(x_0, |z-x_0|)} |z-x|^{(\alpha-n)q} u(x) dx \\
& \leq \left(\sup_{x \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} u(x) \right) \int_{|z-\omega| < 4|z-x_0|} |z-\omega|^{(\alpha-n)q} d\omega \\
& \leq c_7 |z-x_0|^{(\alpha-n)q+n} \left(\sup_{x \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} u(x) \right) \\
& \quad \text{note that } (\alpha-n)q+n > 0 \\
& \leq c_8 |z-x_0|^{(\alpha-n)q} \int_{B(x_0, R_0) \cap C(x_0, c_2|z-x_0|)} u(x) dx.
\end{aligned}$$

PROOF OF PROPOSITION 2.4.

Necessary Part. Suppose that $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$. Then the Hardy conditions (2.3) and (2.3*) are satisfied in view of Theorem 2.2. The fact that (2.16) is a necessary condition can be seen as (2.4).

Sufficient Part. Conversely to derive $I_{\alpha, B_0} : L^p(v) \rightarrow L^q(u)$, by Theorem 2.2, the task remains to check the reinforced Muckenhoupt condition (2.7) or the pointwise inequality (2.10) (if $p < \frac{n}{\alpha}$ and $p \leq q \leq p^*$) since (2.3) and (2.3*) are already assumed by hypothesis.

Condition (2.7) follows readily from $v^{1-p'}(\cdot) \in RI_{\alpha, p, q}(B(x_0, R_0))$, $u(\cdot) \in RI_{\alpha, p, q}(B(x_0, R_0))$ and the Muckenhoupt condition (2.16) since

$$\begin{aligned}
& \int_{x \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} |z-x|^{(\alpha-n)t} \\
& \quad \left[\int_{y \in B(x_0, R_0) \cap C(x_0, |z-x_0|)} |x-y|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx \\
& \leq c_3 \left(|z-x_0|^{(\alpha-n)t} \int_{B(x_0, R_0) \cap B(x_0, 2c_2|z-x_0|)} v^{1-p'}(y) dy \right)^{\frac{q}{p'}} \\
& \quad \times \left(|z-x_0|^{(\alpha-n)t} \int_{x \in B(x_0, R_0) \cap B(x_0, c_2|z-x_0|)} u(x) dx \right) \\
& \leq c_4 R^{(\alpha-n)q} \left(\int_{B(x_0, R_0)} u(x) dx \right) \left(\int_{B(x_0, R)} v^{1-p'}(y) dy \right)^{\frac{q}{p'}} \leq c_4 A^q.
\end{aligned}$$

In this last inequality the real R is such that $R = R_0$ if $R_0 \leq 2c_2|z-x_0|$ else $R = 2c_2|z-x_0|$.

For $p < \frac{n}{\alpha}$ and $p \leq q < p^*$ the point is that $v^{1-p'}(\cdot) \in \mathcal{H}(B(x_0, R_0)) \implies v^{1-p'}(\cdot) \in RI_{\alpha, p, q}(B(x_0, R_0))$ and $u(\cdot) \in \mathcal{H}(B(x_0, R_0)) \implies u(\cdot) \in RI_{\alpha, p, q}(B(x_0, R_0))$. Arguments to get these implications are essentially the same as those used for $u(\cdot) \in \mathcal{H}(B(x_0, R_0)) \implies u(\cdot) \in RI_{\alpha, 1, q}(B(x_0, R_0))$ above.

For $p < \frac{n}{\alpha}$ and $p \leq q = p^*$ (e.g. $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$) the key is now to prove that the pointwise condition (2.10) is implied by the Muckenhoupt condition (2.16) under the assumptions $v^{1-p'}(\cdot) \in \mathcal{H}(B(x_0, R_0))$ and $u(\cdot) \in \mathcal{H}(B(x_0, R_0))$. This is indeed the case because for $|z-x_0| < R_0$:

$$\begin{aligned}
& |z-x_0|^{\alpha+\frac{n}{q}-\frac{n}{p}} \left(u(z) \right)^{\frac{1}{q}} \left(\sup_{y \in B(x_0, R_0) \cap C(x_0, 2|z-x_0|)} v^{1-p'}(y) \right)^{\frac{1}{p'}} \\
& \leq c_5 |z-x_0|^{-n(\frac{1}{q}+\frac{1}{p'})} \\
& \quad \left(\int_{B(x_0, R_0) \cap C(x_0, c_2|z-x_0|)} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R_0) \cap C(x_0, c_2|z-x_0|)} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \\
& \leq c_6 R^{\alpha-n} \left(\int_{B(x_0, R)} u(x) dx \right)^{\frac{1}{q}} \left(\int_{B(x_0, R)} v^{1-p'}(y) dy \right)^{\frac{1}{p'}} \leq c_6 A
\end{aligned}$$

where $R = R_0$ if $R_0 \leq c_2|z-x_0|$ else $R = c_2|z-x_0|$.

PROOF OF EXAMPLE 2.7. To derive $I_\alpha : L^1(v) \rightarrow L^q(u)$, by Proposition 2.3, the task remains to check conditions (2.1) and (2.15) (with $x_0 = 0$ and $R_0 = \infty$) since $q < \frac{n}{n-\alpha}$ and $u(\cdot) \in \mathcal{H}$ (because $u(\cdot)$ is a monotone function, see [Ra2]).

Let $R > 0$. Then

$$(4.1) \quad \int_{2R < |x|} |x|^{(\alpha-n)q} u(x) dx = \int_{2R < |x|} |x|^{(\alpha-n)q + (\gamma-\alpha)q-n} dx \approx R^{q[\gamma-n]}$$

since $\gamma - n < 0$, and

$$(4.2) \quad \int_{|x| < R} u(x) dx = \int_{|x| < R} |x|^{(\gamma-\alpha)q-n} dx \approx R^{q[\gamma-\alpha]}$$

because $(\gamma - \alpha) > 0$. On the other hand

$$(4.3) \quad \operatorname{ess\,sup}_{|y| < R} \left(\frac{1}{v(y)} \right) \approx R^{n-\gamma}$$

since $n - \gamma > 0$.

The Hardy condition (2.1) appears from (4.1) and (4.3). And the Muckenhoupt condition (2.15) is a direct consequence of (4.2) and (4.3).

PROOF OF EXAMPLE 2.8. To get $I_{\alpha, B(x_0, R_0)} : L^1(v) \rightarrow L^q(u)$, by Part B of Theorem 2.1, it remains to check conditions (2.1), (2.1*) and (2.8) since $q < \frac{n}{n-\alpha}$.

Let $R > 0$ with $2R < R_0$. Then

$$(4.4) \quad \int_{2R < |x-x_0| < R_0} |x-x_0|^{(\alpha-n)q} u(x) dx \approx \int_{2R}^{R_0} r^{(\alpha-n)q+\beta-1} dr \leq c_1 R^{q[\alpha-n+\frac{\beta}{q}]}$$

since $(\alpha - n)q + \beta < 0$, and

$$(4.5) \quad \int_{|x-x_0| < R} u(x) dx \approx \int_0^R r^{\beta-1} dr \approx R^{q(\frac{\beta}{q})}$$

because $\beta > 0$. On the other hand

$$(4.6) \quad \operatorname{ess\,sup}_{|y-x_0| < R} \left(\frac{1}{v(y)} \right) \approx R^{n-\gamma}$$

since $n - \gamma > 0$ and

$$(4.7) \quad \operatorname{ess\,sup}_{2R < |y-x_0| < R_0} \left(|y-x_0|^{\alpha-n} \frac{1}{v(y)} \right) \approx R^{\alpha-\gamma}$$

because $\alpha - \gamma < 0$.

The Hardy condition (2.1) (resp. dual Hardy condition (2.1*)) appears from (4.4) and (4.6) (resp. (4.5) and (4.7)). Indeed if $H(R)$ (resp. $H^*(R)$) denotes the first member of (2.1) (resp. (2.1*)) then for $2R < R_0$:

$$(4.8) \quad H(R), H^*(R) \leq c_2 R^{\alpha+\frac{\beta}{q}-\gamma} \leq c_2 R_0^{\alpha+\frac{\beta}{q}-\gamma} \quad \text{because } \alpha + \frac{\beta}{q} - \gamma \geq 0.$$

Finally condition (2.8) follows from $\alpha + \frac{\beta}{q} - \gamma \geq 0$ since, if $A(R)$ denotes the first member of (2.8) then

$$(4.9) \quad A(R) \leq c_3 R^{\alpha + \frac{\beta}{q} - \gamma} \leq c_2 R_0^{\alpha + \frac{\beta}{q} - \gamma}.$$

Estimates (4.8) and (4.9) show that the boundedness $I_{\alpha, B(x_0, R_0)} : L^1(v) \rightarrow L^q(u)$ holds with the bound $C = c R_0^{\alpha + \frac{\beta}{q} - \gamma}$ where c depends only on α, n, q and on the parameters β and γ .

PROOF OF EXAMPLE 2.7. To derive $I_{\alpha, B(x_0, R_0)} : L^1(v) \rightarrow L^q(u)$, as above, we have to check conditions (2.1), (2.1*) and (2.8).

Let $R > 0$ with $2R < R_0$. Then

$$(4.10) \quad \begin{aligned} & \int_{2R < |x - x_0| < R_0} |x - x_0|^{(\alpha - n)q} u(x) dx \\ & \approx \int_{2R}^{R_0} r^{(\alpha - n)q + n - 1} (R_0 - r)^{\beta - n} dr \leq c_1 R_0^{q[\frac{\beta}{q} + (\alpha - n)]} \end{aligned}$$

since $(\alpha - n)q + n > 0$ and $\beta - n \geq 0$. Moreover

$$(4.11) \quad \int_{|x - x_0| < R} u(x) dx \approx \int_0^R r^{n-1} (R_0 - r)^{\beta - n} dr \leq c_2 R_0^{\beta - n} R^n$$

because $\beta - n \geq 0$. On the other hand

$$(4.12) \quad \operatorname{ess\,sup}_{|y - x_0| < R} \left(\frac{1}{v(y)} \right) = \operatorname{ess\,sup}_{|y| < R} \left((R_0 - |y|)^{n - \gamma} \right) \leq R_0^{n - \gamma}$$

since $n - \gamma \geq 0$, and consequently

$$(4.13) \quad \begin{aligned} & \operatorname{ess\,sup}_{2R < |y - x_0| < R_0} \left(|y - x_0|^{\alpha - n} \frac{1}{v(y)} \right) \\ & = \operatorname{ess\,sup}_{2R < |y| < R_0} \left(|y|^{\alpha - n} (R_0 - |y|)^{n - \gamma} \right) \leq R^{\alpha - n} R_0^{n - \gamma}. \end{aligned}$$

The Hardy condition (2.1) (resp. dual Hardy condition (2.1*)) appears from (4.10) and (4.12) (resp. (4.11) and (4.13)). Indeed if $H(R)$ (resp. $H^*(R)$) denotes the first member of (2.1) (resp. (2.1*)) then for $2R < R_0$: $H(R) \leq c_3 R_0^{\alpha + \frac{\beta}{q} - \gamma}$. And similarly $H^*(R) \leq c_3 R_0^{\alpha + \frac{\beta}{q} - \gamma}$ since $\alpha + \frac{n}{q} - n > 0$. Condition (2.8) also results from this last assumption since, if $A(R)$ denotes the first member of (2.8) then $A(R) \leq R_0^{\alpha + \frac{\beta}{q} - \gamma}$.

PROOF OF EXAMPLE 2.8. To get $I_\alpha : L^p(v) \rightarrow L^q(u)$, by Proposition 2.4, the task is reduced to check conditions (2.3), (2.3*) and (2.16) since $u(\cdot) \in \mathcal{H}$ and $v^{1-p'}(\cdot) \in \mathcal{H}$. The Hardy condition (2.3) [resp. dual Hardy condition (2.3*)] holds because $\alpha p < \gamma$ and so $(\alpha - n)q + q(\frac{\gamma}{p} - \alpha) < 0$ [resp. $\alpha - \frac{\gamma}{p} < 0$ and so $q(\frac{\gamma}{p} - \alpha) > 0$]. The details of computations are not given since their are quite similar to those for Example 2.5.

PROOF OF EXAMPLE 2.9. To derive $I_{\alpha, B(x_0, R_0)} : L^p(v) \rightarrow L^q(u)$, by Theorem 2.2, the task remains to check conditions (2.3), (2.3*) and (2.9) (if $\frac{n}{\alpha} \leq p \leq q$ or $p < \frac{n}{\alpha}$ and $p \leq q < p^*$) or (2.10) (if $p < \frac{n}{\alpha}$ and $p \leq q = p^*$).

Let $R > 0$ with $2R < R_0$. Observe that

$$(4.14) \quad \int_{2R < |x-x_0| < R_0} |x-x_0|^{(\alpha-n)q} u(x) dx \approx \int_{2R}^{R_0} r^{(\alpha-n)q+\beta-1} dr \\ \leq c_1 \begin{cases} R^{q[(\alpha-n)+\frac{\beta}{q}]} & \text{if } (\alpha-n)q + \beta < 0 \\ R_0^{q[(\alpha-n)+\frac{\beta}{q}]} & \text{if } (\alpha-n)q + \beta > 0 \end{cases}$$

and

$$(4.15) \quad \int_{|x-x_0| < R} u(x) dx \approx \int_0^R r^{\beta-1} dr \approx R^{q(\frac{\beta}{q})}$$

since $\beta > 0$. On the other hand

$$(4.16) \quad \int_{2R < |y-x_0| < R_0} |y-x_0|^{(\alpha-n)p'} v^{1-p'}(y) dy \approx \int_{2R}^{R_0} r^{p'(\alpha-\frac{\gamma}{p})-1} dr \\ \leq c_1 \begin{cases} R^{p'(\alpha-\frac{\gamma}{p})} & \text{if } \alpha - \frac{\gamma}{p} < 0 \\ R_0^{p'(\alpha-\frac{\gamma}{p})} & \text{if } \alpha - \frac{\gamma}{p} > 0 \end{cases}$$

and

$$(4.17) \quad \int_{|y-x_0| < R} v^{1-p'}(y) dy \approx \int_0^R r^{p'(n-\frac{\gamma}{p})-1} dr \approx R^{p'(n-\frac{\gamma}{p})}$$

since $n - \frac{\gamma}{p} > 0$.

The Hardy condition (2.3) (resp. dual Hardy condition (2.3*)) appears from (4.14) and (4.17) (resp. (4.15) and (4.16)). To justify such a claim observe that

$\alpha + \frac{\beta}{q} - \frac{\gamma}{p} \geq 0$ and $n - \frac{\gamma}{p} > 0$ under one of the assumptions i), ii), iii) and iv). And denote by $H(R)$ (resp. $H^*(R)$) the first member of (2.3) (resp. (2.3*)). Then for some fixed constant $c_2 > 0$ and for $2R < R_0$: $H(R) \leq c_2 R_0^{\alpha + \frac{\beta}{q} - \gamma}$. This last inequality holds in cases of i) and iii) since $\alpha + \frac{\beta}{q} - \frac{\gamma}{p} \geq 0$. For ii) and iv) just the fact that $n - \frac{\gamma}{p} > 0$ is used. Similarly for $2R < R_0$ then $H^*(R) \leq c_2 R_0^{\alpha + \frac{\beta}{q} - \gamma}$. This inequality is satisfied in cases of i) and ii) since $\alpha + \frac{\beta}{q} - \frac{\gamma}{p} \geq 0$. For iii) and iv) the conclusion follows from $\beta > 0$. Finally condition (2.9) (if $\frac{n}{\alpha} \leq p \leq q$ or $p < \frac{n}{\alpha}$ and $p \leq q < p^*$) or (2.10) (if $p < \frac{n}{\alpha}$ and $p \leq q = p^*$ is satisfied. Indeed if $A(R)$ denotes the first member of (2.9) or (2.10) [and this case $R = |x - x_0| (< R_0)$] then $A(R) \leq R_0^{\alpha + \frac{\beta}{q} - \gamma}$. Such a conclusion is a consequence of $\alpha + \frac{\beta}{q} - \frac{\gamma}{p} \geq 0$.

PROOF OF EXAMPLE 2.10. To get $I_\alpha : L^p(v) \rightarrow L^p(u)$, by Proposition 2.4, the task is reduced to check conditions (2.3), (2.3*) and (2.16) because $u(\cdot)$ and $v^{1-p'}(\cdot)$ are radial and monotone weights and consequently $u(\cdot) \in \mathcal{H}$, $v^{1-p'}(\cdot) \in \mathcal{H}$.

Let $R > 0$. Observe that

$$(4.18) \quad \int_{2R < |x|} |x|^{(\alpha-n)p} u(x) dx \approx \int_{2R}^{\infty} r^{(n-np)-1} \varphi^{-\frac{p}{p'}}(r) dr \leq c_1 \left[\varphi^{-\frac{1}{p'}}(R) R^{-\frac{n}{p'}} \right]^p$$

since $\varphi(\cdot)$ is an increasing function. And for the similar reason then

$$(4.19) \quad \int_{|y| < R} v^{1-p'}(y) dy \approx \int_0^R r^{n-1} \varphi(r) dr \leq c_2 \left[\varphi^{\frac{1}{p'}}(R) R^{\frac{n}{p'}} \right]^{p'}.$$

On the other hand since $\varphi(\cdot) \in \Delta_\sigma$ with $(n - \alpha p) - \sigma \frac{p}{p'} > 0$ then

$$(4.20) \quad \int_{|x| < R} u(x) dx \approx \int_0^R r^{(n-\alpha p)-1} \varphi^{-\frac{p}{p'}}(r) dr \leq c_3 \left[\varphi^{-\frac{1}{p'}}(R) R^{\frac{n}{p}-\alpha} \right]^p$$

and

$$(4.21) \quad \begin{aligned} \int_{2R < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) dy &\approx \int_{2R}^{\infty} r^{(\alpha-n)p'+n-1} \varphi(r) dr \\ &\leq c_4 \left[\varphi^{\frac{1}{p'}}(R) R^{\alpha-\frac{n}{p}} \right]^{p'}. \end{aligned}$$

The Hardy condition (2.3) (resp. dual Hardy condition (2.3*)) follows immediately from (4.18) and (4.19) (resp. (4.20) and (4.21)). And the Muckenhoupt condition (2.16) arises from (4.20) and (4.19).

PROOF OF EXAMPLE 2.11. To derive $I_\alpha : L^p(v) \rightarrow L^p(u)$, by Theorem 2.2, the task remains to check conditions (2.3), (2.3*) and (2.9).

Let $R > 0$. Then

$$(4.22) \quad \int_{|x| < R} u(x) dx \leq c_1 \begin{cases} R^{p(\frac{\delta}{p})} & \text{for } R < 2^{-1} \\ R^{p(\frac{\delta}{p})} & \text{for } R > 2^{-1} \end{cases}$$

since $\beta, \delta > 0$. And

$$(4.23) \quad \int_{R < |x|} |x|^{(\alpha-n)p} u(x) dx \leq c_2 \begin{cases} 1 & \text{for } R < 2^{-1} \\ R^{p[(\alpha-n)+\frac{\beta}{p}]} & \text{for } R > 2^{-1} \end{cases}$$

because $(\alpha - n)p + \delta > 0$ and $(\alpha - n)p + \beta < 0$. Moreover

$$(4.24) \quad \int_{|y| < R} v^{1-p'}(y) dy \leq c_3 \begin{cases} \ln^{-\frac{p'}{p}}(R^{-1}) & \text{for } R < 2^{-1} \\ R^{p'(n-\frac{\gamma}{p})} & \text{for } R > 2^{-1} \end{cases}$$

because $n - \frac{\gamma}{p} > 0$. On the other hand,

$$(4.25) \quad \int_{R < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) dy \leq c_4 \begin{cases} R^{p'(\alpha-n)} & \text{for } R < 2^{-1} \\ R^{p'(\alpha-\frac{\gamma}{p})} & \text{for } R > 2^{-1} \end{cases}$$

since $\alpha - \frac{\gamma}{p} < 0$.

The Hardy condition (2.3) (resp. dual Hardy condition (2.3*)) follows immediately from (4.23) and (4.24) (resp. (4.22) and (4.25)). To justify such a claim let us denote by $H(R)$ (resp. $H^*(R)$) the first member of (2.3) (resp. (2.3*)). Then for some fixed constant $c_5 > 0$: $H(R) \leq c_5$. This estimate is trivially true for $R < 2^{-1}$, and for $R > 2^{-1}$ the reason is that $\alpha + \frac{\beta}{p} - \frac{\gamma}{p} \leq 0$. Similarly $H^*(R) \leq c_6$ for some fixed constant $c_6 > 0$. This estimate is true for $R < 2^{-1}$ since $(\alpha - n) + \frac{\delta}{p} \geq 0$ and also for $R > 2^{-1}$ because $\alpha + \frac{\beta}{p} - \frac{\gamma}{p} \leq 0$.

Finally the pointwise condition (2.9) is also satisfied. Indeed if $A(R)$ denotes the first member of (2.7) then $A(R) \leq c_7$ for some fixed constant $c_7 > 0$. This last inequality is true for $2^{-1} < 4^{-1}R$ because $\alpha + \frac{\beta}{p} - \frac{\gamma}{p} \leq 0$. And for $4R < 2^{-1}$ the point is that $(\alpha - n) + \frac{\delta}{p} > 0$. And the conclusion arises immediately when $4^{-1}R \leq 2^{-1} \leq 4R$.

PROOF OF EXAMPLE 2.12. To get $I_\alpha : L^p(v) \rightarrow L^p(u)$, by Theorem 2.2, the task is reduced to check conditions (2.3), (2.3*) and (2.7).

Let $R > 0$. Then

$$(4.26) \quad \int_{|y| < R} v^{1-p'}(y) dy \approx R^{p'(n-\frac{\gamma}{p})}$$

since $n - \frac{\gamma}{p} > 0$ and

$$(4.27) \quad \int_{R < |y|} |y|^{(\alpha-n)p'} v^{1-p'}(y) dy \approx R^{p'(\alpha - \frac{\gamma}{p})}$$

because $\alpha - \frac{\gamma}{p} < 0$. Clearly for $z \neq 0$ and $2^{-1}|z| < |x| < 2|z|$:

$$(4.28) \quad \int_{2^{-1}|x| < |y| < 2|x|} |x-y|^{\alpha-n} v^{1-p'}(y) dy \leq c_1 |z|^{(1-p')(\gamma-n)+\alpha}.$$

We claim that

$$(4.29) \quad \int_{|x| < R} u(x) dx \leq c_2 R^{p[\frac{1}{p}(\beta_1+\beta_2)]},$$

$$(4.30) \quad \int_{2R < |x|} |x|^{(\alpha-n)p} u(x) dx \leq c_3 R^{p[(\alpha-n)+\frac{1}{p}(\beta_1+\beta_2)]},$$

and for $z \neq 0$

$$(4.31) \quad \int_{2^{-1}|z| < |x| < 2|z|} |z-x|^{\alpha-n} u(x) dx \leq c_4 |z|^{(\alpha-n)+(\beta_1+\beta_2)}.$$

The proofs of these last inequalities will be postponed below.

The Hardy condition (2.3) (resp. dual Hardy condition (2.3*)) follows immediately from (4.30) and (4.26) (resp. (4.29) and (4.27)). To justify such a statement let us denote by $H(R)$ (resp. $H^*(R)$) the first member of (2.3) (resp. (2.3*)). Then for some fixed constant $c_5 > 0$: $H(R) \leq c_5$ and $H^*(R) \leq c_5$ since $\alpha + \frac{1}{p}(\beta_1 + \beta_2) - \frac{\gamma}{p} = 0$. And condition (2.7) is also satisfied. Indeed if $B(|z|)$ denotes the first member of (2.7) then for some fixed constant $c_6 > 0$: $B(|z|) \leq c_6$ because $-\gamma + \alpha p + (\beta_1 + \beta_2) = 0$.

To end with the proof of Example 2.12 it is now time to prove claims (4.29), (4.30) and (4.31). Inequality (4.29) is satisfied since for $\beta_1, \beta_2 > 0$ then

$$\int_{|x| < R} u(x) dx \leq \left(\int_{|x_1| < R} |x_1|^{\beta_1-n_1} dx_1 \right) \left(\int_{|x_2| < R} |x_2|^{\beta_2-n_2} dx_2 \right) \leq c_2 R^{(\beta_1+\beta_2)}.$$

And (4.30) arises since $(\alpha-n)p + (\beta_1 + \beta_2) < 0$ and consequently

$$\begin{aligned} & \int_{2R < |x|} |x|^{(\alpha-n)p} u(x) dx \\ & \leq c_5 \sum_{k=1}^{\infty} (2^k R)^{(\alpha-n)p} \int_{2^k R < |x| \leq 2^{k+1} R} |x_1|^{\beta_1-n_1} |x_2|^{\beta_2-n_2} dx \end{aligned}$$

$$\begin{aligned}
&\leq c_6 \sum_{k=1}^{\infty} (2^k R)^{(\alpha-n)p} \left(\int_{|x_1| < 2^{k+1} R} |x_1|^{\beta_1-n_1} dx_1 \right) \left(\int_{|x_2| < 2^{k+1} R} |x_2|^{\beta_2-n_2} dx_2 \right) \\
&\leq c_7 \sum_{k=1}^{\infty} (2^k R)^{(\alpha-n)p} (2^k R)^{(\beta_1+\beta_2)} \\
&= c_7 R^{(\alpha-n)p+(\beta_1+\beta_2)} \sum_{k=1}^{\infty} 2^{k[(\alpha-n)p+(\beta_1+\beta_2)]} = c_8 R^{(\alpha-n)p+(\beta_1+\beta_2)}.
\end{aligned}$$

To prove (4.31), it is sufficient to get

$$(4.32) \quad \int_{|z-x| < 2^{-3}|z|} |z-x|^{\alpha-n} u(x) dx \leq c_{11} |z|^{(\alpha-n)+(\beta_1+\beta_2)}$$

since by (4.29)

$$\begin{aligned}
&\int_{2^{-1}|z| < |x| < 2|z|} |z-x|^{\alpha-n} u(x) dx \\
&\leq c_9 |z|^{\alpha-n} \int_{|x| < 4|z|} u(x) dx + \int_{|z-x| < 2^{-3}|z|} |z-x|^{\alpha-n} u(x) dx \\
&\leq c_{10} |z|^{(\alpha-n)+(\beta_1+\beta_2)} + \int_{|z-x| < 2^{-3}|z|} |z-x|^{\alpha-n} u(x) dx.
\end{aligned}$$

The point for (4.32) is that

$$(4.33) \quad S(m) = \int_{2^{-(m+1)}|z| < |z-x| < 2^{-m}|z|} u(x) dx \leq c_{12} (2^{-m}|z|)^{(\beta_1+\beta_2)}$$

for all integers $m \geq 3$, since $(\alpha-n) + (\beta_1+\beta_2) > 0$ and consequently

$$\begin{aligned}
&\int_{|z-x| < 2^{-3}|z|} |z-x|^{\alpha-n} u(x) dx \\
&\leq c_{13} \sum_{m=3}^{\infty} (2^{-m}|z|)^{(\alpha-n)} \int_{2^{-(m+1)}|z| \leq |z-x| < 2^{-m}|z|} u(x) dx \\
&\leq c_{14} \sum_{m=3}^{\infty} (2^{-m}|z|)^{(\alpha-n)+(\beta_1+\beta_2)} \\
&\leq c_{14} |z|^{(\alpha-n)+(\beta_1+\beta_2)} \sum_{m=3}^{\infty} 2^{-m[(\alpha-n)+(\beta_1+\beta_2)]} = c_{15} |z|^{(\alpha-n)+(\beta_1+\beta_2)}.
\end{aligned}$$

Finally to get (4.33) the following four cases should be considered:

$$(i) \quad 2^{-m}|z| < 2^{-1}|z_1| \quad \text{and} \quad 2^{-m}|z| < 2^{-1}|z_2|,$$

$$(ii) \quad 2^{-1}|z_1| < 2^{-m}|z| \quad \text{and} \quad 2^{-1}|z_2| < 2^{-m}|z|,$$

$$(iii) \quad 2^{-m}|z| < 2^{-1}|z_1| \quad \text{and} \quad 2^{-1}|z_2| < 2^{-m}|z|,$$

$$(iv) \quad 2^{-1}|z_1| < 2^{-m}|z| \quad \text{and} \quad 2^{-m}|z| < 2^{-1}|z_2|.$$

For the case i) the key is that $2^{-1}|z_1| < |x_1| < 2|z_1|$ and $2^{-1}|z_2| < |x_2| < 2|z_2|$ whenever $|z - x| < 2^{-m}|z|$ since

$$\begin{aligned} S(m) &\leq c_{15}|z_1|^{\beta_1-n_1}|z_2|^{\beta_2-n_2} \int_{|z-x|<2^{-m}|z|} dx \\ &\leq c_{16}|z_1|^{\beta_1-n_1}|z_2|^{\beta_2-n_2}(2^{-m}|z|)^n \\ &\leq c_{17}(2^{-m}|z|)^{\beta_1+\beta_2} \quad \text{recall that } n = n_1 + n_2. \end{aligned}$$

This last inequality is satisfied since $22^{-m}|z| < |z_1|, |z_2|$ and $\beta_1 - n_1 < 0, \beta_2 - n_2 < 0$.

For the case ii), observe that $|x_1| < 42^{-m}|z|$ and $|x_2| < 42^{-m}|z|$ whenever $|z - x| < 2^{-m}|z|$. And consequently

$$\begin{aligned} S(m) &\leq \int_{|x_1| \leq 42^{-m}|z|; |x_2| \leq 42^{-m}|z|} |z_1|^{\beta_1-n_1}|x_2|^{\beta_2-n_2} dx \\ &\leq c_{18}(2^{-m}|z_1|)^{\beta_1}(2^{-m}|z_2|)^{\beta_2} \quad \text{since } \beta_1 \text{ and } \beta_2 > 0 \\ &\leq c_{18}(2^{-m}|z|)^{\beta_1+\beta_2}. \end{aligned}$$

For the case iii), the point is that $2^{-1}|z_1| < |x_1| < 2|z_1|$ and $|x_2| < 42^{-m}|z|$ whenever $|z - x| < 2^{-m}|z|$ since

$$\begin{aligned} S(m) &\leq c_{19}|z_1|^{\beta_1-n_1} \int_{|x-z|<2^{-m}|z|} |x_2|^{\beta_2-n_2} dx \\ &\leq c_{19}|z_1|^{\beta_1-n_1} \left(\int_{|z_1-x_1|<2^{-m}|z|} dx_1 \right) \left(\int_{|x_2|<42^{-m}|z|} |x_2|^{\beta_2-n_2} dx_2 \right) \\ &\leq c_{20}|z_1|^{\beta_1-n_1}(2^{-m}|z|)^{n_1}(2^{-m}|z|)^{\beta_2} \quad \text{because } \beta_2 > 0 \\ &\leq c_{21}(2^{-m}|z|)^{\beta_1+\beta_2} \quad \text{since } 22^{-m}|z| < |z_1| \text{ and } \beta_1 - n_1 < 0. \end{aligned}$$

For the case iv), the key is that $2^{-1}|z_2| < |x_2| < 2|z_2|$ and $|x_1| < 42^{-m}|z|$ whenever $|z - x| < 2^{-m}|z|$. And the estimates become similar to those of iii).

5. Proofs of Basic Results

PROOF OF PROPOSITION 3.1.

Necessary Part. Suppose that $I_{\alpha, B(0, R_0)} : L^p(v) \rightarrow L^q(u)$ for some constant $C > 0$. The inequality (3.2) is obviously satisfied since $(\tilde{I}_{\alpha, B(0, R_0)} f)(\cdot) \leq (I_{\alpha, B(0, R_0)} f)(\cdot)$. And the Hardy inequality (3.1) appears since by considering a nonnegative function $g(\cdot)$ whose support is the ball $B(0, 2^{-1}R_0)$ then

$$\begin{aligned} & \int_{|x| < R_0} \left[\int_{|y| < 2^{-1}|x|} g(y) dy \right]^q |x|^{(\alpha-n)q} u(x) dx \\ & \leq 2^{(n-\alpha)q} \int_{|x| < R_0} \left[\int_{|y| < 2^{-1}|x|} |x-y|^{\alpha-n} g(y) dy \right]^q u(x) dx \\ & \leq (2^{n-\alpha} C)^q \left(\int_{|y| < 2^{-1}R_0} g^p(y) v(y) dy \right)^{\frac{q}{p}}. \end{aligned}$$

Similarly the dual Hardy inequality (3.1*) is satisfied since for each nonnegative function $f(\cdot)$ supported by the ball $B(0, R_0)$

$$\begin{aligned} & \int_{|x| < 2^{-1}R_0} \left[\int_{2|x| < |y| < R_0} |y|^{\alpha-n} f(y) dy \right]^q u(x) dx \\ & \leq 2^{(n-\alpha)q} \int_{|x| < 2^{-1}R_0} \left[\int_{2|x| < |y| < R_0} |x-y|^{\alpha-n} f(y) dy \right]^q u(x) dx \\ & \leq (2^{n-\alpha} C)^q \left(\int_{|y| < R_0} f^p(y) v(y) dy \right)^{\frac{q}{p}}. \end{aligned}$$

Sufficient Part. Now assume that inequalities (3.1), (3.1*) and (3.2) are satisfied. Since for $|x| < R_0$

$$\begin{aligned} & (I_{\alpha, B(0, R_0)} f)(x) \\ & = \left(\int_{|y| < 2^{-1}|x|} + \int_{\begin{cases} 2|x| < |y| \\ |y| < R_0 \end{cases}} \right) |x-y|^{\alpha-n} f(y) dy + (\tilde{I}_{\alpha, B(0, R_0)} f)(x) \\ & \leq 2^{n-\alpha} |x|^{\alpha-n} \int_{|y| < 2^{-1}|x|} f(y) dy \\ & \quad + 2^{n-\alpha} \int_{\begin{cases} 2|x| < |y| \\ |y| < R_0 \end{cases}} |y|^{\alpha-n} f(y) dy + (\tilde{I}_{\alpha, B(0, R_0)} f)(x) \end{aligned}$$

then the boundedness $I_{\alpha, B(0, R_0)} : L^p(v) \rightarrow L^q(u)$ holds because

$$\begin{aligned}
& \int_{|x| < R_0} \left(I_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \\
& \leq c \left\{ \int_{|x| < R_0} \left[\int_{|y| < 2^{-1}|x|} f(y) dy \right]^q |x|^{(\alpha-n)q} u(x) dx \right. \\
& \quad + \int_{|x| < 2^{-1}R_0} \left[\int_{2|x| < |y| < R_0} |y|^{(\alpha-n)} f(y) dy \right]^q u(x) dx \\
& \quad \left. + \int_{|x| < R_0} \left(\tilde{I}_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \right\} \\
& \quad \text{where } c = 2^{q-1}(1 + 2^{q-1})(1 + 2^{(n-\alpha)q}) \\
& \leq 3 \times \mathcal{A}^q \left(\int_{|y| < R_0} f^p(y) v(y) dy \right)^{\frac{q}{p}}.
\end{aligned}$$

PROOF OF PROPOSITION 3.2.

Necessary Part. Suppose the Hardy inequality (3.1) is satisfied for some constant $\mathcal{A} > 0$ and first assume that $p = 1$ and let $0 < R < 2^{-1}R_0$. For all functions $g(\cdot)$, supported by the ball $B(0, R)$ and with $\int_{|y| < R} g(y)v(y)dy = 1$, it appears from (3.1) that

$$\int_{|y| < R} g(y)v(y) \left[\left(\int_{2R < |x| < R_0} |x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \frac{1}{v(y)} \right] dy \leq \mathcal{A}.$$

Condition (3.3) appears from this last inequality after using a well-known property of functionals on L^p (see for instance [To], Proposition 2.3 p. 222). Next consider that $p > 1$ and let $0 < R < 2^{-1}R_0$. Then, because of (3.1), for all functions $f(\cdot)$ supported by the ball $B(0, R)$

$$\left(\int_{2R < |x| < R_0} |x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left(\int_{|y| < R} f(y) dy \right) \leq \mathcal{A} \left(\int_{|y| < R} f^p(y)v(y) dy \right)^{\frac{1}{p'}}.$$

Condition (3.4) follows from this last inequality by taking $f(y) = v^{1-p'}(y)$ on its support $B(0, R)$ and by using $(1 - p')p + 1 = 1 - p'$ and the assumption $0 < \int_{|y| < R} v^{1-p'}(y) dy < \infty$. Conditions (3.3*) and (3.4*) can be similarly obtained by using close arguments.

Sufficient Part. Assume that condition (3.3) or (3.4) is satisfied for some constant $\mathcal{A} > 0$. To get the Hardy inequality (3.1) consider a bounded and non-negative function $g(\cdot)$ supported by the ball $B(0, 2^{-1}R_0)$. The result for general functions can be obtained by applying the monotone convergence since the estimates which will be used do not depend on the bound of the function under consideration. Then $2^{m_0} \leq \int_{|y| < 2^{-1}R_0} g(y) dy < 2^{m_0+1}$ for some integer $m_0 \in \mathbb{Z}$.

Since the function $r \in]0, R_0[\rightarrow \int_{|y| < 2^{-1}r} g(y) dy$ is continuous and increasing, there is a sequence $(r_m)_{m \in \{-\infty, \dots, m_0+1\}}$ such that

$$0 < \dots < r_m < r_{m+1} < \dots < r_{m_0} < r_{m_0+1} = R_0$$

and

$$(5.1) \quad 2^m = \int_{|y| < 2^{-1}r_m} g(y) dy = 2 \int_{2^{-1}r_{m-1} \leq |y| < 2^{-1}r_m} g(y) dy$$

for all integers $m \leq m_0$.

Setting

$$E_m = \{x \mid r_m \leq |x| < r_{m+1}\} \quad \text{for all integers } m \leq m_0$$

the E_m 's are pairwise disjoint sets and

$$(5.2) \quad B(0, R_0) = \bigcup_{m=-\infty}^{m_0} E_m$$

and

$$(5.3) \quad \int_{|y| < 2^{-1}|x|} g(y) dy \leq 2^{m+1} \quad \text{for all } x \in E_m.$$

The conclusion follows now immediately from (5.1)–(5.3) and condition (3.3) or (3.4) since

$$\begin{aligned} & \int_{|x| < R_0} \left[\int_{|y| < 2^{-1}|x|} g(y) dy \right]^q |x|^{(\alpha-n)q} u(x) dx \\ &= \sum_{m=-\infty}^{m_0} \int_{x \in E_m} \left[\int_{|y| < 2^{-1}|x|} g(y) dy \right]^q |x|^{(\alpha-n)q} u(x) dx \quad \text{by (5.2)} \\ &\leq 2^{2q} \sum_{m=-\infty}^{m_0} \left(\int_{2^{-1}r_{m-1} \leq |y| < 2^{-1}r_m} g(y) dy \right)^q \left(\int_{r_m \leq |x| < r_{m+1}} |x|^{(\alpha-n)q} u(x) dx \right) \\ &\quad \text{by using (5.3) and (5.1)} \\ &\leq (4A)^q \sum_{m=-\infty}^{m_0} \left(\int_{2^{-1}r_{m-1} \leq |y| < 2^{-1}r_m} g^p(y) v(y) dy \right)^{\frac{q}{p}} \end{aligned}$$

by condition (3.3) if $p = 1$, and by the Hölder inequality and (3.4) if $p > 1$

$$\leq (4A)^q \left(\sum_{m=-\infty}^{m_0} \int_{2^{-1}r_{m-1} \leq |y| < 2^{-1}r_m} g^p(y) v(y) dy \right)^{\frac{q}{p}} \quad \text{since } \frac{q}{p} \geq 1$$

$$= (4A)^q \left(\int_{|y| < 2^{-1}R_0} g^p(y) v(y) dy \right)^{\frac{q}{p}}.$$

The dual inequality (3.1*) can be obtained by a similar way. The details are left to the readers.

PROOF OF PROPOSITION 3.3.

Necessary Part. Assume the restricted boundedness (3.2) is satisfied for some constant $\mathcal{A} > 0$. Then for all nonnegative functions $f(\cdot)$ supported by $\{y; 2^{-1}|z| < |y| < |z|\}$ and for z with $|z| < R_0$

$$\begin{aligned} (5.4) \quad & |z|^{\alpha-n} \left(\int_{2^{-1}|z| < |y| < |z|} f(y) dy \right) \left(\int_{2^{-1}|z| < |x| < |z|} u(x) dx \right)^{\frac{1}{q}} \\ & \leq 2^{n-\alpha} \left(\int_{2^{-1}|z| < |x| < |z|} \left(\tilde{I}_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \right)^{\frac{1}{q}} \\ & \leq 2^{n-\alpha} \mathcal{A} \left(\int_{2^{-1}|z| < |y| < |z|} f^p(y) v(y) dy \right)^{\frac{1}{p}}. \end{aligned}$$

Considering (5.4) for $p = 1$ and functions $f(\cdot)$ such that $\int_{2^{-1}|z| < |y| < |z|} f(y) v(y) dy = 1$, then

$$\int_{2^{-1}|z| < |y| < |z|} f(y) v(y) \left[|z|^{\alpha-n} \left(\int_{2^{-1}|z| < |x| < |z|} u(x) dx \right)^{\frac{1}{q}} \frac{1}{v(y)} \right] dy \leq 2^{n-\alpha} \mathcal{A}.$$

Condition (3.5) is implied by this last inequality. And for $p > 1$ condition (3.6) arises from inequality (5.4) by taking the function $f(\cdot)$ such that $f(y) = v^{1-p'}(y)$ on its support $\{y; 2^{-1}|z| < |y| < |z|\}$.

Before we handle with the sufficient part, let us first prove that (3.7) $\implies$ (3.5) and (3.8) $\implies$ (3.6). The first implication is true since for $|z| < R_0$

$$\begin{aligned} & |z|^{(\alpha-n)q} \int_{2^{-1}|z| < |x| < |z|} u(x) dx \\ & \leq 2^{(n-\alpha)q} \int_{2^{-1}|z| < |x| < 2|z|; |x| < R_0} |z-x|^{(\alpha-n)q} u(x) dx. \end{aligned}$$

It is also the case for the second implication because if $|z| < R_0$ then

$$\begin{aligned} & |z|^{(\alpha-n)q} \left(\int_{2^{-1}|z| < |x| < |z|} u(x) dx \right) \left(\int_{2^{-1}|z| < |y| < |z|} v^{1-p'}(y) dy \right)^{\frac{q}{p'}} \\ & = \int_{2^{-1}|z| < |x| < |z|} |z|^{(\alpha-n)t} \left[\int_{2^{-1}|z| < |y| < |z|} |z|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx \end{aligned}$$

$$\begin{aligned}
&\leq 2^{(n-\alpha)q} \int_{2^{-1}|z| < |x| < |z|} |z-x|^{(\alpha-n)t} \\
&\quad \times \left[\int_{2^{-1}|z| < |y| < |z|} |x-y|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx \\
&\leq 2^{(n-\alpha)q} \int_{2^{-1}|z| < |x| < 2|z|; |x| < R_0} |z-x|^{(\alpha-n)t} \\
&\quad \times \left[\int_{2^{-1}|x| < |y| < 2|x|; |y| < R_0} |x-y|^{(\alpha-n)t} v^{1-p'}(y) dy \right]^{\frac{q}{p'}} u(x) dx.
\end{aligned}$$

Sufficient Part. First assume that $p = 1 \leq q < \infty$ and the condition (3.7) is satisfied. Then the restricted boundedness (3.2) appears as follows

$$\begin{aligned}
&\left(\int_{x \in B(0, R_0)} \left(\tilde{I}_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \right)^{\frac{1}{q}} \\
&= \left(\int_{x \in B(0, R_0)} \left[\int_{z \in B(0, R_0) \cap C(0, |x|)} |x-z|^{\alpha-n} f(z) dz \right]^q u(x) dx \right)^{\frac{1}{q}} \\
&\leq \int_{z \in B(0, R_0)} f(z) \left[\left(\int_{x \in B(0, R_0) \cap C(0, |z|)} |z-x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \right] dz \\
&\quad \text{by the Minkowski inequality if } q > 1 \\
&\leq \int_{z \in B(0, R_0)} f(z) v(z) \\
&\times \left[\left(\int_{x \in B(0, R_0) \cap C(0, |z|)} |z-x|^{(\alpha-n)q} u(x) dx \right)^{\frac{1}{q}} \left(\operatorname{ess\,sup}_{y \in B(0, R_0) \cap C(0, |z|)} \frac{1}{v(y)} \right) \right] dz \\
&\leq A \int_{y \in B(0, R_0)} f(z) v(z) dz \quad \text{by condition (3.7).}
\end{aligned}$$

Next consider the case $1 < p \leq q < \infty$ and assume the condition (3.8) is satisfied. Observe that for $x \in B(0, R_0)$

$$\begin{aligned}
\left(\tilde{I}_{\alpha, B(0, R_0)} f \right)(x) &= \int_{z \in B(0, R_0) \cap C(0, |x|)} |x-z|^{(\alpha-n)t[\frac{1}{p'} + \frac{1}{q}]} f(z) dz \\
&\leq \Theta^{\frac{1}{p'}}(x) \left(\int_{z \in B(0, R_0) \cap C(0, |x|)} |x-z|^{(\alpha-n)t\frac{p}{q}} f^p(z) v(z) dz \right)^{\frac{1}{p}}
\end{aligned}$$

where

$$\Theta(x) = \int_{y \in B(0, R_0) \cap C(0, |x|)} |x-y|^{(\alpha-n)t} v^{1-p'}(y) dy.$$

Therefore condition (3.8) can be written as

$$\int_{B(0, R_0) \cap C(0, |z|)} |z - x|^{(\alpha-n)t} \Theta^{\frac{q}{p'}}(x) u(x) dx \leq A^q \quad \text{for all } z \in B(0, R_0).$$

And the restricted boundedness (3.2) now follows immediately since

$$\begin{aligned} & \left(\int_{x \in B(0, R_0)} \left(\tilde{I}_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \right)^{\frac{p}{q}} \\ & \leq \left(\int_{x \in B(0, R_0)} \left[\int_{z \in B(0, R_0) \cap C(0, |x|)} |x - z|^{(\alpha-n)t} f^p(z) v(z) dz \right]^{\frac{q}{p}} \Theta^{\frac{q}{p'}}(x) u(x) dx \right)^{\frac{p}{q}} \\ & \leq \int_{z \in B(0, R_0)} f^p(z) v(z) \left[\left(\int_{x \in B(0, R_0) \cap C(0, |z|)} |z - x|^{(\alpha-n)t} \Theta^{\frac{q}{p'}}(x) u(x) dx \right)^{\frac{p}{q}} \right] dz \\ & \quad \text{by the Minkowski inequality since } \frac{q}{p} \geq 1 \\ & \leq A^p \int_{y \in B(0, R_0)} f^p(z) v(z) dz \quad \text{by condition (3.8).} \end{aligned}$$

PROOF OF PROPOSITION 3.4. The implication (3.9) $\implies$ (3.7) is true since for $q < \frac{n}{n-\alpha}$ then $(\alpha - n)q + n > 0$ and consequently

$$\begin{aligned} & \int_{x \in B(0, R_0) \cap C(0, |z|)} |x - z|^{(\alpha-n)q} u(x) dx \\ & \leq \left(\sup_{x \in B(0, R_0) \cap C(0, |z|)} u(x) \right) \int_{|\omega - z| < 3|z|} |\omega - z|^{(\alpha-n)q} d\omega \\ & \leq c_1 |z|^{(\alpha-n)q+n} \left(\sup_{x \in B(0, R_0) \cap C(0, |z|)} u(x) \right) \quad c_1 = c_1(n, q). \end{aligned}$$

The implication (3.10) $\implies$ (3.8) is also true since by hypothesis $(\alpha - n)t + n > 0$ and consequently for $x \in C(0, |z|)$

$$\begin{aligned} & \int_{y \in B(0, R_0) \cap C(0, |x|)} |y - x|^{(\alpha-n)t} v^{1-p'}(y) dy \\ & \leq \left(\sup_{y \in B(0, R_0) \cap C(0, 2|z|)} v^{1-p'}(y) \right) \int_{|\omega - x| < 6|z|} |\omega - x|^{(\alpha-n)t} d\omega \\ & \leq c_2 |z|^{(\alpha-n)t+n} \left(\sup_{y \in B(0, R_0) \cap C(0, 2|z|)} v^{1-p'}(y) \right) \quad c_2 = c_2(n, t) \end{aligned}$$

and

$$\begin{aligned} & \int_{x \in B(0, R_0) \cap C(0, |z|)} |x - z|^{(\alpha-n)t} u(x) dx \\ & \leq c_1 |z|^{(\alpha-n)q+n} \left(\sup_{x \in B(0, R_0) \cap C(0, |z|)} u(x) \right) \quad c_1 = c_1(n, t). \end{aligned}$$

To derive the boundedness (3.2) e.g. $\tilde{I}_{\alpha, B(0, R_0)} : L^p(v) \rightarrow L^q(u)$, from (3.11) (or (3.11')) when $1 < p < \frac{n}{\alpha}$ with $p \leq q \leq p^*$, some preliminaries are first given. Remind that the Hardy-Littlewood Sobolev inequality asserts that

$$(i) \quad I_\alpha : L^p(1) \rightarrow L^{p^*}(1).$$

For convenience let us introduce

$$E_k = \{x; 2^{-(k+1)}R_0 \leq |x| < 2^{-k}R_0\}, \quad \mathcal{U}_k = \sup_{x \in E_k} (u(x))$$

and

$$F_k = \{y \in B(0, R_0); 2^{-(k+2)}R_0 \leq |y| < 2^{-(k-1)}R_0\}$$

where k is a integer with $k \geq 0$. Observe that

$$(ii) \quad \left[(2^{-k}R_0)^{n(\frac{1}{q} - \frac{1}{p^*})} \mathcal{U}_k^{\frac{1}{q}} \right]^p \leq A^p v(y) \quad \text{for each } y \in F_k.$$

This inequality can be deduced from (3.11') since for $y \in F_k$

$$\begin{aligned} & (2^{-k}R_0)^{n(\frac{1}{q} - \frac{1}{p^*})} \mathcal{U}_k^{\frac{1}{q}} \\ & \leq c_1 |y|^{n(\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p})} \left(\sup_{x \in B(0, R_0) \cap C(0, 2|y|)} u(x) \right)^{\frac{1}{q}} \left(v^{1-p'}(y) \right)^{\frac{1}{p}} \left(v(y) \right)^{\frac{1}{p}} \\ & \leq c_1 A \left(v(y) \right)^{\frac{1}{p}}. \end{aligned}$$

Similarly inequality (ii) can be deduced from (3.11) since for $y \in F_k$

$$\begin{aligned} & (2^{-k}R_0)^{n(\frac{1}{q} - \frac{1}{p^*})} \mathcal{U}_k^{\frac{1}{q}} \\ & \leq c_1 \sup_{2^{-(k+1)}R_0 \leq |x| < 2^{-k}R_0} \left\{ |x|^{n[\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}]} \left(u(x) \right)^{\frac{1}{q}} \left(v^{1-p'}(y) \right)^{\frac{1}{p}} \left(v(y) \right)^{\frac{1}{p}} \right\} \\ & \leq c_2 \sup_{2^{-(k+1)}R_0 \leq |x| < 2^{-k}R_0} \left\{ |x|^{n[\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}]} \left(u(x) \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left(\sup_{z \in B(0, R_0) \cap C(0, 2|x|)} v^{1-p'}(z) \right)^{\frac{1}{p'}} \left(v(y) \right)^{\frac{1}{p}} \right\} \end{aligned}$$

$$\leq c_2 A \left(v(y) \right)^{\frac{1}{p}}.$$

The boundedness (3.2) can be derived now by using i) and ii) as follows

$$\begin{aligned} & \int_{B(0, R_0)} \left(\tilde{I}_{\alpha, B(0, R_0)} f \right)^q(x) u(x) dx \\ & \leq \sum_{k=0}^{\infty} \int_{E_k} (I_{\alpha} f \mathbb{I}_{F_k})^q(x) u(x) dx \\ & \leq c_1 \sum_{k=0}^{\infty} (2^{-k} R_0)^{n(1-\frac{q}{p^*})} \mathcal{U}_k \left(\int_{\mathbb{R}^n} (I_{\alpha} f \mathbb{I}_{F_k})^{p^*}(x) dx \right)^{\frac{q}{p^*}} \\ & \quad \text{by the Hölder inequality if } \frac{q}{p^*} > 1 \\ & \leq c_2 \sum_{k=0}^{\infty} (2^{-k} R_0)^{n(1-\frac{q}{p^*})} \mathcal{U}_k \left(\int_{F_k} f^p(y) dy \right)^{\frac{q}{p}} \quad \text{by i)} \\ & = c_2 \sum_{k=0}^{\infty} \left(\int_{F_k} f^p(y) \left[(2^{-k} R_0)^{n(\frac{1}{q}-\frac{1}{p^*})} \mathcal{U}_k^{\frac{1}{q}} \right]^p dy \right)^{\frac{q}{p}} \\ & \leq (c_3 A)^q \sum_{k=0}^{\infty} \left(\int_{F_k} f^p(y) v(y) dy \right)^{\frac{q}{p}} \quad \text{by ii)} \\ & \leq (c_3 A)^q \left(\sum_{k=0}^{\infty} \int_{F_k} f^p(y) v(y) dy \right)^{\frac{q}{p}} \quad \text{since } \frac{q}{p} \geq 1 \\ & \leq (c_4 A)^q \left(\int_{\mathbb{R}^n} f^p(y) v(y) dy \right)^{\frac{q}{p}}. \end{aligned}$$

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