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INTUITIONISTIC LOGIC WITH A DUAL OPERATION

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Abstract

In this paper we study the Intuitionistic Logic with a weak negative modal operator. We introduce the variety of Heyting algebras with a negation operator. We determine the simple and subdirectly irreducible algebras, and we develop a duality between these algebras and certain relational structures. We also prove that some extensions of this logic are canonical and frame complete. Finally, we study the interpolation property for some canonical logics.

1. Introduction

In [4] Kosta Dosen introduces extensions of the intuitionistic logic with negative modal operators. Motivated by his studies in intuitionistic modal logics with modal operators $\Box$ and/or $\Diamond$, he observed that it is possible to define operators $\Box' = \neg\Box$ and $\Diamond' = \neg\Diamond$ by means of the intuitionistic negation $\neg$. In classical modal logic if any of the above operators is taken, the others can be defined. In the context of the intuitionistic logic, the operator $\Diamond'$ is interesting because it is a type of weak negation that is dual to the intuitionistic negation $\neg$. On the other hand, in [8] H. P. Sankkappanavar introduces the variety of Heyting algebras with a dual pseudocomplementation, i.e., Heyting algebras with a new unary operation $+$ that verifies the dual properties of the negation $\neg$. This variety of algebras defines in the usual way an intuitionistic logic with an operator that is an extension of the intuitionistic logic $H\Diamond'$ with the operator $\Diamond'$ introduced by Dosen.

In this paper we shall study the logic $H\Diamond'$ and some extensions. We shall adapt known methods of intuitionistic logic and classical modal logics to study the $H\Diamond'$ logic.

In the following section we shall introduce the necessary definitions and notions for the development of this work. We will define the logic $\mathcal{I}^+$ (or $H\Diamond'$ in the notation of [4]) and we shall recall its semantics. The basic results on the duality between Heyting algebras and descriptive intuitionistic frames are given

(see [3]). In Section 3 we shall define the variety of $\mathcal{H}^+$ -algebras and we shall give some important examples of these algebras. We shall also prove that there exists a bijective correspondence between congruences of $\mathcal{H}^+$ -algebras and a particular type of filter, and we will determinate the simple and subdirectly irreducible algebras. In Section 4 we shall give a duality between $\mathcal{H}^+$ -algebras and certain relational structures. In Section 5 we shall characterize some subvarieties of $\mathcal{H}^+$ -algebras by means of first-order conditions defined in the associated frames. These varieties correspond to canonical extensions of the basic $\mathcal{I}^+$. We shall also prove that these extensions are frame-complete. We will finish this work proving that all canonical extension of the logic $\mathcal{I}^+$, such that the class of all its frames is closed under bisimulation-products has the interpolation property. We shall follow the lines of the analogous results for classical modal logic proved in [7].

2. Preliminaries

The language $\mathcal{L}$ of propositional logic that we assume in the paper has the connectives $\{\wedge, \vee, \rightarrow\}$ and has in addition one propositional constant $\perp$. The set of propositional variables is denoted by Var . The negation $\neg$ and the constant $\top$ are defined by $\neg p = p \rightarrow \perp$ and $\top = \neg \perp$, respectively. Fm will denote the set of formulas. Let $\mathcal{L}^+$ be the language $\mathcal{L}$ with a new unary connective $^+$.

The intuitionistic logic is denoted by **Int**. The intuitionistic modal logic $\mathcal{I}^+$ is the logic in the language $\mathcal{L}^+$ with the following sets of axioms and the following rules:

1. Any axiomatization for **Int** (see [3]).
2. $(\varphi \wedge \psi)^+ \rightarrow \varphi^+ \vee \psi^+$.
3. $(\neg \top)^+$.
4.
$$\frac{\varphi \rightarrow \psi}{\psi^+ \rightarrow \varphi^+}.$$

An $\mathcal{I}^+$ -logic is any logic $\mathcal{I}$ such that $\mathcal{I}^+ \subseteq \mathcal{I}$.

An intuitionistic frame, or frame for short, is a relational structure $\langle X, \leq \rangle$ such that $\leq$ is an partial order defined on X .

DEFINITION 1. An $\mathcal{I}^+$ -frame is a relational structure $\mathcal{F} = \langle X, \leq, R \rangle$ where $\langle X, \leq \rangle$ is a frame, and R is a binary relation on X such that $\leq^{-1} \circ R \subseteq R$.

Let $\mathcal{F} = \langle X, \leq, R \rangle$ be an $\mathcal{I}^+$ -frame. For $Y \subseteq X$, we put $[Y] = \{x \in X : y \leq x, \text{ for some } y \in Y\}$ and $(Y) = \{x \in X : x \leq y, \text{ for some } y \in Y\}$. A subset Y of X is *increasing* if $Y = [Y]$ and is *decreasing* if $Y = (Y)$. The set of all increasing sets of X will be denoted by $\mathcal{P}_i(X)$.

A *valuation* on an $\mathcal{I}^+$ -frame $\mathcal{F}$ is a function $V : Var \rightarrow \mathcal{P}_i(X)$. Any valuation

V can be extended recursively to Fm by means of the following clauses:

1. $V(\perp) = \emptyset$,
2. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
3. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$,
4. $V(\varphi \rightarrow \psi) = \{x \in X : [x] \cap V(\varphi) \subseteq V(\psi)\}$.
5. $V(\varphi^+) = \{x \in X : R(x) \subseteq V(\varphi)\}$.

An $\mathcal{I}^+$ -model is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is an $\mathcal{I}^+$ -frame and V is a valuation defined on $\mathcal{F}$.

We define the semantic notions of truth and validity in an $\mathcal{I}^+$ -model and validity in an $\mathcal{I}^+$ -frame for formulas. Given an $\mathcal{I}^+$ -model $\langle \mathcal{F}, V \rangle$ and a point $x \in X$, we say that a formula φ is *true* at x in $\langle \mathcal{F}, V \rangle$, in symbols $\langle \mathcal{F}, V \rangle \models_x \varphi$, if $x \in V(\varphi)$. A formula φ is *valid in an $\mathcal{I}^+$ -model* $\langle \mathcal{F}, V \rangle$, in symbols $\langle \mathcal{F}, V \rangle \models \varphi$, if it is true at every point in X . A formula φ is *valid in an $\mathcal{I}^+$ -frame* $\mathcal{F}$, in symbols $\mathcal{F} \models \varphi$, if for any valuation V on $\mathcal{F}$, φ is valid in the $\mathcal{I}^+$ -model $\langle \mathcal{F}, V \rangle$.

Let $\mathcal{I}$ be any logic that is an extension of $\mathcal{I}^+$. We shall write $\vdash_{\mathcal{I}} \varphi$ for $\varphi \in \mathcal{I}$. We will denote by $\text{Fr}(\mathcal{I})$ the class of all $\mathcal{I}^+$ -frames where every formula of $\mathcal{I}$ is valid. Now let $\mathbf{F}$ be a class of $\mathcal{I}^+$ -frames. $\text{Th}(\mathbf{F})$ denotes the class of all formulas that are valid in every $\mathcal{I}^+$ -frame in $\mathbf{F}$. A logic $\mathcal{I}$ is *characterized* by a class $\mathbf{F}$ of $\mathcal{I}^+$ -frames, or it is *complete* relative to a class $\mathbf{F}$ of $\mathcal{I}^+$ -frames, *F-complete* for short, if $\text{Th}(\mathbf{F}) = \mathcal{I}$. In [4] it was proved that the logic $\mathcal{I}^+$ is characterized by the class $\mathbf{F}^+$ of all $\mathcal{I}^+$ -frames, i.e., $\mathcal{I}^+ = \text{Th}(\mathbf{F}^+)$. Let us recall that the intuitionistic logic **Int** is characterized by the class $\mathbf{F}$ of all frames.

A *Heyting algebra* is an algebra $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ where $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice such that for any $x, y, z \in A : x \wedge y \leq z \Leftrightarrow x \leq y \rightarrow z$.

If $\mathcal{F} = \langle X, \leq \rangle$ is a frame, then it is known that $\langle \mathcal{P}_i(X), \cup, \cap, \Rightarrow, \emptyset, X \rangle$ is a Heyting algebra where the operation $\Rightarrow$ is defined by $U \Rightarrow V = \{x \in X : [x] \cap U \subseteq V\}$.

If A is a Heyting algebra and $Fp(A)$ is the set of all prime filters of A , then $\mathcal{F}(A) = \langle Fp(A), \subseteq \rangle$ is the associated frame of A . The algebra

$$A(\mathcal{F}(A)) = \langle \mathcal{P}_i(Fp(A)), \cup, \cap, \Rightarrow, \emptyset, Fp(A) \rangle$$

is called the canonical extension of A .

A *general frame* is a structure $\mathcal{F}_g = \langle \mathcal{F}, D \rangle$ where $\mathcal{F} = \langle X, \leq \rangle$ is a frame and D is a Heyting subalgebra of $\mathcal{P}_i(X)$, i.e., D is a bounded sublattice of $\mathcal{P}_i(X)$ such that for every $U, V \in D$, $U \Rightarrow V \in D$. The algebra D associated to the general frame $\mathcal{F}_g$ will be denoted by $A(\mathcal{F}_g)$.

By means of the Stone representation theorem for the Heyting algebras we

shall define the general frame of a Heyting algebra A . Let us consider the map

$$\varphi_A : A \rightarrow \mathcal{P}_i(Fp(A))$$

given by

$$\varphi_A(a) = \{P \in Fp(A) : a \in P\}.$$

The map φ_A is an injective homomorphism of Heyting algebras. Thus, $A \simeq \varphi_A(A) = \{\varphi_A(a) : a \in A\}$. The general frame associated to A is $\mathcal{F}_g(A) = \langle \mathcal{F}(A), \varphi_A(A) \rangle$.

Let $\mathcal{F}_{g_1} = \langle \mathcal{F}_1, D_1 \rangle$ and $\mathcal{F}_{g_2} = \langle \mathcal{F}_2, D_2 \rangle$ be two general frames. A map $f : \mathcal{F}_{g_1} \rightarrow \mathcal{F}_{g_2}$ is a p-morphism if and only if

1. If $x \leq_1 y$ then $f(x) \leq_2 f(y)$.
2. If $f(x) \leq_2 y$, then there exists $z \in X_1$ such that $x \leq_1 z$ and $f(z) = y$.
3. For every $U \in D_2$, $f^{-1}(U) \in D_1$.

An *isomorphism* between general frames is a bijective p-morphism.

Let $f : \mathcal{F}_{g_1} \rightarrow \mathcal{F}_{g_2}$ be a p-morphism. Then the map

$$A(f) : A(\mathcal{F}_{g_2}) \rightarrow A(\mathcal{F}_{g_1}) \text{ defined by } A(f)(U) = f^{-1}(U), U \in D_2,$$

is an homomorphism of Heyting algebras. If $h : A_1 \rightarrow A_2$ is a homomorphism of Heyting algebras, then the map

$$\mathcal{F}_g(h) : \mathcal{F}_g(A_2) \rightarrow \mathcal{F}_g(A_1) \text{ defined by } \mathcal{F}_g(h)(P) = h^{-1}(P), P \in Fp(A_2),$$

is a p-morphism.

Let $\mathcal{F}_g = \langle \mathcal{F}, D \rangle$ general frame. Let us consider the map

$$F_X : X \rightarrow Fp(D)$$

given by

$$F_X(x) = \{U \in D : x \in U\}.$$

The general frame $\mathcal{F}_g = \langle \mathcal{F}, D \rangle$ is *descriptive* if and only if

1. The map F_X is an order-isomorphism, i.e., $x \leq y \Leftrightarrow F_X(x) \subseteq F_X(y)$, and
2. F_X is surjective.

THEOREM 2 [3]. *Let A be a Heyting algebra and let $\mathcal{F}_g$ be a general frame. Then*

1. *The general frame $\mathcal{F}_g(A)$ is descriptive and $A \simeq A(\mathcal{F}_g(A))$.*
2. *$\mathcal{F}_g$ is descriptive if and only if the map $F_X : X \rightarrow \text{Fp}(D)$ defines an isomorphism between the general frames $\mathcal{F}_g$ and $\mathcal{F}_g(D)$.*

Let $\mathcal{F}_g = \langle \mathcal{F}, D \rangle$ be a general frame. Let us consider the set $D^c = \{X - U : U \in D\}$. We can consider a topology $\mathcal{T}_D$ having as a subbasis the set $D \cup D^c$, obtaining an ordered topological space $\langle X, \leq, \mathcal{T}_D \rangle$. It is easy to show that $\mathcal{F}_g$ is a descriptive frame if and only if

1. D is a Heyting subalgebra of $\mathcal{P}_i(X)$.
2. F_X is an order-isomorphism, and
3. $\langle X, \leq, \mathcal{T}_D \rangle$ is a compact ordered topological space.

In topological terms, it is easy to see that for all $x \in X$, $[x]$ is a closed subset of $\langle X, \leq, \mathcal{T}_D \rangle$.

We denote by $\mathcal{O}_i(X)$ the set of all increasing open subsets of $\langle X, \leq, \mathcal{T}_D \rangle$, and we denote by $\mathcal{C}_i(X)$ the set of all increasing closed subsets of $\langle X, \leq, \mathcal{T}_D \rangle$.

3. $\mathcal{H}^+$ -algebras

DEFINITION 3. An $\mathcal{H}^+$ -algebra is an algebra $\langle A, \vee, \wedge, \rightarrow, +, 0, 1 \rangle$ where $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $+$ is a unary operation that verifies the conditions:

$$\mathbf{H1} \quad (x \wedge y)^+ = x^+ \vee y^+.$$

$$\mathbf{H2} \quad 1^+ = 0.$$

It is clear that the class of $\mathcal{H}^+$ -algebras is a variety. We shall denote by $\mathcal{H}^+$ this variety.

Now, we shall give some important examples of $\mathcal{H}^+$ -algebras.

1. Let $\mathcal{F} = \langle X, \leq, R \rangle$ be an $\mathcal{I}^+$ -frame, or frame for short. Let us consider the operation $*$ defined by

$$U^* = \{x \in X : R(x) \subsetneq U\}.$$

This operation is well defined, i.e., for all $U \in \mathcal{P}_i(X)$, $U^* \in \mathcal{P}_i(X)$. Indeed, let $x, y \in X$ such that $x \leq y$ and $x \in U^*$. Then, since $R(x) \subsetneq U$, there exists $z \in X$ such that $(x, y) \in R$ and $z \notin U$. So, $(y, z) \in \leq^{-1} \circ R \subseteq R$. Thus, $R(y) \subsetneq U$. Consequently,

$$A(\mathcal{F}) = \langle \mathcal{P}_i(X), \cup, \cap, \Rightarrow, *, \emptyset, X \rangle \in \mathcal{H}^+.$$

2. A *Heyting algebra with dual pseudocomplementation* is an algebra $\langle A, \vee, \wedge, \rightarrow, +, 0, 1 \rangle$ such that $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $+$ verifies the axioms:
- (a) $x \vee (x \vee y)^+ = x \vee y^+$.
 - (b) $x \vee 1^+ = x$.
 - (c) $1^{++} = 1$.

These algebras were introduced and studied by H.P. Sankappanavar in [8].

3. A *Symmetrical Heyting algebra* is an algebra $\langle A, \vee, \wedge, \rightarrow, \sim, 0, 1 \rangle$ such that $\langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a Heyting algebra and $\langle A, \vee, \wedge, \sim, 0, 1 \rangle$ is a De Morgan algebra. It is easy to see that the operation $a^+ = \sim \neg(\sim a)$ is the dual pseudocomplement of $\neg a = a \rightarrow 0$. Thus, $\langle A, \vee, \wedge, \rightarrow, \sim, 0, 1 \rangle$ is also an $\mathcal{H}^+$ -algebra.
4. Let Y be a set of variables and let $Fm(Y)$ be the set of all formulas in the language $\mathcal{L}^+$ on the set Y . Let $\mathcal{I}$ be an $\mathcal{I}^+$ -logic on Y . Let us consider on the set $Fm(Y)$ the equivalence relation $\equiv$ defined by

$$\varphi \equiv \psi \Leftrightarrow \vdash_{\mathcal{I}} \varphi \rightarrow \psi \ \& \ \vdash_{\mathcal{I}} \psi \rightarrow \varphi.$$

Let $|\varphi| = \{\psi \in Fm : \varphi \equiv \psi\}$ be the equivalence class of φ and $Fm(Y) / \equiv$ be the quotient set. Then the algebra

$$\mathcal{LA}(Y) = \langle Fm(Y) / \equiv, \vee, \wedge, \rightarrow, +, \top \rangle,$$

where

$$\begin{aligned} |\varphi| \wedge |\psi| &= |\varphi \wedge \psi| & |\varphi| \vee |\psi| &= |\varphi \vee \psi| \\ |\varphi| \rightarrow |\psi| &= |\varphi \rightarrow \psi| & |\varphi^+| &= |\varphi|^+, \quad |\top| = \top, \end{aligned}$$

is the *Lindenbaum-Tarski algebra* of $\mathcal{I}$. It is clear that $\mathcal{LA}(Y)$ is an $\mathcal{H}^+$ -algebra.

Usually, we write $\mathcal{LA}$ by $\mathcal{LA}(Y)$. For a set $\Sigma \subseteq Fm$, we denote by $\mathcal{I}(\Sigma)$ the $\mathcal{I}^+$ -logic generated by Σ , i.e., the least extension of $\mathcal{I}^+$ by means of the set Σ . In this case the Lindenbaum-Tarski algebra we will denote by $\mathcal{LA}_{\Sigma}$.

Let $A \in \mathcal{H}^+$. A *valuation* on A is a homomorphism $V : Fm \rightarrow A$. A formula φ is true in A under a valuation V if $V(\varphi) = 1$, and φ is valid in A , in symbols $A \models \varphi$, if it is true under any valuation. A logic $\mathcal{I} \supseteq \mathcal{I}^+$ is characterized by a class $\mathbf{K}$ of $\mathcal{H}^+$ -algebras if $\mathcal{I} = \{\varphi : \forall A \in \mathbf{K} \ A \models \varphi\}$. It is clear that $\mathcal{I}^+ = \{\varphi : \mathcal{LA} \models \varphi\}$.

Let $A \in \mathcal{H}^+$. We denote by $Fi(A)$ and $Id(A)$ the lattice of filters and ideals of A , respectively. Let $Y \subseteq A$. The filter (ideal) generated by Y is denoted by $F(Y)$ ($I(Y)$).

3.1. Congruences

Recall that the congruences of a Heyting algebra are determined by filters, i.e., there exists a bijective correspondence between filters and congruences on a Heyting algebra. Now, we shall see that the congruences of $\mathcal{H}^+$ -algebras are determined by filters closed under the operation $\Box x = \neg(x^+)$.

Let $A \in \mathcal{H}^+$. We shall define auxiliary operations. Let us consider the operator $\Box : A \rightarrow A$ given by $\Box a = \neg(a^+) = a^+ \rightarrow 0$. By recursion define $\Box^n a$ and $t_n(a)$, by $\Box^0 a = a$, $\Box^{n+1} a = \Box \Box^n a$, and $t_n(a) = a \wedge \Box a \wedge \dots \wedge \Box^n a$.

LEMMA 4. *Let $A \in \mathcal{H}^+$. Then*

1. $\Box(a \wedge b) = \Box a \wedge \Box b$.
2. $\Box 1 = 1$.
3. $\Box(a \rightarrow b) \wedge \Box a \leq \Box b$.
4. $t_n(a \wedge b) = t_n(a) \wedge t_n(b)$.
5. $t_n(a \rightarrow b) \wedge t_n(a) \leq t_n(b)$.
6. $t_n(a) \leq a$.
7. $t_n(1) = 1$.

PROOF. The proof is easy taking into account that $\neg(a \vee b) = \neg a \wedge \neg b$ and that $\neg 0 = 1$. ■

DEFINITION 5. Let $A \in \mathcal{H}^+$ and let $F \subseteq A$ be a filter. We say that F is a normal filter if for each $a \in F$, we get $\Box a \in F$.

The lattice of all normal filters of $A \in \mathcal{H}^+$ we shall denote by $Fn(A)$, and the lattice of congruences of A we will denote by $Con(A)$.

THEOREM 6. *Let $A \in \mathcal{H}^+$. Then $Con(A) \cong Fn(A)$.*

PROOF. Let $F \in Fn(A)$. Then,

$$\Theta(F) = \{(a, b) \in A^2 : \exists f \in F, a \wedge f = b \wedge f\},$$

is a congruence of Heyting algebras. We prove that $\Theta(F)$ preserves the operation $^+$. Let $a, b \in A$ such that $a \wedge f = b \wedge f$ for some $f \in F$. So,

$$\begin{aligned} (a \wedge f)^+ &= (b \wedge f)^+ \Rightarrow (a^+ \vee f^+) \wedge \Box f = (b^+ \vee f^+) \wedge \Box f \\ &\Rightarrow (a^+ \wedge \Box f) \vee (f^+ \wedge \Box f) = (b^+ \wedge \Box f) \vee (f^+ \wedge \Box f) \\ &\Rightarrow a^+ \wedge \Box f = b^+ \wedge \Box f. \end{aligned}$$

Since $\Box f \in F$, then $(a^+, b^+) \in \Theta(F)$.

Let $\Theta \in \text{Con}(A)$. Then

$$F(\Theta) = \{a \in A : (a, 1) \in \Theta\},$$

is a filter. Let $(a, 1) \in \Theta$. Then

$$(a^+, 1^+) = (a^+, 0) \in \Theta \Rightarrow (\neg a^+, \neg 0) = (\Box a, 1) \in \Theta.$$

Thus, $F(\Theta)$ is a normal filter. It is easy to see that $\Theta = \Theta(F(\Theta))$ and $F = F(\Theta(F))$, for $\Theta \in \text{Con}(A)$ and $F \in \text{Fn}(A)$. Thus, $\text{Con}(A) \cong \text{Fn}(A)$. ■

Next, we shall determine the simple and subdirectly irreducible algebras in the variety $\mathcal{H}^+$. For this purpose, we shall first characterized the normal filter generated by a subset.

LEMMA 7. *Let $A \in \mathcal{H}^+$ and let $H \subseteq A$. Then the normal filter generated by H is the filter*

$$\text{Fn}(H) = \{a \in A : \exists \{y_0, y_1, \dots, y_n\} \subseteq H, \ t_{n_0}(y_0) \wedge \dots \wedge t_{n_k}(y_n) \leq a\}.$$

PROOF. The proof is easy and left to the reader. ■

Let us recall that an algebra A is simple if it has only two congruences, and A is subdirectly irreducible if there exists a $\Theta \in \text{Con}(A)$ non trivial such that $\Theta = \bigcap_{\Theta_i \in \text{Con}(A)} \Theta_i$, with Θ_i non trivial. If $A \in \mathcal{H}^+$, since $\text{Con}(A) \cong \text{Fn}(A)$, then A is simple if and only if $\text{Fn}(A) = \{\{1\}, A\}$, and A is subdirectly irreducible if and only if there exists $F \in \text{Fn}(A)$ such that $F \neq \{1\}$ and $F = \bigcap_{F_i \in \text{Fn}(A) - \{1\}} F_i$.

THEOREM 8. *Let $A \in \mathcal{H}^+$. Then*

1. *A is simple if and only if for every $x \in A - \{1\}$ there exists $n \geq 0$ such that $t_n(x) = 0$.*
2. *A is subdirectly irreducible if and only if there exists $a \in A - \{1\}$ such that for every $x \in A - \{1\}$ there exists $n \geq 0$ such that $t_n(x) \leq a$.*

PROOF. 1. $\Rightarrow$) Let $x \in A - \{1\}$. Let us consider the normal filter $\text{Fn}(x)$. Since $x \neq 1$ and $x \in \text{Fn}(x)$, then $\text{Fn}(x) \neq \{1\}$. Thus, $\text{Fn}(x) = A$. This implies that $0 \in \text{Fn}(x)$, i.e., there exists $n \geq 0$ such that $t_n(x) = 0$.

$\Leftarrow$). Suppose that A is not simple. Then there exists $F \in \text{Fn}(A)$ such that $F \neq \{1\}$ and $F \neq A$. It follows that there exists $x \in F$ such that $x \neq 1$. As $0 \notin F$ and $\text{Fn}(x) \subseteq F$, then for every $n \geq n$, we get $t_n(x) \neq 0$.

2. $\Rightarrow$) As A is subdirectly irreducible then, $\bigcap_{F_i \in \text{Fn}(A) - \{1\}} F_i \neq \{1\}$. So, there exists $a \neq 1$ such that $a \in \bigcap_{F_i \in \text{Fn}(A) - \{1\}} F_i$. Let $x \in A - \{1\}$ and let us consider the normal filter $\text{Fn}(x)$. Then $a \in \text{Fn}(x)$, i.e., there exists $n \geq 0$ such that $t_n(x) \leq a$.

$\Leftarrow$) It is left to the reader. ■

4. Duality Theory

In this section we shall develop a duality between $\mathcal{H}^+$ -algebras and relational structures. First, we shall give a representation theorem for the $\mathcal{H}^+$ -algebras.

Let $A \in \mathcal{H}^+$. For each $P \in Fp(A)$ let us consider the following subsets of A :

$$(P^+)^{-1} = \{a \in A : a^+ \in P\},$$

$$g(P) = A - (P^+)^{-1} = \{a \in A : a^+ \notin P\}.$$

On $Fp(A)$ define the binary relation R_A by

$$(P, Q) \in R \Leftrightarrow (P^+)^{-1} \cup Q = A.$$

This relation was defined by K. Dosen in [4] on the canonical frame for $\mathcal{I}^+$.

With the operator $\Box$ we define an auxiliary relation $R_\Box$ on $Fp(A)$ by:

$$(P, Q) \in R_\Box \Leftrightarrow \Box^{-1}(P) \subseteq Q,$$

where $\Box^{-1}(P) = \{a \in A : \Box a \in P\}$.

LEMMA 9. *Let $A \in \mathcal{H}^+$. Then*

1. *For each $P \in Fp(A)$, $g(P) \in Fi(A)$.*
2. *$(P, Q) \in R_A \Leftrightarrow g(P) \subseteq Q$.*
3. *$a^+ \in P \Leftrightarrow$ there exists $Q \in Fp(A)$ such that $g(P) \subseteq Q$ and $a \notin Q$.*
4. *$\Box^{-1}(P) \subseteq Q \Leftrightarrow$ there exists $D \in Fp(A)$ such that $P \subseteq D$ and $g(D) \subseteq Q$, i.e., $R_\Box = \subseteq \circ R$.*
5. *$\Box a \in P \Leftrightarrow$ for every $Q \in R_\Box(P)$, $a \in Q$.*

PROOF. 1. Let $a, b \in g(P)$. Then a^+ and $b^+ \notin P$. So $a^+ \vee b^+ = (a \wedge b)^+ \notin P$, i.e., $a \wedge b \in g(P)$. Let us suppose that $a \leq b$ and $a \in g(P)$. Since the operation $+$ is anti-monotonic, $b^+ \leq a^+$. Then, $b \in g(P)$. Since $1^+ = 0$, $1 \in g(P)$. Thus, $g(P)$ is a filter.

2. It is immediate. We prove 3. Let $a^+ \in P$. We show that $g(P) \cap I(a) = \emptyset$. Assume the contrary. Then there exists $b \in g(P)$ such that $b \leq a$. So, $a^+ \leq b^+$, and as $b^+ \notin P$, $a^+ \notin P$, which is a contradiction. Thus, there exists $Q \in Fp(A)$ such that $g(P) \subseteq Q$ and $a \notin Q$.

4. Suppose that $\Box^{-1}(P) \subseteq Q$. Let us consider the filter $g(Q)$. We prove that the filter $F(P \cup g(Q))$ is proper. Suppose the contrary. Then there exists $p \in P$ and there exists $a \in g(Q)$ such that $p \wedge a^+ = 0$. Thus, $p \leq \neg(a^+) = \Box a \in P$. It follows that, $a \in Q$, which is absurd. As $F(P \cup g(Q))$ is proper then we can assert that there exists $D \in Fp(A)$ such that $P \subseteq D$ and $g(Q) \subseteq D$.

Let us suppose that there exists $D \in Fp(A)$ such that $P \subseteq D$ and $g(D) \subseteq Q$. If $\Box a \in P$, then $\Box a \in D = \neg(a^+) \in D$. So, $a^+ \notin D$, i.e., $a \in g(D)$. Thus, $a \in Q$.

5. Let $\Box a \notin P$. By properties of the operation $\neg$ we can assert that there exists $D \in Fp(A)$ such that $P \subseteq D$ and $a^+ \in D$. By Assertion 3, it follows that there exists $Q \in Fp(A)$ such that $g(D) \subseteq Q$ and $a \notin Q$. Thus, there exists $Q \in Fp(A)$ such that $\Box^{-1}(P) \subseteq Q$ and $a \notin Q$. The other direction is easy and left to the reader. ■

DEFINITION 10. Let A be an $\mathcal{H}^+$ -algebra. The structure $\mathcal{F}(A) = \langle Fp(A), \subseteq, R_A \rangle$ is called the $\mathcal{I}^+$ -frame of A , and the algebra $A(\mathcal{F}(A)) = \langle \mathcal{P}_i(Fp(A)), \cup, \cap, \Rightarrow, *, Fp(A) \rangle$ is called the canonical extension of A .

THEOREM 11. Every $\mathcal{H}^+$ -algebra A is isomorphic to a subalgebra of $A(\mathcal{F}(A))$.

PROOF. Let $A \in \mathcal{H}^+$. Let us consider the frame $\mathcal{F}(A) = \langle Fp(A), \subseteq, R_A \rangle$. By Example 1, $A(\mathcal{F}(A)) = \langle \mathcal{P}_i(Fp(A)), \cup, \cap, \Rightarrow, +, \emptyset, Fp(A) \rangle$ is an $\mathcal{H}^+$ -algebra. Let us consider the map

$$\varphi_A : A \rightarrow \mathcal{P}_i(Fp(A))$$

defined by

$$\varphi_A(a) = \{P \in Fp(A) : a \in P\}.$$

The map φ_A is an injective homomorphism of Heyting algebras. Then we only need to prove that $\varphi_A(a^+) = (\varphi_A(a))^*$. But this equality follows by the Assertion 3 of Lemma 9. ■

DEFINITION 12. Let $A \in \mathcal{H}^+$. The structure $\mathcal{F}_g(A) = \langle Fp(A), \subseteq, R_A, \varphi_A(A) \rangle$ is called the general $\mathcal{I}^+$ -frame of A . The algebra $\varphi_A(A)$ is called the associated algebra of $\mathcal{F}_g(A)$ and we will be denoted by $A(\mathcal{F}_g(A))$.

Now, we shall extended the duality between Heyting algebras and general descriptive frames to a duality between $\mathcal{H}^+$ -algebras and $\mathcal{I}^+$ -descriptive frames.

DEFINITION 13. A general $\mathcal{I}^+$ -frame $\mathcal{F}_g = \langle X, \leq, R, D \rangle$ is descriptive if

1. $\langle X, \leq, D \rangle$ is a descriptive frame,
2. For all $x, y \in X$, if $(F_X(x), F_X(y)) \in R_D$, then $(x, y) \in R$.

THEOREM 14. Let $A \in \mathcal{H}^+$ and let $\mathcal{F}_g$ be a general $\mathcal{I}^+$ -frame. Then:

1. $A \cong A(\mathcal{F}_g(A))$.
2. $\mathcal{F}_g$ is descriptive if and only if $\mathcal{F}_g \cong \mathcal{F}_g(A(\mathcal{F}_g))$.

PROOF. The proof is standard. ■

DEFINITION 15. Let $\mathcal{F}_{g_1} = \langle X_1, \leq_1, R_1, D_1 \rangle$ and $\mathcal{F}_{g_2} = \langle X_2, \leq_2, R_2, D_2 \rangle$ be two general $\mathcal{I}^+$ -frames. A map $f : \mathcal{F}_{g_1} \rightarrow \mathcal{F}_{g_2}$ is an $\mathcal{I}^+$ -morphism if is a p-morphism and

- P1** If $(x, y) \in R_1$, then $(f(x), f(y)) \in R_2$.
- P2** If $(f(x), z) \in R_2$, then there exists $y \in X_1$ such that $(x, y) \in R_1$ and $f(y) \leq_2 z$.
- P3** For every $U \in D_2$, $f^{-1}(U) \in D_1$.

An $\mathcal{I}^+$ -morphism between two $\mathcal{I}^+$ -frames $\mathcal{F}_{g_1} = \langle X_1, \leq_1, R_1 \rangle$ and $\mathcal{F}_{g_2} = \langle X_2, \leq_2, R_2 \rangle$ is a map $f : X_1 \rightarrow X_2$ that verifies **P1** and **P2** of the definition above.

THEOREM 16. Let $\mathcal{F}_g$ be a general $\mathcal{I}^+$ -frame. Then, $\mathcal{F}_g$ is descriptive if and only if $\mathcal{F}_g \cong \mathcal{F}_g(A(\mathcal{F}_g))$.

PROOF. The proof is straight forward. ■

THEOREM 17. Let $f : \mathcal{F}_{g_1} \rightarrow \mathcal{F}_{g_2}$ be an $\mathcal{I}^+$ -morphism. Then the map $A(f) : A(\mathcal{F}_{g_2}) \rightarrow A(\mathcal{F}_{g_1})$ defined by $A(f)(U) = f^{-1}(U)$, for every $U \in D_2$, is a homomorphism of $\mathcal{H}^+$ -algebras.

PROOF. It is clear that $A(f)$ is a homomorphism of Heyting algebras. It remains to show that $f^{-1}(U^*) = (f^{-1}(U))^*$, for every $U \in D_2$. Suppose that $x \in f^{-1}(U^*)$, i.e., there exist $z \in X_2$ such that $(f(x), z) \in R_2$ and $z \notin U$. By the condition **P2**, there exist $y \in X_1$ such that $(x, y) \in R_1$ and $f(y) \leq_2 z$. Since $U \in \mathcal{P}_i(X_2)$, $y \notin f^{-1}(U)$, i.e., $R_1(x) \subseteq f^{-1}(U)$. Thus, $x \in (f^{-1}(U))^*$.

The converse inclusion follows by the condition **P1**. ■

THEOREM 18. Let $h : A_1 \rightarrow A_2$ be a homomorphism of $\mathcal{H}^+$ -algebras. Then the map $\mathcal{F}(h) : \mathcal{F}_g(A_2) \rightarrow \mathcal{F}_g(A_1)$ given by $\mathcal{F}(h)(P) = h^{-1}(P)$, for every $P \in Fp(A_2)$, is an $\mathcal{I}^+$ -p-morphism.

PROOF. It is known that $\mathcal{F}(h)$ is a p-morphism. We show that $\mathcal{F}(h)$ verifies the condition **P2**. Let $P \in Fp(A_1)$, $Q \in Fp(A_2)$ such that $(h^{-1}(P), Q) \in R_{A_2}$. Let us consider the ideal $I(h(Q^c))$. We prove that

$$g(P) \cap I(h(Q^c)) = \emptyset. \quad (1)$$

Suppose that (1) does not hold. Then there exists $p \in g(P)$ and $q \notin Q$ such that

$$h(q^+) = h(q)^+ \leq p^+.$$

Since $p^+ \notin P$, we get $h(q^+) \notin P$, i.e., $q \in g(h^{-1}(P)) \subseteq Q$, which is absurd.

Thus, (1) holds. Then there exists $D \in Fp(A_1)$ such that $g(P) \subseteq D$ and $h(Q^c) \cap D = \emptyset$. This implies that $(P, D) \in R_{A_1}$ and $h^{-1}(D) \subseteq Q$. ■

The following result gives an alternative characterization of Condition 2 of Definition 13 in topological terms.

PROPOSITION 19. *Let $\mathcal{F}_g = \langle X, \leq, R, D \rangle$ be a general $\mathcal{I}^+$ -frame such that $\langle X, \leq, D \rangle$ is descriptive. Then the following conditions are equivalent:*

1. *For all $x, y \in X$, if $(F_X(x), F_X(y)) \in R_D$, then $(x, y) \in R$.*
2. *For all $x \in X$, $R(x) \in \mathcal{C}_i(X)$.*

PROOF. $1 \Rightarrow 2$. Let $y \leq z$ and $(x, y) \in R$. Then, $F_X(y) \subseteq F_X(z)$ and $(F_X(x), F_X(y)) \in R_D$. It is easy to see that $(F_X(x), F_X(y)) \in R_D$. Thus, $(x, y) \in R$, i.e., $R(x)$ is an increasing subset of X .

Let $y \in \overline{R(x)}$, where $\overline{R(x)}$ is the closure of $R(x)$. Let us suppose that $y \notin R(x)$. Then $(F_X(x), F_X(y)) \notin R_D$. So, there exists $O \in D$ such that $R(x) \subseteq O$ and $y \notin O$. Then, $R(x) \cap O^c = \emptyset$ and $y \in O^c$, which is a contradiction, because $y \in \overline{R(x)}$. Therefore, $R(x)$ is a closed subset of X .

$2 \Rightarrow 1$. It is easy and left to the reader. ■

5. Some subvarieties of $\mathcal{H}^+$

Now, we shall characterize some subvarieties of $\mathcal{H}^+$ by means of the first-order conditions defined on the frame associated to the algebras.

Let $\mathcal{F} = \langle X, \leq, R \rangle$ be a frame. Define R^n inductively as follows: R^0 is the identity $=$, and $R^{n+1} = R \circ R^n$. For the relation $R_\square = \leq \circ R$ define $R_\square^0 = \leq \circ R^0 = \leq$, and $R_\square^{n+1} = R_\square \circ R_\square^n$.

LEMMA 20. *Let $A \in \mathcal{H}^+$. Then for all $n \geq 1$*

$$\Box^n a \in P \Leftrightarrow \forall Q \in R_\square^n(P), \quad a \in Q. \quad (2)$$

PROOF. The proof is by induction on n . The case $n = 1$ is Assertion 5 of Lemma 9. Suppose that (2) is valid for all $k \leq n$. We prove that (2) is valid for $n + 1$. Let us assume that $\Box^{n+1}a = \Box\Box^n a \notin P$. Then there exists a prime filter D such that $(P, D) \in R_\square$ and $\Box^n a \notin D$. By inductive hypothesis, there exists a prime filter Q such that $(D, Q) \in R_\square^n$ and $a \notin Q$. Thus, $(P, Q) \in R_\square^{n+1}$ and $a \notin Q$. The other direction is easy and left to the reader. ■

We shall say that a relation R on a set X is a *weakly n -transitive* relation if and only if for any $x, y \in X$, if $(x, y) \in R^{n+1}$ then there exists $j \leq n$ such that $(x, y) \in R^j$.

THEOREM 21. *Let $A \in \mathcal{H}^+$. Then*

1. $A \models \neg\varphi \rightarrow \varphi^+ \Leftrightarrow R_A(P) \cap [P] \neq \emptyset$, for every $P \in Fp(A)$.
2. $A \models \varphi^{++} \rightarrow \varphi \Leftrightarrow R_A$ is symmetrical.
3. $A \models \neg(\varphi^+ \wedge \varphi^{++}) \approx 0 \Leftrightarrow R_A$ is euclidean.
4. $A \models \varphi \vee \varphi^+ \approx 1 \Leftrightarrow R_A$ is reflexive.
5. $A \models t_n(\varphi) \rightarrow \Box^{n+1}(\varphi) \Leftrightarrow R_\Box$ is weakly n -transitive.

PROOF. 1. $\Rightarrow$) Let $P \in Fp(A)$. We prove that the filter $F(P \cup g(P))$ is proper. If we suppose the contrary, then there exists $p \in P$ and $q \in g(P)$ such that $p \wedge q = 0$. This implies that $p \leq \neg q \leq q^+ \in P$, i.e., $q \notin g(P)$, which is absurd.

$\Leftarrow$) Suppose that $\neg a \not\leq a^+$ for some $a \in A$. Then there exists $P \in Fp(A)$ such that $\neg a \in P$ and $a \in g(P)$. Since $R_A(P) \cap [P] \neq \emptyset$ then there exists $Q \in Fp(A)$ such that $g(P) \subseteq Q$ and $P \subseteq Q$. So, $\neg a \in Q$ and $a \in Q$, which implies that $\neg a \wedge a = 0 \in Q$, which is a contradiction.

2. $\Rightarrow$) Let $P, Q \in Fp(A)$ such that $g(P) \subseteq Q$ and $g(Q) \subsetneq P$. Then, there exists $q^+ \notin Q$ and $q \in P$. So, $q^{++} \notin P$. But this implies that $q^+ \in g(P) \subseteq Q$, which is a contradiction.

$\Leftarrow$) It is easy.

3. $\Rightarrow$) Let $P, Q, D \in Fp(A)$ such that $g(P) \subseteq Q$, $g(P) \subseteq D$ and $g(Q) \subsetneq D$. Then, there exists $q \in A$ such that $q^+ \notin Q$ and $q \notin D$. Then, $q^{++} \in P$ and $q^+ \in P$. So, $q^{++} \wedge q^+ \in P$, which is a contradiction.

$\Leftarrow$) It is easy.

5. $\Rightarrow$) Let $P, Q \in Fp(A)$ such that $(P, Q) \in R_\Box^{n+1}$. Suppose that for all $j \leq n$, $(P, Q) \notin R_\Box^j$. Then, there exist $p_0, p_1, \dots, p_n \in A$ such that $p_0 \in P$ and $p_0 \notin Q$, $\Box p_1 \in P$ and $p_1 \notin Q$, $\dots$, $\Box^n p_n \in P$ and $p_n \notin Q$. Since $\Box$ is a modal operator it is easy to check that for all $j \leq n$, $p_0 \wedge \Box p_1 \wedge \dots \wedge \Box^n p_n \leq \Box^j(p_0 \vee p_1 \vee \dots \vee p_n)$. Since $A \models t_n(\varphi) \rightarrow \Box^{n+1}(\varphi)$, then $p_0 \wedge \Box p_1 \wedge \dots \wedge \Box^n p_n \leq \Box^{n+1}(p_0 \vee p_1 \vee \dots \vee p_n) \in P$. Since, $(P, Q) \in R_\Box^{n+1}$, $p_0 \vee p_1 \vee \dots \vee p_n \in Q$, which is absurd. The other direction is easy.

The assertion 4. is easy and left to the reader. ■

5.1. Correspondence results

Now we give some correspondence results, i.e., results on the correspondence between valid formulas in frames and properties of the frames. Also, we shall deduce that the extension of the logic $\mathcal{I}^+$ by means of these formulas are d-persistent, and thus are canonical.

Let $\langle \mathcal{F}, V \rangle = \langle X, \leq, R, V \rangle$ be a model. Let us consider the relations $R^+ = R \circ \leq$ and $R'_\Box = \leq \circ R \circ \leq$.

Let $A \in \mathcal{H}^+$ -algebra. It is easy to see that $\subseteq \circ R_A \circ \subseteq = \subseteq \circ R_A$ and that $R_A = R_A \circ \subseteq$.

LEMMA 22. *Let $\langle \mathcal{F}, V \rangle$ be a model.*

1. *For each $x \in X$, $R^+(x), R'_\square(x) \in \mathcal{P}_i(X)$.*
2. *For each $\varphi \in Fm$,*
 - (a) $V(\varphi^+) = \{x \in X : R^+(x) \subsetneq V(\varphi)\}$.
 - (b) $V(\Box\varphi) = \{x \in X : R'_\square(x) \subseteq V(\varphi)\}$.

PROOF. 1. Let $y \leq z$ and $y \in R^+(x)$. Then there exists $w \in X$ such that $(x, w) \in R$ and $w \leq y$. Since $w \leq y \leq z$, then $z \in R^+(x)$.

The proof of that $R'_\square(x) \in \mathcal{P}_i(X)$ is similar.

2. a) Let $x \in V(\varphi^+)$. Then $R(x) \subsetneq V(\varphi)$. Since $R(x) \subseteq R^+(x)$, then $R^+(x) \subsetneq V(\varphi)$.

Suppose that $R^+(x) \subsetneq V(\varphi)$. Then there exist $y, z \in X$ such that $(x, y) \in R$, $y \leq z$, and $z \notin V(\varphi)$. Since $V(\varphi)$ is increasing, $y \notin V(\varphi)$, i.e., $x \in V(\varphi^+)$.

b) Let $x \in V(\Box\varphi)$ and $(x, z) \in R'_\square$. Then $[x] \cap V(\varphi^+) = \emptyset$ and there exists $y \in X$ such that $x \leq y$ and $(y, z) \in R^+$. Then $y \notin V(\varphi^+)$, and by a) it follows that $z \in V(\varphi)$, i.e., $R'_\square(x) \subseteq V(\varphi)$.

Suppose that $R'_\square(x) \subseteq V(\varphi)$ and suppose that $x \notin V(\Box\varphi)$. Then, $[x] \cap V(\varphi^+) \neq \emptyset$. So, there exist $y, z \in X$ such that $x \leq y$, $(y, z) \in R^+$ and $z \notin V(\varphi)$. Then, $(x, z) \in R'_\square$, and by assumption $y \in V(\varphi)$, which is a contradiction. Thus, $x \in V(\Box\varphi)$. ■

THEOREM 23. *Let $\mathcal{F}$ be an $\mathcal{I}^+$ -frame. Then*

1. $\mathcal{F} \models \neg\varphi \rightarrow \varphi^+ \Leftrightarrow R^+(x) \cap [x] \neq \emptyset$.
2. $\mathcal{F} \models \varphi \vee \varphi^+ \Leftrightarrow R$ is reflexive.
3. $\mathcal{F} \models \varphi^{++} \rightarrow \varphi \Leftrightarrow R^+$ is symmetrical.
4. $\mathcal{F} \models \neg(\varphi^+ \wedge \varphi^{++}) \Leftrightarrow R^+$ is euclidean.
5. $\mathcal{F} \models t_n(\varphi) \rightarrow \Box^{n+1}\varphi \Leftrightarrow R'_\square$ is weakly n -transitive.

PROOF. 1. Assume that $\mathcal{F} \models \neg\varphi \rightarrow \varphi^+$. Let $x \in X$ and let V be the valuation defined by $V(p) = R^+(x)$. Suppose that $R^+(x) \cap [x] = \emptyset$. Then $[x] \cap V(p) = \emptyset$, i.e., $x \in V(\neg p) \subseteq V(p^+)$. It follows that $R^+(x) \subsetneq V(p)$, which is a contradiction.

Suppose that $R^+(x) \cap [x] \neq \emptyset$ for all $x \in X$. Let $x \in V(\neg p)$. Then, $[x] \cap V(p) = \emptyset$. By hypothesis, there exists $y \in X$ such that $y \in R(x)$ and $x \leq y$. This implies that $y \notin V(p)$, i.e., $x \in V(p^+)$.

4. Assume that $\mathcal{F} \models \neg(\varphi^+ \wedge \varphi^{++})$. Let $x, y, z \in X$ such that $(x, y) \in R^+$, $(x, z) \in R^+$. Let us suppose that $(y, z) \notin R^+$. Let us consider the valuation

V defined by $V(p) = R^+(y)$. Then, $z \notin V(p)$, and since $(x, z) \in R^+$, we get $x \in V(p^+)$. Since $y \notin V(p^+)$ and $(x, y) \in R^+$, then $x \in V(p^{++})$. So, $x \in V(p^+ \wedge p^{++})$. Since $\leq$ is reflexive, $[x] \cap V(p^+ \wedge p^{++}) \neq \emptyset$, which is a contradiction.

Reciprocally. Suppose that $[x] \cap V(p^+ \wedge p^{++}) \neq \emptyset$. Then there exists $y \in X$ such that $x \leq y$, $y \in V(p^+)$, and $y \in V(p^{++})$. This implies that there exists $z, k \in X$ such that $(y, z) \in R^+$, $(y, k) \in R^+$, $z \notin V(p)$, and $k \notin V(p^+)$. So, since $(k, z) \in R^+$ and $R^+(k) \subseteq V(p)$, $z \in V(p)$, which is a contradiction.

5. Assume that $\mathcal{F} \models t_n \varphi \rightarrow \Box^{n+1} \varphi$ and let $a, b \in X$ such that $(a, b) \in R_{\Box}^{n+1}$. Let us consider the function $V(p) = \{x \in X : \text{there exists } j \leq n \text{ such that } (a, x) \in (R'_{\Box})^j\}$, with $p \in \text{Var}$. By Lemma 22, $V(p) \in \mathcal{P}_i(X)$ and consequently V is a valuation. Since $a \in V(t_n(p))$, then $a \in V(\Box^{n+1} p)$. It follows $b \in V(p)$, i.e., there exists $j \leq n$ such that $(a, b) \in (R'_{\Box})^j$. The other direction is easy. ■

An $\mathcal{I}^+$ -logic $\mathcal{I}$ is said to be *d-persistent* if when $\langle \mathcal{F}, D \rangle$ is a general descriptive $\mathcal{I}^+$ -frame such that $\langle \mathcal{F}, D \rangle \models \mathcal{I}$ then $\mathcal{F} \models \mathcal{I}$. It is clear that any d-persistent logic is canonical.

Let $\mathcal{I}$ be any extension of $\mathcal{I}^+$ by means of any subsets of the following sets of formulas

$$\{\neg \varphi \rightarrow \varphi^+, \varphi \vee \varphi^+, \varphi^{++} \rightarrow \varphi, \neg(\varphi^+ \wedge \varphi^{++}), t_n(\varphi) \rightarrow \Box^{n+1} \varphi\}.$$

THEOREM 24. *The logic $\mathcal{I}$ is d-persistent and frame-complete.*

PROOF. It follows by Theorems 23 and 21. ■

6. Interpolation Property

The Interpolation Property (**IP**) is a known property of numerous logics. For example, the classical logic and the intuitionistic logic has this property. Now we shall prove that this property also holds for some canonical extensions of $\mathcal{I}^+$. For this purpose, we shall introduce an adequate notion of bisimulation for the language $\mathcal{L}^+$. We shall adapt the techniques given in [7], where it is proved by means of bisimulations, that any canonical extension of the modal logic **K** such that its class of all frames is closed under bisimulation-product has the **IP**.

We shall say that an $\mathcal{I}^+$ -logic $\mathcal{I}$ has the Interpolation Property if for any $\varphi, \psi \in Fm$ whenever $\vdash_{\mathcal{I}} \varphi \rightarrow \psi$ there exists $\alpha \in Fm$ such that $\vdash_{\mathcal{I}} \varphi \rightarrow \alpha$, $\vdash_{\mathcal{I}} \alpha \rightarrow \psi$, and $\text{Var}(\alpha) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$.

The following definition is a modification of the definition given in [6].

DEFINITION 25. Let $\mathcal{F}_1 = \langle X_1, \leq_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle X_2, \leq_2, R_2 \rangle$ be two $\mathcal{I}^+$ -frames. A bisimulation between $\mathcal{F}_1$ and $\mathcal{F}_2$ is a pair of subsets B_1, B_2 of $X_1 \times X_2$,

in symbols $B = (B_0, B_1)$, such that the following conditions are verified:

- B1** If $(a, b) \in B_0$ and $b \leq_2 y$, then there exists $x \in X_1$ such that $a \leq_1 x$ and $(x, y) \in B_0 \cap B_1$.
- B2** If $(a, b) \in B_1$ and $a \leq_1 x$, then there exists $y \in X_2$ such that $b \leq_2 y$ and $(x, y) \in B_0 \cap B_1$.
- B3** If $(a, b) \in B_0$ and $(a, x) \in R_1$, then there exists $y \in X_2$ such that $(b, y) \in R_2$ and $(x, y) \in B_0 \cap B_1$.
- B4** If $(a, b) \in B_1$ and $(b, y) \in R_2$, then there exists $x \in X_1$ such that $(a, x) \in R_1$ and $(x, y) \in B_0 \cap B_1$.

Let $B = (B_0, B_1)$ be a bisimulation between the frames $\mathcal{F}_1 = \langle X_1, \leq_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle X_2, \leq_2, R_2 \rangle$. We shall say that $B = (B_0, B_1)$ is *total* if $\text{dom}(B_0 \cap B_1) = X_1$ and $\text{rang}(B_0 \cap B_1) = X_2$.

We note that an $\mathcal{I}^+$ -morphism is a bisimulation.

DEFINITION 26. Let $\mathcal{M}_1 = \langle \mathcal{F}_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2 \rangle$ be two $\mathcal{I}^+$ -models. A bisimulation between the $\mathcal{I}^+$ -models $\mathcal{M}_1$ and $\mathcal{M}_2$ is a bisimulation $B = (B_0, B_1)$ between the $\mathcal{I}^+$ -frames $\mathcal{F}_1$ and $\mathcal{F}_2$ such that:

- V1** If $(a, b) \in B_0$ and $a \in V_1(p)$, then $b \in V_2(p)$, for any $p \in \text{Var}$.
- V2** If $(a, b) \in B_1$ and $b \in V_2(p)$, then $a \in V_1(p)$, for any $p \in \text{Var}$.

LEMMA 27. Let $B = (B_0, B_1)$ be a bisimulation between the models $\mathcal{M}_1$ and $\mathcal{M}_2$. For all $a \in X_1$, $b \in X_2$, and for all $\varphi \in \text{Fm}$:

1. If $(a, b) \in B_0$ and $a \in V_1(\varphi)$, then $b \in V_2(\varphi)$.
2. If $(a, b) \in B_1$ and $b \in V_2(\varphi)$, then $a \in V_1(\varphi)$.

PROOF. The proof is by induction on the complexity of the formulas. We treat only the case $\varphi = \alpha^+$. For the intuitionistic connectives see [6].

1. Suppose that $(a, b) \in B_0$ and $a \in V_1(\alpha^+)$. Then there exists $x \in X_1$ such that $(a, x) \in R_1$ and $x \notin V_1(\alpha)$. By the condition **B3**, there exists $y \in X_2$ such that $(b, y) \in R_2$ and $(x, y) \in B_0 \cap B_1$. As $x \notin V_1(\alpha)$, we get $y \notin V_2(\alpha)$. Thus, $b \in V_2(\alpha^+)$.

The proof of 2, is similar and left to the reader. ■

DEFINITION 28. Let $\mathcal{F}_1 = \langle X_1, \leq_1, R_1 \rangle$ and $\mathcal{F}_2 = \langle X_2, \leq_2, R_2 \rangle$ be two $\mathcal{I}^+$ -frames and let $B = (B_0, B_1)$ be a total bisimulation between them. The structure $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle$, where

$$X_B = \{(x, y) \in X_1 \times X_2 : (x, y) \in B_0 \cap B_1\},$$

$$(x, y) \leq_B (x', y') \Leftrightarrow x \leq_1 x' \text{ and } y \leq_2 y',$$

and

$$(x, y) R_B (x', y') \Leftrightarrow x R_1 x' \text{ and } y R_2 y',$$

is called the **bisimulation-product** of $\mathcal{F}_1$ and $\mathcal{F}_2$.

It is clear that every bisimulation-product of $\mathcal{I}^+$ -frames is an $\mathcal{I}^+$ -frame.

LEMMA 29. *Let B be a total bisimulation between the $\mathcal{I}^+$ -frames $\mathcal{F}_1$ and $\mathcal{F}_2$. Then the bisimulation-product $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle$ is a subdirected-product and the projections are $\mathcal{I}^+$ -morphisms.*

PROOF. By definition $B = (B_0, B_1)$ is total, so the projections π_i are surjective. We prove that the projection π_1 is an $\mathcal{I}^+$ -morphism. The condition **P1** is immediate. Let $x_1, y_1 \in X_1$ and $x_2 \in X_2$ such that $(x_1, x_2) \in X_B = B_0 \cap B_1$ and $\pi_1(x_1, y_1) = x_1 R y_1$. Since $(x_1, x_2) \in B_0 \cap B_1$, $(x_1, y_1) \in R$, and B is a bisimulation, there exists $y_2 \in X_2$ such that $(x_2, y_2) \in S$ and $(y_1, y_2) \in B_0 \cap B_1$. So, $(x_1, y_1) R^B (x_2, y_2)$ and $\pi_1(y_1, y_2) = y_1$. ■

DEFINITION 30. Let $\mathbf{K}$ be a class of frames. We shall say $\mathbf{K}$ is closed under bisimulation-products if for every $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{K}$ and for every total bisimulation $B = (B_0, B_1)$ between $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle \in \mathbf{K}$.

LEMMA 31. *The class $\mathbf{F}^+$ of all $\mathcal{I}^+$ -frames is closed under bisimulation-products.*

PROOF. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}^+$. Given a bisimulation between $\mathcal{F}_1$ and $\mathcal{F}_2$, the relational structure $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle$ is also an $\mathcal{I}^+$ -frame. Then, $\mathcal{F}_B \in \mathbf{F}^+$. ■

Let $\varphi, \psi \in Fm$. We denote by $Fm(\varphi)$ and $Fm(\varphi, \psi)$ the set of all formulas generated by $Var(\varphi)$ and $Var(\varphi) \cap Var(\psi)$, respectively. We denote by A_φ the Lindenbaum-Tarski algebra of $Fm(\varphi)$. Let $\mathcal{F}_\varphi = \langle Fp(A_\varphi), \leq_\varphi, R_\varphi \rangle$ be the $\mathcal{I}^+$ -frame associated to the algebra A_φ . Let us consider the $\mathcal{I}^+$ -model $\mathcal{M}_\varphi = \langle \mathcal{F}_\varphi, V_\varphi \rangle$ on $\mathcal{F}_\varphi$ where the valuation V_φ is defined by

$$V_\varphi(a) = \{P \in Fp(A_\varphi) : a \in P\}.$$

This model we will be called the *canonical model* over A_φ .

THEOREM 32. *Let $\varphi, \psi \in Fm$ and let A_1, A_2 and A_0 be the Lindenbaum-Tarski algebras on the sets $Var(\varphi)$, $Var(\psi)$ and $Var(\varphi) \cap Var(\psi)$, respectively. Let us consider the subsets $B_0, B_1 \subseteq Fp(A_1) \times Fp(A_2)$ defined by*

$$B_0 = \{(P, Q) \in Fp(A_1) \times Fp(A_2) : P \cap A_0 \subseteq Q\},$$

and

$$B_1 = \{(P, Q) \in Fp(A_1) \times Fp(A_2) : Q \cap A_0 \subseteq P\},$$

respectively. Then $B = (B_0, B_1)$ is a total bisimulation between the $\mathcal{I}^+$ -frames $\mathcal{F}(A_1)$ and $\mathcal{F}(A_2)$.

PROOF. We prove that B is total. Let $Q \in Fp(A_2)$ and let us consider the filter in A_1 , $F(Q \cap A_0)$, and the ideal $I(Q^c \cap A_0)$. As, $F(Q \cap A_0) \cap I(Q^c \cap A_0) = \emptyset$, then there exists a prime filter $P \in Fp(A_1)$ such that

$$Q \cap A_0 \subseteq P \text{ and } A_0 \cap P \subseteq Q.$$

This implies that $(P, Q) \in B_0 \cap B_1$. A similar argument shows that if $P \in Fp(A_1)$ then there exists $Q \in Fp(A_2)$ such that $(P, Q) \in B_0 \cap B_1$. Thus $B = (B_0, B_1)$ is total.

We prove that B verifies the condition **B1** of Definition 25. Let $P \in Fp(A_1)$ and $Q, D \in Fp(A_2)$ such that $P \cap A_0 \subseteq Q$ and $Q \subseteq D$. Let T be the filter generated by $P \cup \{D \cap A_0\}$ in A_1 , and let I be the ideal generated by the set $\Delta = D^c \cap A_0$ in A_1 . Then

$$T \cap I = \emptyset,$$

because in the contrary case there exists $p \in P$, $d \in D \cap A_0$, and $q \in \Delta$ such that $p \wedge d \leq q$. So, we have $p \leq d \rightarrow q \in P \cap A_0 \subseteq Q \subseteq D$. Then, $q \in D$, which is a contradiction. Thus, $T \cap I = \emptyset$. Then, by the Prime Filter theorem, there exists $Z \in Fp(A_1)$ such that

$$P \subseteq Z \text{ and } D \cap A_0 = Z \cap A_0,$$

i.e., $P \subseteq Z$ and $(Z, D) \in B_0 \cap B_1$.

We prove **B4**. Let $P \in Fp(A_1)$, $Q, Z \in Fp(A_2)$ such that $Q \cap A_0 \subseteq P$ and $g(Q) \subseteq Z$. Let us consider in A_1 the filter $F(g(P) \cup (Z \cap A_0))$ and the ideal $I(Z^c \cap A_0)$. We prove that

$$F(g(P) \cup (Z \cap A_0)) \cap I(Z^c \cap A_0) = \emptyset.$$

Suppose the contrary. Then, there exists $p \in g(P)$, $z \in Z \cap A_0$, and $a \in Z^c \cap A_0$ such that $p \wedge z \leq a$. Thus, $p \leq z \rightarrow a$. So, $(z \rightarrow a)^+ \leq p^+ \notin P$. Since $z \rightarrow a \in A_0$, $(z \rightarrow a)^+ \notin A_0 \cap Q$, i.e., $z \rightarrow a \in g(Q) \subseteq Z$. Thus, $z \wedge (z \rightarrow a) \leq a \in Z$, which is a contradiction. Therefore, there exists $D \in Fp(A_1)$ such that

$$g(P) \subseteq D \text{ and } D \cap A_0 = Z \cap A_0.$$

The condition **B2** is similar to **B1** and **B3** is similar to **B4**. $\blacksquare$

The following result is known (see for example [3]), but we give a short proof for the sake of completeness.

LEMMA 33. *Let A_0, A_1 and A_2 be Heyting algebras such that A_0 is a subalgebra of A_1 and A_2 . Suppose that for all $a \in A_1$ and for all $b \in A_2$ there exists no $c \in A_0$ such that $a \leq_1 c$ or $c \leq_2 b$. Then there exists $P \in Fp(A_1)$, $Q \in Fp(A_2)$ such that*

$$a \in P, b \notin Q \text{ and } P \cap A_0 = Q \cap A_0.$$

PROOF. Let $F(a)$ be the filter generated by a in A_1 , $I(b)$ the ideal generated by b in A_2 , and let $F(F(a) \cap A_0)$ be the filter generated by $F(a) \cap A_0$ in A_2 . By the hypothesis, we have that

$$F(F(a) \cap A_0) \cap I(b) = \emptyset.$$

Thus, there exists $Q \in Fp(A_2)$ such that $b \notin Q$ and $F(a) \cap A_0 \subseteq Q$. Let us consider in A_1 the filter $F = F(\{a\} \cup Q \cap A_0)$ and the ideal $I = I(Q^c \cap A_0)$. We prove that $F \cap I = \emptyset$. Suppose the contrary. Then there exists $q \in Q \cap A_0$ and $c \in Q^c \cap A_0$ such that $a \leq_1 q \rightarrow c$. So, $q \rightarrow_1 c \in F(a) \cap A_0 \subseteq Q$. It follows that $c \in Q$, which is absurd. Thus, there exists $P \in Fp(A_1)$ such that $a \in P$ and $P \cap A_0 = Q \cap A_0$. $\blacksquare$

THEOREM 34 (of Interpolation). *Let $\mathcal{I} = Th(\mathbf{K})$ be a canonical $\mathcal{I}^+$ -logic. If $\mathbf{K}$ is closed under bisimulation-products, then $\mathcal{I}$ has the interpolation property.*

PROOF. Suppose that there exists $\varphi, \psi \in Fm$ such that for every $\alpha \in Fm$ with $Var(\alpha) \subseteq Var(\varphi) \cup Var(\psi)$, $\Vdash_{\mathbf{K}} \varphi \rightarrow \alpha$ or $\Vdash_{\mathbf{K}} \alpha \rightarrow \psi$. Let us consider the Lindenbaum-Tarski algebras A_1, A_2 , and A_0 generated by $Var(\varphi)$, $Var(\psi)$ and $Var(\varphi) \cup Var(\psi)$, respectively. It is clear that A_0 is a subalgebra of A_1 and A_2 , and that $|\varphi| \not\leq_1 |\alpha|$ or $|\alpha| \not\leq_2 |\psi|$ for any $|\alpha| \in A_0$. By the above Lemma, there exists $P \in Fp(A_1)$ and $Q \in Fp(A_2)$ such that

$$|\varphi| \in P, |\psi| \notin Q \text{ and } P \cap A_0 = Q \cap A_0. \quad (3)$$

Let us consider the total bisimulation $B = (B_0, B_1)$ between $\mathcal{F}(A_1)$ and $\mathcal{F}(A_2)$ defined in Theorem 32. Let us consider the frame $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle$, where we recall that

$$X_B = \{(P, Q) \in Fp(A_1) \times Fp(A_2) : P \cap A_0 = Q \cap A_0\}.$$

Note that by (3) $X_B \neq \emptyset$. Define on $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle$ the map

$$V_B : Var(\varphi) \cup Var(\psi) \rightarrow \mathcal{P}_i(X_B)$$

by

$$V_B(p) = \{(P, Q) \in X_B : P \in V_\varphi(p) \text{ or } Q \in V_\psi(p)\},$$

where $V_\varphi(p) = \{P \in Fp(A_1) : |p| \in P\}$ and $V_\psi(p) = \{P \in Fp(A_2) : |p| \in P\}$. It is clear that V_B is well defined because if $(P, Q) \in V_B(p)$ and $p \in Fm(\varphi) \cap Fm(\psi)$, then

$$P \in V_\varphi(p) \text{ if and only if } Q \in V_\psi(p).$$

This condition is extended to the set of all formulas because $B = (B_0, B_1)$ by (32) it is a total bisimulation between $\mathcal{F}(A_1)$ and $\mathcal{F}(A_2)$. Thus, V_B is really a valuation defined on $\mathcal{F}_B$.

Since $Th(\mathbf{K})$ is canonical, then $\mathcal{F}_\varphi, \mathcal{F}_\psi \in \mathbf{K}$, and as $\mathbf{K}$ is closed under bisimulation-products, $\mathcal{F}_B = \langle X_B, \leq_B, R_B \rangle \in \mathbf{K}$. But, by (3), there exists $(P, Q) \in X_B$ such that

$$\langle \mathcal{F}_B, V_B \rangle \models_{(P, Q)} \varphi \text{ and } \langle \mathcal{F}_B, V_B \rangle \not\models_{(P, Q)} \psi,$$

i.e., $\langle \mathcal{F}_B, V_B \rangle \not\models_{(P, Q)} \varphi \rightarrow \psi$. Thus, $\not\models_{\mathbf{K}} \varphi \rightarrow \psi$. ■

We note that as the logic $\mathcal{I}^+$ is canonical and by the Lemma 31 the class $\mathbf{F}^+$ of all frames of $\mathcal{I}^+$ is closed under bisimulation-products, then $\mathcal{I}^+$ has the interpolation property. We also note that by the Theorem 23 the class $\mathbf{F}_r^+$ of all frames of the logic $\mathcal{I}_r^+ = \mathcal{I}^+ + \{\varphi \vee \varphi^+\}$ is the class of all frames $\mathcal{F} = \langle X, \leq, R \rangle$ where the relation R is reflexive. It is easy to see that the class $\mathbf{F}_r^+$ is closed under bisimulation-products, and since $\mathcal{I}_r^+$ is canonical, then has the interpolation property. It remains to study if the class of frames of the other canonical logics presented in this paper are closed under bisimulation-products.

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