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(Citation)
Kobe Journal of Mathematics,18(1):23-49

(Issue Date)
2001-06

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0026795

(URL)
https://hdl.handle.net/20.500.14094/E0026795

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KOBE J. MATH.,
18 (2001) 23—49

THE YAMADA POLYNOMIAL OF SPATIAL GRAPHS WITH $\mathbb{Z}_n$-SYMMETRY

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(Received February 4, 2000)
(Revised April 6, 2001)

1. Introduction

Throughout this paper, we work in the piecewise-linear category. Denote by $\Lambda = \mathbb{Z}[A, A^{-1}]$ the Laurent polynomial ring in the indeterminate A over $\mathbb{Z}$ and put $\sigma = A + 1 + A^{-1} \in \Lambda$. Let G be a graph, i.e., a 1-dimensional simplicial complex, whose maximal degree is less than 4 and let $\widetilde{G}$ be a spatial graph obtained from G by embedding into the 3-dimensional Euclidean space $\mathbb{R}^3$. A diagram of $\widetilde{G}$ is defined as the image of a projection of $\widetilde{G}$ into the plane. Let g be such a diagram. Then the Yamada polynomial $\mathbf{R}(g) \in \Lambda$ is defined for g, and $\mathbf{R}(g)$ is an ambient isotopy invariant of $\widetilde{G}$ up to multiplying $(-A)^k$ for some integer k ([2]).

In this paper we consider spatial graphs with $\mathbb{Z}_n$-symmetries and with non-zero wrapping numbers. By a $\mathbb{Z}_n$-symmetry we mean a faithful $\mathbb{Z}_n$-action on $\mathbb{R}^3$ generated by a homeomorphism f of $\mathbb{R}^3$ such that $f(\widetilde{G}) = \widetilde{G}$. The wrapping number w of a spatial graph with a $\mathbb{Z}_n$-symmetry is the minimum number of intersection points of $\widetilde{G}$ and P where P runs through all half-planes bounded by the axis of the $\mathbb{Z}_n$-symmetry.

Our main theorems concern the cases when the wrapping number are 1 and 2. In the case when the wrapping number is 1, we obtain the following result.

THEOREM 3.1. *Let $n \geq 1$ be an integer. Let g be a diagram of a spatial graph with a $\mathbb{Z}_n$-symmetry and with wrapping number* 1. *Let $g/\mathbb{Z}_n$ be the quotient diagram obtained from g by the $\mathbb{Z}_n$-symmetry, then the Yamada polynomial of g is given by*

$$\mathbf{R}(g) = \frac{\mathbf{R}(g/\mathbb{Z}_n)^n}{\sigma^{n-1}}.$$

In the case when the wrapping number is 2, we realize g as the following diagram. There are n tetragons with the same shape.

g

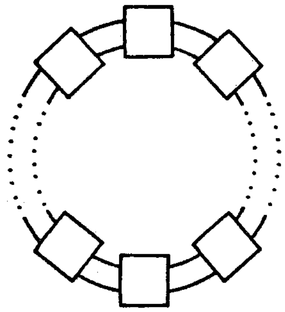
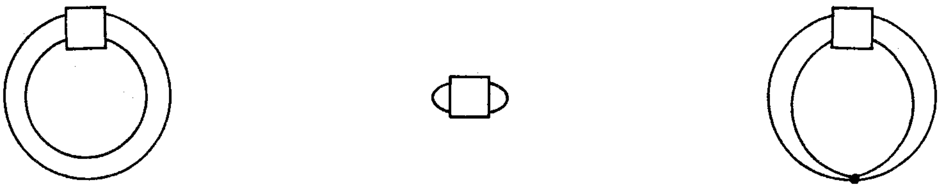


Fig. 1.

Note that the center point of the diagram is the axis of the symmetry.

We call the following three diagrams g_1, g_2, and g_3 the quotient diagrams of g.



g_1 g_2 g_3

Fig. 2.

Then, we have the following theorem.

THEOREM 3.2. *Let $n \geq 1$ be an integer. Let g be a diagram of a spatial graph with a $\mathbb{Z}_n$-symmetry and with wrapping number 2. As above, let g_1, g_2 and g_3 denote the three quotient diagrams of g. Define Δ, a, b, and c as follows*:

$$\Delta = (-1+\sigma)^2\sigma^3(-1-\sigma+\sigma^2),$$

$$a = \frac{\sigma^3 - 2\sigma^4 + \sigma^5}{\Delta}\mathbf{R}(g_1) + \frac{-\sigma^2+\sigma^3}{\Delta}\mathbf{R}(g_2) + \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_3),$$

$$b = \frac{-\sigma^2+\sigma^3}{\Delta}\mathbf{R}(g_1) + \frac{\sigma^3 - 2\sigma^4 + \sigma^5}{\Delta}\mathbf{R}(g_2) + \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_3),$$

$$c = \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_1) + \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_2) + \frac{-\sigma^2+\sigma^4}{\Delta}\mathbf{R}(g_3).$$

Then, the Yamada polynomial $\mathbf{R}(g)$ of g is given by the formula

$$\mathbf{R}(g) = (a+(b-c)\sigma)^n + \sigma(a+c-c\sigma)^n + a^n(1-\sigma-\sigma^2).$$

After checking some detailed cases, the above theorem implies the following result.

THEOREM 7.1. *Let p be an odd prime number. The notation $f(x) \equiv g(x)$ (mod p, $A^p \equiv 1$) means that $f(x) - g(x)$ is an element of the ideal generated by p and A^p. Let $\widetilde{G}$ be a spatial graph with a $\mathbb{Z}_p$-symmetry and with wrapping number 2, and g a diagram of $\widetilde{G}$. Then, for some integer k ($0 \le k < p$), we have the following form:*

$$(-A)^k\mathbf{R}(g) \equiv a_2A^2 + a_1A + a_0 + a_1A^{-1} + a_2A^{-2} \pmod{p,\ A^p \equiv 1}.$$

Note that this gives a necessary condition for a spatial graph with wrapping number 2 to have a $\mathbb{Z}_p$-symmetry. In Example 7.2, we show a certain graph cannot have a $\mathbb{Z}_5$-symmetry.

The paper is organized as follows. In Section 2, we give the definition of the Yamada polynomial and describe some of its properties. In addition, we define a $\mathbb{Z}_n$-symmetry for a spatial graph and the wrapping number. In Section 3, we restate the above two theorems after our setups. We prove those theorems in Section 4. For any wrapping number w, we define irreducible diagrams and count them in Section 5. Furthermore, we give a generating function that counts the number of irreducible diagrams in Section 6. We specialize to the cases when wrapping number 2 in Section 7, and describe, for an odd prime number p, a necessary condition for such a spatial graph to have a $\mathbb{Z}_p$-symmetry. Using the above result we give an example of a spatial graph which cannot have any $\mathbb{Z}_n$-symmetries.

The author would like to thank Prof. S. Yamada for his help during the preparation of this paper.

2. The Yamada Polynomial

All propositions in this section are given in [1] or in [2].

We give the definitions of a spatial graph and an ambient isotopy between two spatial graphs.

DEFINITION 2.1. Let G be a graph. Let $i : G \longrightarrow \mathbb{R}^3$ be an embedding of G into $\mathbb{R}^3$. Then the image $\widetilde{G} = i(G)$ is said to be a *spatial graph* of G.

DEFINITION 2.2. A diagram is defined as the image of a projection of the spatial graph $\widetilde{G}$ into the plane. It is assumed that the multiple points of the image are ordinary double points produced by transversally intersecting line segments. At each intersection, two line segments are distinguished by drawing one over-crossing segment and one under-crossing segment. We indicate the under-crossing line by breaking it at the intersection.

DEFINITION 2.3. Let X be a topological space and A and B subspaces of X. Then (X, A) is *ambient isotopic* to (X, B) in X if and only if there exists a homeomorphism $\Phi : X \times I \to X \times I$ satisfying the following conditions:

1. $\Phi(x, t) \in X \times \{t\}$,
2. $\Phi(x, 0) = (x, 0)$,
3. $\Phi(A, 1) = \Phi(B, 1)$.

In particular, for two spatial graphs $\widetilde{G_1}$ and $\widetilde{G_2}$ of the same graph G, if $(\mathbb{R}^3, \widetilde{G_1})$ is ambient isotopic to $(\mathbb{R}^3, \widetilde{G_2})$ in $\mathbb{R}^3$, then $\widetilde{G_1}$ is said to be ambient isotopic to $\widetilde{G_2}$.

DEFINITION 2.4. We define the following deformations of diagrams as in Fig. 3, which we call the *extended Reidemeister moves*.

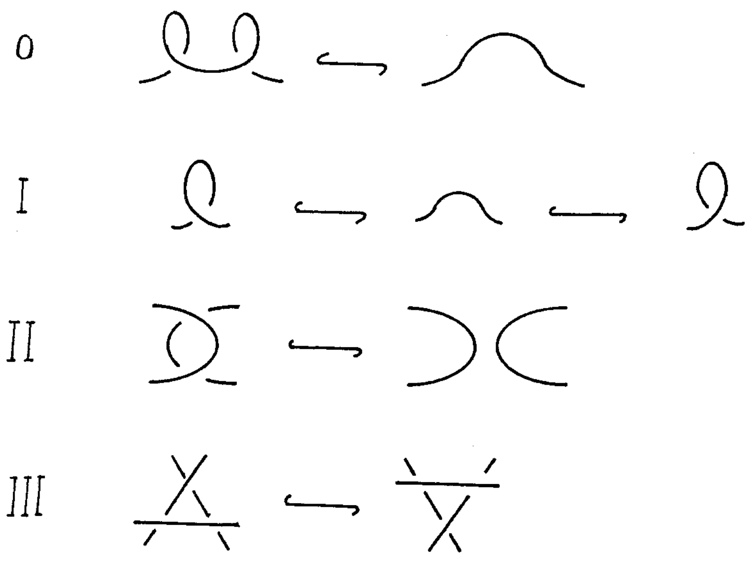


Fig. 3.

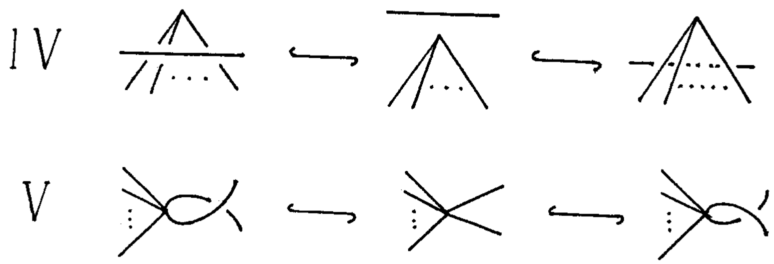


Fig. 3 (continued).

We call any deformation obtained from any combination of the deformations *I*, *II*, *III*, *IV*, and *V* an *ambient deformation.* We call any deformation obtained from any combination of the deformations *0*, *II*, *III*, and *IV* a *regular deformation.*

The following property holds for spatial graphs:

PROPOSITION 2.5 ([1]). *Let $\widetilde{G_1}$ and $\widetilde{G_2}$ be two spatial graphs of G, and g_1 and g_2 their diagrams. Then $\widetilde{G_1}$ is ambient isotopic to $\widetilde{G_2}$ in $\mathbb{R}^3$ if and only if g_1 can be transformed into g_2 by a finite sequence of extended Reidemeister moves.*

Let $G = (V, E)$ be a graph, where V is the point set of G and E the edge set. We remark that $G - e$ means that the edge e is deleted from G without deleting the end points of e. Let $b_0(G)$ and $b_1(G)$ be the 0-dimensional and 1-dimensional Betti numbers of G, respectively. Let x and y be indeterminate elements. Put

$$f(G) = f(G)(x, y) = x^{b_0(G)} y^{b_1(G)},$$

and define a 2-variable Laurent polynomial by

$$h(G) = h(G)(x, y) = \sum_{F \subset E} (-x)^{-\sharp F} f(G - F),$$

where F ranges over the family of all subsets of E, and $\sharp F$ means the number of elements of F. We define that $h(\emptyset) = 1$.

PROPOSITION 2.6 ([2]). *Let e be a nonloop edge of a graph G. Then,*

$$h(G) = h(G/e) - \frac{1}{x} h(G - e),$$

where G/e is the graph obtained from G by contracting e.

For two graphs G_1 and G_2, denote by $G_1 \amalg G_2$ the disjoint union of G_1 and G_2, and denote by $G_1 \vee G_2$ a wedge of G_1 and G_2, i.e., $G_1 \vee G_2 = G_1 \cup G_2$ and

$G_1 \cap G_2 = \{\text{one point}\}$. We say that a graph G has a *cut edge* e if the number of the connected components of $G - e$ is greater than that of G. Then the following proposition holds.

PROPOSITION 2.7 ([2]).
The polynomial $h(G)$ has the following properties:

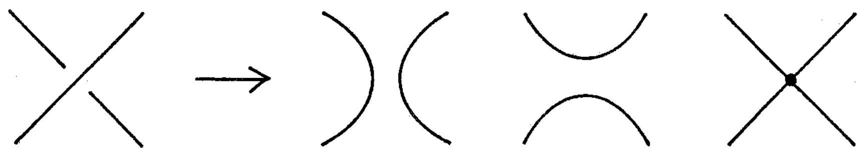
1. $h(G_1 \amalg G_2) = h(G_1)h(G_2)$,
2. $h(G_1 \vee G_2) = \frac{1}{x}\, h(G_1)h(G_2)$,
3. *if G has a cut edge, then $h(G) = 0$.*

We obtain the following theorem.

THEOREM 2.8 ([2]). *The polynomial $h(G)$ is a topological invariant of a graph G, i.e., if G is homeomorphic to a graph G', then $h(G) = h(G')$.*

We define the Yamada polynomial ([2]).

DEFINITION 2.9. Let g be a diagram of a spatial graph $\widetilde{G}$. For a crossing c of g, we define S_+, S_-, and S_0, called the *spin* of $+1$, -1, and 0, as shown in Figure 4.



C $\quad S_+ \quad S_- \quad S_0$

Fig. 4.

Let S be a plane graph obtained from g by replacing each crossing with a spin. S is called a *state* on g. Denote by $S(g)$ the set of all states on g.

Define $\{g \mid S\} = A^{p-q}$, where p and q are the number of crossings with spin of $+1$ and that of -1 in S, respectively. We define a 1-variable Laurent polynomial $\mathbf{R}(g)(A)$ as follows:

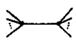
$$\mathbf{R}(g) = \mathbf{R}(g)(A) = \sum_{S \subset S(g)} \{g \mid S\} H(S)$$

where $H(S) = h(S)(-1, -A-2-A^{-1})$. We define that $\mathbf{R}(\emptyset) = 1$. We call the above $\mathbf{R}(g)$ the *Yamada polynomial*.

We obtain the following properties.

PROPOSITION 2.10 ([2]).

(1) $\mathbf{R}($  $) = A\mathbf{R}($  $) + A^{-1}\mathbf{R}($  $) + \mathbf{R}($  $)$,

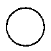
(2) $\mathbf{R}($  $) = \mathbf{R}($  $) + \mathbf{R}($  $)$,

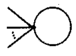
(3) $\mathbf{R}(g_1 \amalg g_2) = \mathbf{R}(g_1)\mathbf{R}(g_2)$,

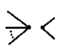
(4) $\mathbf{R}(g_1 \vee g_2) = -\mathbf{R}(g_1)\mathbf{R}(g_2)$,

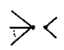
(5) *if g has a cut edge, then* $\mathbf{R}(g) = 0$.

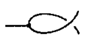
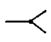
PROPOSITION 2.11 ([2]). *Let $\sigma = A + 1 + A^{-1}$, then we have*

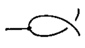
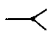
(1) $\mathbf{R}($  $) = \sigma$,

(2) $\mathbf{R}($  $) = -\sigma\mathbf{R}($  $)$,

(3) $\mathbf{R}($  $) = -A\mathbf{R}($  $) - (A^2 + A)\mathbf{R}($  $)$,

(4) $\mathbf{R}($  $) = -A^{-1}\mathbf{R}($  $) - (A^{-2} + A^{-1})\mathbf{R}($  $)$,

(5) $\mathbf{R}($  $) = -A\mathbf{R}($  $)$,

(6) $\mathbf{R}($  $) = -A^{-1}\mathbf{R}($  $)$,

(7) $\mathbf{R}($  $) = A^2\mathbf{R}($  $)$,

(8) $\mathbf{R}($  $) = A^{-2}\mathbf{R}($  $)$.

PROPOSITION 2.12 ([2]). $\mathbf{R}(g)$ *is an invariant under any regular deformation of the diagram g.*

We define a $\mathbb{Z}_n$-symmetry for a spatial graph and the wrapping numbers.

DEFINITION 2.13. Let $\widetilde{G}$ be a spatial graph. We say $\widetilde{G}$ *has a $\mathbb{Z}_n$-symmetry* if there exists a homeomorphism $r : (\mathbb{R}^3, \widetilde{G}) \longrightarrow (\mathbb{R}^3, \widetilde{G})$ such that $r^n = id$ and $r^m \neq id$ for m $(1 \leq m \leq n-1)$.

Since the Smith conjecture holds in the piecewise-linear category, this homeomorphism r is equivalent to a rotation r_0 centered at one axis, i.e., there exists a homeomorphism f such that $r = f^{-1} \circ r_0 \circ f$. Hence we may assume that the transformation r of the $\mathbb{Z}_n$-symmetry is a $\frac{2\pi}{n}$-rotation.

From now on, a $\mathbb{Z}_n$-symmetry means the symmetry of the $\frac{2\pi}{n}$-rotation around the perpendicular axis of $\mathbf{R}^3$.

DEFINITION 2.14. The *wrapping number* w of a spatial graph with a $\mathbb{Z}_n$-symmetry is the minimum number of intersection points of $\widetilde{G}$ and P where P runs through all half-planes bounded by the axis of the $\mathbb{Z}_n$-symmetry.

We can assume that a diagram of a spatial graph with a $\mathbb{Z}_n$-symmetry and with wrapping number m has the following shape.

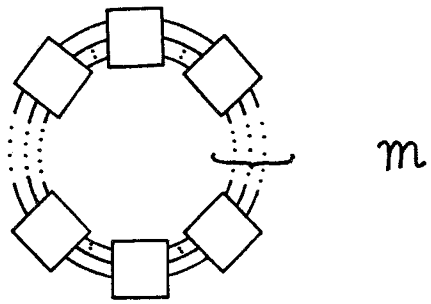


Fig. 5.

Remark that the center point of the diagram is the axis of the rotation, and there are n tetragons with the same shape.

DEFINITION 2.15. We call the above diagram the *standard diagram* of the spatial graph $\widetilde{G}$ with a $\mathbb{Z}_n$-symmetry and with wrapping number m.

3. The main theorems

Let $\widetilde{G}$ be a spatial graph with a $\mathbb{Z}_n$-symmetry r and with wrapping number 1. From Definition 2.15, we can describe a standard diagram g for $\widetilde{G}$ as in Fig. 6.

We regard the graph $\widetilde{G}/\mathbb{Z}_n$ as the spatial graph by the following identification.

$$\widetilde{G}/\mathbb{Z}_n = (\mathbb{R}^3, \widetilde{G})/r = (\mathbb{R}^3/r, \widetilde{G}/r) = (\mathbb{R}^3, \widetilde{G}/r).$$

We define the diagram $g/\mathbb{Z}_n$ obtained from g by the $\mathbb{Z}_n$-symmetry as in Fig. 7.

Then we have the following theorem.

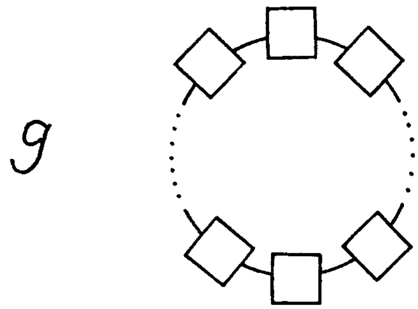


Fig. 6.

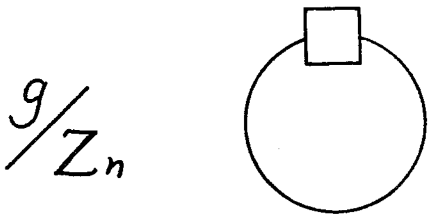


Fig. 7.

THEOREM 3.1. *Let $n \geq 1$ be an integer. Let g be a diagram of a spatial graph with a $\mathbb{Z}_n$-symmetry and with wrapping number 1. Then $\mathbf{R}(g) = \frac{\mathbf{R}(g/\mathbb{Z}_n)^n}{\sigma^{n-1}}$.*

In the case when the wrapping number is 2, we obtain the following theorem.

THEOREM 3.2. *Let $n \geq 1$ be an integer. Let g be a diagram of a spatial graph with a $\mathbb{Z}_n$-symmetry and with wrapping number 2. Let g_1, g_2 and g_3 be the quotient diagrams of g. Let Δ, a, b, and c be as follows:*

$$\begin{aligned}
\Delta &= (-1+\sigma)^2\sigma^3(-1-\sigma+\sigma^2),\\
a &= \frac{\sigma^3 - 2\sigma^4 + \sigma^5}{\Delta}\mathbf{R}(g_1) + \frac{-\sigma^2+\sigma^3}{\Delta}\mathbf{R}(g_2) + \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_3),\\
b &= \frac{-\sigma^2+\sigma^3}{\Delta}\mathbf{R}(g_1) + \frac{\sigma^3 - 2\sigma^4 + \sigma^5}{\Delta}\mathbf{R}(g_2) + \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_3),\\
c &= \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_1) + \frac{-\sigma^3+\sigma^4}{\Delta}\mathbf{R}(g_2) + \frac{-\sigma^2+\sigma^4}{\Delta}\mathbf{R}(g_3).
\end{aligned}$$

Then, the Yamada polynomial $\mathbf{R}(g)$ of g is given by the following formula.

$$\mathbf{R}(g) = (a + (b-c)\sigma)^n + \sigma(a + c - c\sigma)^n + a^n(1 - \sigma - \sigma^2).$$

4. Proof of theorems

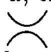
Put $I=[0,1]$. For an integer $w \geq 1$, put a_i and b_i as

$$a_i=\left(0, \frac{i}{w+1}\right), \quad b_i=\left(1, \frac{i}{w+1}\right), \quad i=1, \ldots, w.$$

Let d be a diagram on $I \times I$ satisfying $\partial(I \times I) \cap d=\{a_1, \ldots, a_w, b_1, \ldots, b_w\}$ and assume that the degree of d at each of these $2w$ points are 1. Denote by $\mathcal{D}^w$ the set of such diagrams. Let A be an indeterminate and set $\Lambda=\mathbb{Z}[A, A^{-1}]$. We denote by $\overline{\mathcal{D}^w}$ a free-Λ-module generated by $\mathcal{D}^w$.

Define an equivalence relation $\sim$ on $\overline{\mathcal{D}^w}$ generated by the following relations.

i. For two diagrams d and d', if d can be deformed into d' by a regular deformation, then $d \sim d'$.

ii. For a crossing  in a diagram d, three diagrams can be obtained by splicing the crossing as;  ,  , and  . Denote them by d_+, d_- and d_0 respectively. Then $d \sim A d_+ + A^{-1} d_- + d_0$.

iii. Let e be a nonloop edge of d such that e has no crossings and that e has no end points on $\partial(I \times I)$. Then $d \sim (d-e)+(d/e)$, where $d-e$ (d/e, respectively) is the diagram obtained from d by deleting e (contracting e, respectively).

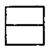
iv. Let l be a loop of d such that l has no crossings. Then $d \sim \sigma(d-l)$.

v. If there is an isolated point v, then $d \sim -(d-v)$, where $d-v$ is the diagram obtained from d by deleting v.

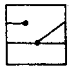
Define $\widetilde{\mathcal{D}^w}$ as the quotient Λ-module of $\overline{\mathcal{D}^w}$ by the above equivalence relation $\sim$.

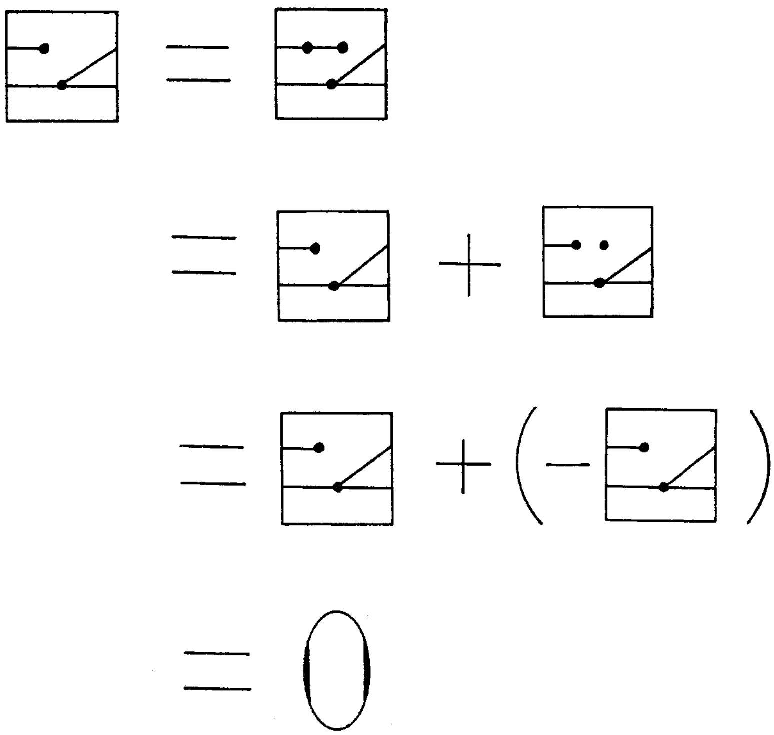
PROPOSITION 4.1. *Let e_1 and e_2 be two edges of a diagram d which incident to a point of degree 2. Then, from the equivalence relation* iii, *we have $d \sim d/e_2$.*

PROOF. Let P be the point of degree 2 which is incident to both e_1 and e_2. From the equivalence relation iii, we have $d \sim (d-e_2)+(d/e_2)$. Put $d'=d-e_2$. Then $d' \sim (d'-e_1)+(d'/e_1)$. Put $d''=d'/e_1$. From the equivalence relation v, we have $d'-e_1 \sim -(d'-e_1-\{p\})=-d'/e_1$. Hence we obtain the above proposition. $\square$

PROPOSITION 4.2. (1) *Let e be the diagram*  . *Then $\widetilde{\mathcal{D}^1}$ is a free-Λ-module whose base is $\{[e]\}$.*

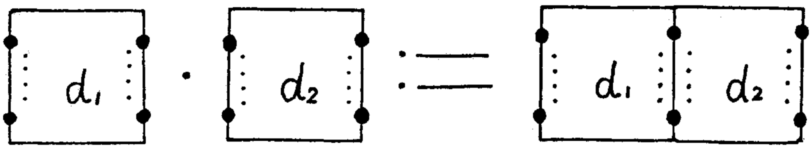
(2) *Let e_1, e_2 and e_3 be three diagrams*  ,  ,  , *respectively. Then $\widetilde{\mathcal{D}^2}$ is a free-Λ-module with a basis $\{[e_1],[e_2],[e_3]\}$.*

PROOF. Since the proof of the first part is parallel to the proof of the second part, we show the second part. From the equivalence relation ii every diagram is represented by a linear combination of some diagrams without crossing points. From the equivalence relation iii and iv, every diagram with crossing points is represented by a linear combination of some diagrams whose edges incident to one of a_0, a_1, b_0 and b_1. From the equivalence relation v, every diagram is represented by a linear combination of some diagrams without isolated points. Finally, we show that a diagram with an edge one of whose end points is not on $\partial(I \times I)$ and the degree of the end point is 1, as we can see below where  vanishes in $\widetilde{\mathcal{D}}^2$.



Thus, we complete the proof. □

For two diagrams d_1 and d_2 of $\mathcal{D}^w$, we define the products $d_1 \cdot d_2 \in \mathcal{D}^w$ as follows:



We extend canonically the above product on $\overline{\mathcal{D}^w}$.

For four diagrams d_1, d_2, d_1' and d_2', if $d_1 \sim d_1'$ and $d_2 \sim d_2'$, then we have $d_1 \cdot d_2 \sim d_1' \cdot d_2'$. Hence, we can introduce the following definition.

DEFINITION 4.3. Define the product $\cdot$ on $\widetilde{\mathcal{D}^w}$ by $[d_1] \cdot [d_2] = [d_1 \cdot d_2]$.

By the equivalence relations i–v, in the case when $w = 2$, we have the following propositions.

PROPOSITION 4.4. *The following relations hold in* $\widetilde{\mathcal{D}^2}$.

(1) $[e_1] \cdot [e_1] = [e_1]$,

(2) $[e_1] \cdot [e_2] = [e_2] \cdot [e_1] = [e_2]$,

(3) $[e_1] \cdot [e_3] = [e_3] \cdot [e_1] = [e_3]$,

(4) $[e_2] \cdot [e_2] = \sigma[e_2]$,

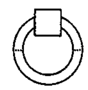
(5) $[e_2] \cdot [e_3] = [e_3] \cdot [e_2] = -\sigma[e_2]$,

(6) $[e_3] \cdot [e_3] = [e_2] + (1 - \sigma)[e_3]$.

PROPOSITION 4.5. $\widetilde{\mathcal{D}^2}$ *is a* Λ*-algebra whose rank is* 3 *and the identity element is* $[e_1]$.

PROOF. The proof is clear by the definition of the product that $\widetilde{\mathcal{D}^2}$ is a Λ-algebra. It follows from Propositions 4.2 and 4.3 that $[e_1]$ is the identity element. □

We define a map $\varphi_w : \mathcal{D}^w \longrightarrow \Lambda$ as follows:

$\varphi_w(d) = \mathbf{R}($  $)$.

We extend the above map onto $\overline{\mathcal{D}^w}$ linearly, and denote it by the same notation φ_w, i.e.,

$$\varphi_w(\lambda_1 d_1 + \cdots + \lambda_k d_k) = \lambda_1 \varphi_w(d_1) + \cdots + \lambda_k \varphi_w(d_k).$$

PROPOSITION 4.6. *If $d \sim d'$ in $\mathcal{D}^w$, then $\varphi_w(d) = \varphi_w(d')$.*

PROOF. We will check the identity $\varphi_w(d) = \varphi_w(d')$ for each equivalence relation i–v.

i. Since the Yamada polynomial is a regular deformation invariant, it is clear that $\varphi_w(d) = \varphi_w(d')$.

ii. Using Proposition 2.11 (1), we can prove that

$$\begin{aligned} \varphi_w(d) &= \mathbf{R}(\quad) \\ &= A\mathbf{R}(\quad) + A^{-1}\mathbf{R}(\quad) + \mathbf{R}(\quad) \\ &= A\varphi_w(d_+) + A^{-1}\varphi_w(d_-) + \varphi_w(d_0) \\ &= \varphi_w(Ad_+ + A^{-1}d_- + d_0) \\ &= \varphi_w(d'). \end{aligned}$$

iii. Remark that e is a nonloop edge. Denote $d' = d - e$, and $d'' = d/e$. Using Proposition 2.11 (2), we can prove that

$$\begin{aligned} \varphi_w(d) &= \mathbf{R}(\quad) \\ &= \mathbf{R}(\quad) + \mathbf{R}(\quad) \\ &= \varphi_w(d') + \varphi_w(d'') \\ &= \varphi_w(d' + d''). \end{aligned}$$

iv. Note that the value of the Yamada polynomial of a point is -1, and from Proposition 2.11 (3) and (4), we obtain the following. Denote by $v(l)$ the point on the loop l.
If the loop l is a connected component, we have

$$\begin{aligned} \varphi_w(d) &= \mathbf{R}(l)\varphi_w(d - l - v(l)) \\ &= \sigma\varphi_w(d - l - v(l)) \\ &= -\sigma\varphi_w(d - l) \\ &= \varphi_w(-\sigma(d - l)). \end{aligned}$$

Otherwise,

$$\varphi_w(d) = -\mathbf{R}(l)\varphi_w(d - l)$$

$$\begin{aligned} &= -\sigma\varphi_w(d-l) \\ &= \varphi_w(-\sigma(d-l)). \end{aligned}$$

v. We can obtain the following:

$$\begin{aligned} \varphi_w(d) &= \varphi_w(d-v)\cdot \mathbf{R}(v) \\ &= \varphi_w(d-v)(-1) \\ &= \varphi_w(-(d-v)). \end{aligned}$$

□

From Proposition 4.5 we can define a map $\varphi_w : \widetilde{\mathcal{D}^w} \longrightarrow \Lambda$ by

$$\varphi_w([d]) = \varphi_w(d).$$

Simplify the notation by writing $d = [d]$.

We will prove Theorem 3.1. Let g be a diagram as in Figure 6.

From Proposition 4.2, $d \in \widetilde{\mathcal{D}^1}$ is represented as $d = ae$ $(a \in \Lambda)$.

Then, by noting that $\mathbf{R}(g) = \varphi_1(d^n) = a^n\varphi_1(e^n) = a^n\varphi_1(e) = a^n\sigma$ and $\mathbf{R}(g/\mathbb{Z}_n) = \varphi_1(d) = a\varphi_1(e) = a\sigma$, we can easily verify Theorem 3.1.

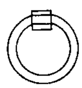
In the case when $w = 2$, we take a diagram g as in Figure 1.

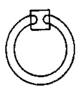
Then the three quotient diagrams g_1, g_2, and g_3 of g are as illustrated in Figure 2, and we obtain the following lemma.

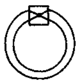
LEMMA 4.7. *For the map* φ_2, *it satisfies the properties*:

1. $\varphi_2(e_1) = \sigma^2$,
2. $\varphi_2(e_2) = \sigma$,
3. $\varphi_2(e_3) = -\sigma^2$.

PROOF. We obtain the above results directly by acting φ_2 to e_1, e_2 and e_3 respectively.

1. $\varphi_2(e_1) = \mathbf{R}($  $) = \sigma^2$,

2. $\varphi_2(e_2) = \mathbf{R}($  $) = \sigma$,

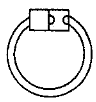
3. $\varphi_2(e_3) = \mathbf{R}($  $) = -\sigma^2.$

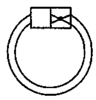
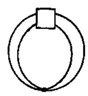
□

LEMMA 4.8. *For the map φ_2, we have*

1. $\mathbf{R}(g_1) = \varphi_2(d) = \varphi_2(d \cdot e_1)$,
2. $\mathbf{R}(g_2) = \varphi_2(d \cdot e_2)$,
3. $\mathbf{R}(g_3) = \varphi_2(d \cdot e_3)$.

PROOF. We have the first result by the definition of the map φ_2. The second and the third identities are proved as follows:

$\varphi_2(d \cdot e_2) = \mathbf{R}($  $) = \mathbf{R}($  $),$

$\varphi_2(d \cdot e_3) = \mathbf{R}($  $) = \mathbf{R}($  $).$

□

Using Lemmas 4.7 and 4.8, we easily obtain the following proposition.

PROPOSITION 4.9. *Let d be an element of $\widetilde{\mathcal{D}^2}$. Using the basis $\{e_1, e_2, e_3\}$, d is represented as $d = ae_1 + be_2 + ce_3$. Then we obtain the following matrix representation*:

$$\begin{pmatrix} \mathbf{R}(g_1) \\ \mathbf{R}(g_2) \\ \mathbf{R}(g_3) \end{pmatrix} = \begin{pmatrix} \sigma^2 & \sigma & -\sigma^2 \\ \sigma & \sigma^2 & -\sigma^2 \\ -\sigma^2 & -\sigma^2 & -\sigma^2(1-\sigma) \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Take an element $d \in \widetilde{\mathcal{D}^2}$ and fix it, we define a map $\psi_d : \widetilde{\mathcal{D}^2} \longrightarrow \widetilde{\mathcal{D}^2}$ by $\psi_d(v) = d \cdot v$.

LEMMA 4.10. *For $\lambda, \lambda' \in \Lambda$, we have $\psi_d(\lambda v + \lambda' v') = \lambda \psi_d(v) + \lambda' \psi_d(v')$.*

Denote by Ψ_d the matrix representation of the above map $\psi_d : \widetilde{\mathcal{D}^2} \longrightarrow \widetilde{\mathcal{D}^2}$, related to the basis $\{e_1, e_2, e_3\}$. Then

$$\Psi_{e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Psi_{e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \sigma & -\sigma \\ 0 & 0 & 0 \end{pmatrix}, \quad \Psi_{e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma & 1 \\ 1 & 0 & 1-\sigma \end{pmatrix}.$$

Hence, for $d = ae_1 + be_2 + ce_3 \in \widetilde{\mathcal{D}^2}$, the matrix Ψ_d is represented as follows:

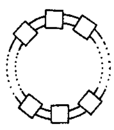
$$\begin{aligned} \Psi_d &= a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 1 & \sigma & -\sigma \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma & 1 \\ 1 & 0 & 1-\sigma \end{pmatrix} \\ &= \begin{pmatrix} a & 0 & 0 \\ b & a+(b-c)\sigma & -b\sigma + c \\ c & 0 & a+c-c\sigma \end{pmatrix}. \end{aligned}$$

From Lemma 4.7, we obtain the following formula:

$$\varphi_2(xe_1 + ye_2 + ze_3) = (\sigma^2, \sigma, -\sigma^2)\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The eigenvalues of Ψ_d are a, $a + (b - c)\sigma$ and $a + c - c\sigma$, and corresponding eigenvectors are $(-1 + \sigma, 1/\sigma, 1)$, $(0,1,1)$ and $(0,1,0)$ respectively. Put $s = a + (b - c)\sigma$, and $t = a + c - c\sigma$.

Now the following computation of $\mathbf{R}(g)$ implies Theorem 3.2.

$\mathbf{R}(g) = \mathbf{R}($  $)$

$$\begin{aligned} &= \varphi_2(d^n) \\ &= \varphi_2({\psi_d}^n(e_1)) \\ &= (\sigma^2, \sigma, -\sigma^2)\Psi_d^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$= (\sigma^2, \sigma, -\sigma^2) \begin{pmatrix} a^n & 0 & 0 \\ \frac{a^n - s^n + \sigma(-t^n + s^n)}{(-1+\sigma)\sigma} & s^n & t^n - s^n \\ \frac{a^n - t^n}{-1+\sigma} & 0 & t^n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= (a + (b-c)\sigma)^n + \sigma(a + c - c\sigma)^n + a^n(1 - \sigma - \sigma^2),$$

where a, b, c and Δ are given in Theorem 3.2.

5. Enumeration of irreducible diagrams

We define an irreducible diagram. Let D be a diagram of $\mathcal{D}^w$. If a point of D does not lie on $\partial(I \times I)$, we call the point an *inner point* of D. If an edge of D has no end points on $\partial(I \times I)$, we call the edge an *inner edge* of D. If D has no crossings, no inner edges, and no inner points whose degrees are 1 or 2, then we call D an *irreducible diagram*. Denote by $\mathcal{ID}^w$ the set of irreducible diagrams in $\mathcal{D}^w$.

From the equivalence relation ii, any element of $\widetilde{\mathcal{D}^w}$ is represented as a linear combination of some diagrams without crossings. From the equivalence relations iii, iv, and v, a diagram without crossings is represented as a linear combination of some diagrams without inner edges. If a diagram has isolated points, it can be represented by a linear combination of some diagrams without isolated points.

Furthermore, if a diagram has inner points whose degrees are 1, the diagram represents the zero element of $\widetilde{\mathcal{D}^w}$. From Proposition 4.1, if a diagram has an inner point of degree 2, the inner point can be eliminated in $\widetilde{\mathcal{D}^w}$. Hence $\mathcal{ID}^w$ becomes the base of $\widetilde{\mathcal{D}^w}$.

We count the number of elements of $\mathcal{ID}^w$.

Take a diagram D of $\mathcal{ID}^w$. Let $I \times I$ be the square where the diagram D lies on. Recall that we specified $2w$ points $a_1, \ldots, a_w, b_1, \ldots, b_w$ on $\{0,1\} \times I$, so that $D \cap \partial(I \times I) = \{a_1, \ldots, a_w, b_1, \ldots, b_w\}$.

Consider a homeomorphism γ from $I \times [0,1)$ onto the closed upper half-plane $\mathbb{R}^2{}_+ = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$, such that $a_1, \ldots, a_w, b_w, \ldots, b_1$ are mapped to $P_1, \ldots, P_w, P_{w+1}, \ldots, P_{2w}$ respectively, where $P_i = (i, 0)$ on $\partial\mathbb{R}^2{}_+$. For simplicity, we assume that the diagram $\gamma(D)$ lies on $\mathbb{R}^2{}_+$ instead of D on $I \times I$.

For example, take the following diagram D as in Fig. 8 in the case when $w = 6$.

Then we obtain an illustration of $\gamma(D)$ as in Fig. 9.

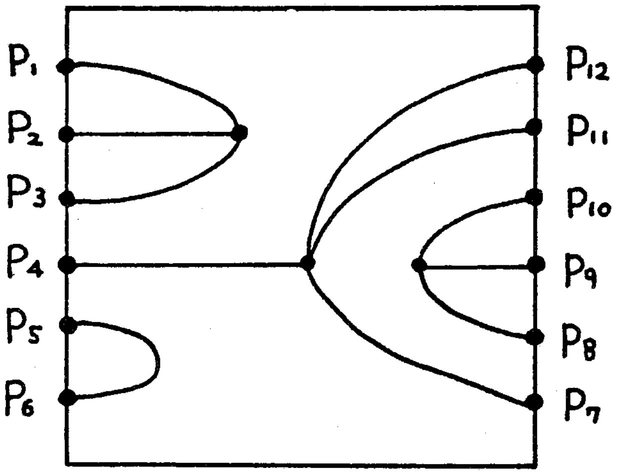


Fig. 8.

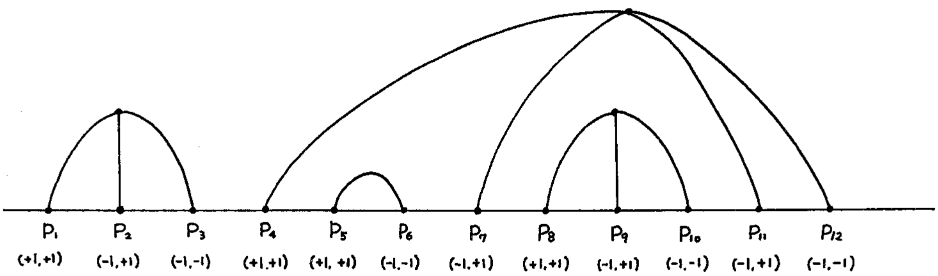


Fig. 9.

To count the number of irreducible diagrams, put $\{P_i\} = \{P_1, P_2, \ldots, P_{2w-1}, P_{2w}\}$. We assign the label of ordered pair $(+1,+1)$, $(-1,-1)$ or $(-1,+1)$ to each P_i as follows:

- If P_i is not connected to P_i for all $j < i$, we let $(+1,+1)$ correspond to P_i.
- If P_i is not connected to P_i for all $j > i$, we let$(-1,-1)$ correspond to P_i.
- Otherwise we let $(-1,+1)$ correspond to P_i.

For a given diagram D, we therefore obtain an ordered pair for each point P_i $(1 \leq i \leq 2w)$ and call it the label of the point P_i. We define the sequence $\{A_1, A_2, \ldots, A_{2w-1}, A_{2w}\}$ by setting (A_{2i-1}, A_{2i}) to be the ordered pair as given the above corresponding to P_i. Call the sequence the *label sequence* of $\{P_i\}$.

Then this sequence has the following conditions.

1. $(A_{2i-1}, A_{2i}) = (+1,+1), (-1,-1)$ or $(-1,+1)$ for all $1 \le i \le 2w$,

2. $\sum_{i=1}^{4w} A_i = 0$,

3. $\sum_{i=1}^{k} A_i \ge 0$ for all $1 \le k \le 4w$.

Conversely, if a sequence $\{A_i\}$ of length $4w$ which satisfies the above given conditions **1** $\sim$ **3**, a diagram D is uniquely determined. Hence we have a one-to-one correspondence between irreducible diagrams and sequences $\{A_i\}$ of length $4w$ satisfying **1** $\sim$ **3**.

We denote by $[P_i, P_j]$ a closed interval on $\mathbb{R} = \partial{\mathbb{R}_+}^2$, and put

$$\mathcal{I}^* = \{[P_i, P_j] \mid P_i \text{ and } P_j \text{ are connected in } D\}.$$

Denote by $[P_{i1}, P_{j1}], [P_{i2}, P_{j2}], \ldots, [P_{ik}, P_{jk}]$ the maximal elements of $\mathcal{I}^*$ as subsets of $\mathbb{R}$, and we call them *maximal strands* of D, and denote them by $l_1, l_2, \ldots, l_k$ respectively.

Let a_r be the number of points P_q $(i_r \le q \le j_r)$ corresponding to $(+1,+1)$ in l_r, and put $a^* = \sum_{r=1}^{k} a_r$. For example, in Figure 9 we can see there are two maximal strands l_1 and l_2, $a_1 = 1$, $a_2 = 3$, and $a^* = a_1 + a_2 = 4$.

We shall consider irreducible diagrams with a points corresponding to $(+1,+1)$.

Let $\{P_i'\} = \{P_1', P_2', \ldots, P_{2a}'\}$ be a subset of $\{P_i\}$ obtained by dropping $2w - 2a$ points labeled $(-1,+1)$. Then any point $P_i' \in \{P_i'\}$ is labeled $(+1,+1)$ or $(-1,-1)$.

To simplify the notation, we denote by $+1$ instead of $(+1,+1)$, and -1 instead of $(-1,-1)$. From the above subset $\{P_i'\}$, we have the label sequence denoted by $\{A_i'\} = \{A_1', A_2', \ldots, A_{2a}'\}$.

The sequence $\{A_i'\}$ $(1 \le i \le 2a)$ satisfies

1′. $A_i' = +1$ or -1 for all $1 \le i \le 2a$,

2′. $\sum_{i=1}^{2a} A_i' = 0$,

3′. $\sum_{i=1}^{k} A_i' \ge 0$ for all $1 \le k \le 2a$.

Since there are $2a$ points in $\{P_i'\}$, we can assign the label $+1$ on a points in $\binom{2a}{a}$ ways. Note that $\binom{2a}{a}$ is the binomial coefficient. We wish to subtract the number of ways of assigning the label which do not satisfy **3′** from $\binom{2a}{a}$. This can be done by the following consideration.

For a sequence $\{A_i'\}$, pick up the first term A_k' with $\sum_{i=1}^{k} A_i' < 0$. Then for all terms A_i' $(i > k)$, alternate all labels as the following operation α

$$\alpha : +1 \longrightarrow -1, \qquad -1 \longrightarrow +1.$$

By this operation α, the number of terms labeled $+1$ is $a-1$, and the number of terms labeled -1 is $a+1$.

Conversely, take a sequence $\{A'_i\}$ that consists of $a-1$ terms labeled $+1$, and $a+1$ terms labeled -1. For this sequence, pick up the first term A'_k with $\sum_{i=1}^{k} A'_i < 0$. After alternating labels of all terms A'_i with $i > k$ by the above operation α, we obtain the former label sequence.

Hence the number of ways of assigning the label which does not satisfy **3′** is equal to the number of repeated permutations obtained by $a-1$ terms labeled $+1$ and $a+1$ terms labeled -1, i.e., $\frac{(2a)!}{(a+1)!(a-1)!}$.

Therefore the number of ways of assigning the label $+1$ on a points in $2a$ points is equal to

$$\begin{aligned}\binom{2a}{a} - \frac{(2a)!}{(a+1)!(a-1)!} &= \frac{(2a)!}{a!a!} - \frac{(2a)!}{(a+1)!(a-1)!}\\ &= \left(1 - \frac{a}{a+1}\right)\frac{(2a)!}{a!a!}\\ &= \frac{1}{a+1}\binom{2a}{a}.\end{aligned}$$

Now, we count the number of irreducible diagrams with a single maximal strand l. There are $2a$ points in $\{P'_i\}$, and it is clear that P'_1 is labeled $+1$ and P'_{2a} is labeled -1. Thus the number of ways to assign the label $+1$ on $a-1$ points is equal to

$$\frac{1}{a}\binom{2(a-1)}{a-1}.$$

We count the number of ways to assign the label $(+1,+1)$ on $2w-2a$ points in $\{P_i\}$. To do it, count the number of ways to add $2w-2a$ points in $\{P'_i\}\setminus\{P'_1, P'_{2a}\}$, then assign the label $(-1,+1)$ on the added points in $\frac{1}{a}\binom{2(a-1)}{a-1}$ ways.

In the same way, we count irreducible diagrams with k maximal strands $l_1,\dots,l_k$, and a^* points labeled $(+1,+1)$ in $\{P_i\}_{i=1}^{2w}$. Recall $a^* = \sum_{r=1}^{k} a_r$, then $\{P'_i\}$ consists of $2a^*$ points.

For any two maximal strands $[P_{i_{r-1}}, P_{j_{r-1}}]$ and $[P_{ir}, P_{jr}]$, put $P'_i = P_{j_{r-1}}$, then $P'_{i+1} = P_{ir}$. We cannot add any points between P'_i and P'_{i+1} in $\{P'_i\}$. Thus we can add $2w-2a^*$ points labeled $(-1,+1)$ with repetition to $\{P'_i\}$ in

$$\binom{2w-k-1}{2w-2a^*}$$

ways.

Hence if there are $2w$ points $P_1, P_2, \ldots, P_{2w-1}, P_{2w}$ in D, the number of irreducible diagrams is equal to

$$\sum_{a^*=1}^{w} \sum_{k=1}^{a^*} \binom{2w-k-1}{2w-2a^*} \sum_{a_1+a_2+\cdots+a_k=a^*} \prod_{r=1}^{k} \frac{1}{a_r} \binom{2(a_r-1)}{a_r-1}.$$

6. The generating function

We consider the generating function of the above counting formula.

For an integer n ($n \geq 1$), denote by $\mathcal{A}_n$ the set of finite sequences $\{A_i\}$ of length $2n$, satisfying the following conditions **A-1** $\sim$ **A-3**:

A-1. $(A_{2i-1}, A_{2i}) = (+1,+1), (-1,-1)$ or $(-1,+1)$ for all $1 \leq i \leq n$.

A-2. $\sum_{i=1}^{2n} A_i = 0$.

A-3. $\sum_{i=1}^{k} A_i \geq 0$ for $1 \leq k \leq 2n$.

We denote by $\mathcal{A}_n{}^+$ the subset of $\mathcal{A}_n$ which consists of the sequences satisfying

A-3′. $\sum_{i=1}^{k} A_i > 0$ for all $1 \leq k < 2n$.

We define $\widetilde{A_i} = (A_{2i-1}, A_{2i})$ and call it the i-block of $\{A_i\}$. The above conditions imply the following properties.

LEMMA 6.1. 1. $\mathcal{A}_1 = \mathcal{A}_1{}^+ = \emptyset$.

2. *If* $n \geq 2$, *and* $\{A_i\} \in \mathcal{A}_n$, *then* $\widetilde{A_1} = (+1,+1)$, $\widetilde{A_n} = (-1,-1)$.
3. *If* $n \geq 2$, *and* $\{A_i\} \in \mathcal{A}_n{}^+$, *then* $\sum_{i=1}^{2k} A_i \geq 2$ *for all* $1 \leq k \leq n-1$.

For an integer n ($n \geq 1$), denote by $\mathcal{B}_n$ the set of finite sequences $\{B_i\}$ of length $2n$, satisfying the following conditions **B-1** $\sim$ **B-3**:

B-1. $(B_{2i-1}, B_{2i}) = (+1,+1), (-1,-1)$ or $(+1,-1)$ for all $1 \leq i \leq n$.

B-2. $\sum_{i=1}^{2n} B_i = 0$.

B-3. $\sum_{i=1}^{k} B_i \geq 0$ for all $1 \leq k \leq 2n$.

We denote by $\mathcal{B}_n{}^+$ the subset of $\mathcal{B}_n$ which consists of the sequences satisfying

B-3′. $\sum_{i=1}^{k} B_i > 0$ for all $1 \leq k < 2n$.

We define $\widetilde{B_i} = (B_{2i-1}, B_{2i})$ and call it the i-block of $\{B_i\}$.

The following lemma is verified.

LEMMA 6.2. 1. $\mathcal{B}_1 = \mathcal{B}_1{}^+ = \{(+1,-1)\}$.

2. *If* $n \geq 2$, *and* $\{B_i\} \in \mathcal{B}_n{}^+$, *then* $\widetilde{B_1} = (+1,+1)$, $\widetilde{B_n} = (-1,-1)$.

3. *If* $n \geq 2$, *and* $\{B_i\} \in \mathcal{B}_n{}^+$, *then* $\sum_{i=1}^{2k} B_i \geq 2$ *for all* $1 \leq k \leq n-1$.
4. *If* $n \geq 3$, *and* $\{B_i\} \in \mathcal{B}_n{}^+$, *then* $\sum_{i=1}^{2k+1} B_i \geq 3$ *for all* $1 \leq k \leq n-2$, *and* $\sum_{i=1}^{2n-1} B_i = 1$.

Let $n \geq 2$. For $\{A_i\} \in \mathcal{A}_n{}^+$, by alternating all blocks of type $(-1, +1)$ with blocks of type $(+1, -1)$, then an element of $\mathcal{B}_n{}^+$ is obtained. Define this correspondence by $up : \mathcal{A}_n{}^+ \longrightarrow \mathcal{B}_n{}^+$.

LEMMA 6.3. *The above map* $up : \mathcal{A}_n{}^+ \longrightarrow \mathcal{B}_n{}^+$ *is a bijection.*

PROOF. For $\{A_i\} \in \mathcal{A}_n{}^+$, we define $\{B_i\} = up(\{A_i\})$. By the construction, it is clear that $\{B_i\}$ satisfies **B-1**, **B-2** and **B-3′**.

Conversely when $n \geq 2$, for $\{B_i\} \in \mathcal{B}_n{}^+$, by alternating all blocks of type $(+1, -1)$ with the same numbers of blocks of type $(-1, +1)$, it is clear that this sequence satisfies **A-1** and **A-2**. It is shown that this sequence satisfies **A-3′**.

When $n \geq 2$, for all $1 \leq k \leq n-1$, from Lemma 6.2, $\sum_{i=1}^{2k} A_i = \sum_{i=1}^{2k} B_i \geq 2$. In addition, $\sum_{i=1}^{2k+1} A_i = \sum_{i=1}^{2k} A_i + A_{2k+1} \geq 1$. This correspondence is clearly an inverse map of the map $up : \mathcal{A}_n{}^+ \longrightarrow \mathcal{B}_n{}^+$, hence the above lemma is shown. $\square$

Consider the case when $n \geq 3$. For $\{B_i\} \in \mathcal{B}_n{}^+$, we define a sequence $int(\{B_i\})$ whose length is $2n-4$ obtained from $\mathcal{A}_n$ by dropping the first two terms and the last two terms as follows, $(int(\{B_i\}))_j = B_{j+2}$ for all $1 \leq j \leq 2n-4$.

LEMMA 6.4. *For the map* $int(\{B_i\}) \in \mathcal{B}_{n-2}$, $int : \mathcal{B}_n{}^+ \longrightarrow \mathcal{B}_{n-2}$ *is a bijection.*

PROOF. It is clear that the above map satisfies **B-1**. Let $B'_j = (int(\{B_i\}))_j$, then from Lemma 6.2 we have

$$\sum_{j=1}^{2n-4} B'_j = \sum_{i=1}^{2n} B_i - (B_1 + B_2 + B_{2n-1} + B_{2n}) = \sum_{i=1}^{2n} B_i = 0.$$

For k with $1 \leq k \leq 2n-4$, from Lemma 6.2, we have

$$\sum_{j=1}^{k} B'_j = \sum_{i=1}^{k+2} B_i - (B_1 + B_2) = \sum_{i=1}^{k+2} B_i - 2 \geq 0.$$

Hence we obtain $int(\{\mathcal{B}_n{}^+\}) \subset \mathcal{B}_{n-2}$. Conversely, for $\{B'_j\} \in \mathcal{B}_{n-2}$, define a sequence $cl(\{B'_j\})$ whose length is $2n$ as follows:

$$(cl(\{B'_j\}))_i = \begin{cases} +1 & (i = 1, 2) \\ B'_{i-2} & (3 \leq i \leq 2n-2) \\ -1 & (i = 2n-1, 2n) \end{cases}.$$

Then, it is easily shown that $cl(\{B'_j\}) \subset \mathcal{B}_n{}^+$. Furthermore, it is clear that $cl : \mathcal{B}_{n-2} \longrightarrow \mathcal{B}_n{}^+$ is an inverse map of *int*. Hence we complete the lemma. $\square$

For $\{A_i\} \in \mathcal{A}_l$ and $\{A'_j\} \in \mathcal{A}_m$, we define the product, $\{A''_n\} = \{A_i\} \cdot \{A'_j\}$ as follows,

$$A''_n = \begin{cases} A_n & (\text{for } \ 1 \leq n \leq 2l) \\ A'_{n-2l} & (\text{for } \ 2l+1 \leq n \leq 2(l+m)) \end{cases}.$$

Then it is clear that $\{A''_n\} \in \mathcal{A}_{l+m}$. Define

$$\mathcal{A}_l \cdot \mathcal{A}_m = \{\{A_i\} \cdot \{A'_j\} \mid \{A_i\} \in \mathcal{A}_l, \{A'_j\} \in \mathcal{A}_m\}.$$

Similarly we define the product on $\mathcal{B}_{l+m}$, then $\mathcal{B}_n = \bigcup_{i=0}^{n-1} \mathcal{B}_i \cdot \mathcal{B}_{n-i}{}^+$.

We denote by $\sharp X$ the number of elements of the set X. Let $b_n = \sharp\mathcal{B}_n$ and $b_n{}^+ = \sharp\mathcal{B}_n{}^+$, then from Lemma 6.4, $b_n{}^+ = b_{n-2}$ $(n \geq 2)$, and define $b_0 = 1$. We obtain

$$b_n = \sum_{i=0}^{n-2} b_i \cdot b_{n-2-i} + b_{n-1}.$$

We define the generating function of the sequence $\{b_n\}$ as follows:

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots.$$

Note that $b_0 = b_1 = 1$, then

$$\begin{aligned} x^2 b(x)^2 + x b(x) + 1 &= \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j} b_i b_{j-i} \Big) x^{j+2} + \sum_{i=0}^{\infty} b_i x^{i+1} + 1 \\ &= 1 + b_0 x + \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j} b_i b_{j-i} + b_{j+1} \Big) x^{j+2} \\ &= b_0 + b_1 x + \sum_{j=0}^{\infty} b_{j+2} x^{j+2} \\ &= b(x). \end{aligned}$$

Consequently, we obtain

$$b(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

When $n \geq 2$, similarly we have

$$\mathcal{A}_n = \bigcup_{i=0}^{n-2} \mathcal{A}_i \cdot \mathcal{A}_{n-i}{}^{+} \bigcup \mathcal{A}_{n-1} \cdot \mathcal{A}_1{}^{+} = \bigcup_{i=0}^{n-2} \mathcal{A}_i \cdot \mathcal{B}_{n-i}{}^{+} \bigcup \mathcal{A}_{n-1} \cdot \mathcal{A}_1{}^{+}.$$

Note that $\sharp \mathcal{A}_1{}^{+} = 0$, then we define

$$a_n = \begin{cases} 1 & (n = 0) \\ \sharp \mathcal{A}_n & (n \geq 1) \end{cases}.$$

From Lemmas 6.1 and 6.3, we have

$$a_n = \sum_{i=0}^{n-2} a_i \cdot b_{n-2-i}.$$

We define the generating function of the sequence $\{a_n\}$ as follows:

$$a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

Since $a_0 = 1$ and $a_1 = 0$, then

$$\begin{aligned} x^2 a(x) b(x) + 1 &= \sum_{j=0}^{\infty} \Big(\sum_{i=0}^{j} a_i b_{j-i} \Big) x^{j+2} + 1 \\ &= a_0 + a_1 x + \sum_{j=0}^{\infty} a_{j+2} x^{j+2} \\ &= a(x). \end{aligned}$$

Hence we obtain,

$$\begin{aligned} a(x) &= \frac{1}{1 - x^2 b(x)} \\ &= \frac{1}{2} \left(\frac{1}{x} - \frac{\sqrt{1-3x}}{x\sqrt{1+x}} \right) \\ &= \frac{1}{2} \left\{ x^{-1} - \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-3)^i \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} x^{i+j-1} \right\}. \end{aligned}$$

A general term of $\{a_n\}$ is given by

$$-\frac{1}{2} \sum_{i+j-1=n} \binom{\frac{1}{2}}{i} (-3)^i \binom{-\frac{1}{2}}{j}.$$

Note that $n = w$, then the above formula coincides with the counting formula obtained in Section 5.

7. A criterion of $\mathbb{Z}_n$-symmetry

Given a spatial graph $\widetilde{G}$ with the wrapping number 2, we consider if $\widetilde{G}$ has a $\mathbb{Z}_n$-symmetry or not for $n \geq 3$.

From the preceding sections, if a spatial graph $\widetilde{G}$ has a $\mathbb{Z}_n$-symmetry $(n \geq 3)$, then the value of the Yamada polynomial of a diagram g for $\widetilde{G}$ is the following:

$$\mathbf{R}(g) = \mathbf{R}(g)(A) = (a + (b - c)\sigma)^n + \sigma(a + c - c\sigma)^n + a^n(1 - \sigma - \sigma^2).$$

Putting $f(A) = a + (b - c)\sigma$, $g(A) = a + c - c\sigma$ and $h(A) = a$ implies,

$$\begin{aligned}\mathbf{R}(g) &= f(A)^n + (A + 1 + A^{-1})g(A)^n \\ &\quad + (-A^2 - 3A - 3 - 3A^{-1} - A^{-2})h(A)^n.\end{aligned}$$

We now consider $n = p$ where p is an odd prime integer.

The notation $f(x) \equiv g(x)$ ((mod p), A^p) means that $f(x) - g(x)$ is an element of the ideal generated by p and A^p. We apply the well-known fact that $f(x)^p \equiv f(x^p)$ (mod p). Then we have,

$$\begin{aligned}\mathbf{R}(g) &= f(A)^p + (A + 1 + A^{-1})g(A)^p \\ &\quad + (-A^2 - 3A - 3 - 3A^{-1} - A^{-2})h(A)^p \\ &\equiv f(A^p) + (A + 1 + A^{-1})g(A^p) \\ &\quad + (-A^2 - 3A - 3 - 3A^{-1} - A^{-2})h(A^p) \pmod{p}.\end{aligned}$$

In the case when $p = 3$, put $f(1) = \alpha$, $g(1) = \beta$, and $h(1) = \gamma$, then we have, under mod 3 and $A^3 \equiv 1$,

$$\begin{aligned}\mathbf{R}(g) &= \mathbf{R}(g)(A) \\ &\equiv f(A^3) + (A + 1 + A^{-1})g(A^3) + (-A^2 - A^{-2})h(A^3) \\ &\equiv f(1) + (A + 1 + A^{-1})g(1) + (-A - A^{-1})h(1) \\ &= \alpha + (A + 1 + A^{-1})\beta + (-A - A^{-1})\gamma \\ &= (\beta - \gamma)A + (\alpha + \beta) + (\beta - \gamma)A^{-1}.\end{aligned}$$

Therefore two coefficients of A and A^{-1} coincide under mod 3 and $A^3 \equiv 1$.
Similarly, in the case when $p \geq 5$, we have,under mod p and $A^p \equiv 1$.

$$\begin{aligned}\mathbf{R}(g) &= \mathbf{R}(g)(A) \\ &\equiv f(A^p) + (A + 1 + A^{-1})g(A^p) \\ &\quad + (-A^2 - 3A - 3 - 3A^{-1} - A^{-2})h(A^p)\end{aligned}$$

$$\begin{aligned}
&\equiv f(1)+(A+1+A^{-1})g(1)\\
&\quad +(-A^2-3A-3-3A^{-1}-A^{-2})h(1)\\
&= \alpha+(A+1+A^{-1})\beta+(-A^2-3A-3-3A^{-1}-A^{-2})\gamma\\
&= -\gamma A^2+(\beta-3\gamma)A+(\alpha+\beta-3\gamma)+(\beta-3\gamma)A^{-1}-\gamma A^{-2}.
\end{aligned}$$

Hence we have the Yamada polynomial in the following form:

$$\mathbf{R}(g)\equiv a_2A^2+a_1A+a_0+a_1A^{-1}+a_2A^{-2}.$$

For a spatial graph $\widetilde{G}$, the value of the Yamada polynomial $\mathbf{R}(g)$ is uniquely determined up to multiplying $(-A)^k$ for some integer $k\geq 0$. Then we have the following theorem.

THEOREM 7.1. *Let p be an odd prime number. The notation $f(x)\equiv g(x)$ (mod p, $A^3\equiv 1$) means that $f(x)-g(x)$ is an element of the ideal generated by p and A^p. Let $\widetilde{G}$ be a spatial graph with a $\mathbb{Z}_p$-symmetry and with wrapping number 2, and g a diagram of $\widetilde{G}$. Then, for some integer k $(0\leq k<p)$, we have the following form:*

$$(-A)^k\mathbf{R}(g)\equiv a_2A^2+a_1A+a_0+a_1A^{-1}+a_2A^{-2}\quad (\text{mod } p,\ A^p\equiv 1).$$

EXAMPLE 7.2. The spatial graph as in Figure 10 $\widetilde{G}$ cannot have a $\mathbb{Z}_5$-symmetry.

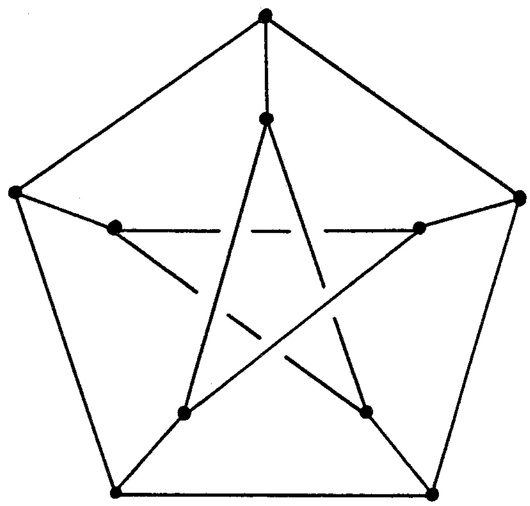


Fig. 10.

Calculating the value of the Yamada polynomial of $\widetilde{G}$, we have

$$\begin{aligned}\mathbf{R}(g) = & -A^{-6} - A^{-5} + 5A^{-4} - 15A^{-3} + 25A^{-2} - 35A^{-1} + 40 \\ & - 40A + 35A^2 - 25A^3 + 15A^4 - 5A^5 + A^6 + A^7.\end{aligned}$$

Under mod 5 and $A^5 \equiv 1$,

$$\mathbf{R}(g) \equiv 4A^{-1} + 4 + A + A^2.$$

Hence the above spatial graph $\widetilde{G}$ does not have a $\mathbb{Z}_5$-symmetry.

References

[1] L.H. Kauffman: Invariants of graphs in three-space. Preprint.
[2] S. Yamada: An Invariant of Spatial Graphs, Journal of Graph Theory **13** (1989), 537–551.

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