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ON A PSEUDOHYPERBOLIC EQUATION WITH A NONLOCAL BOUNDARY CONDITION

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Abstract

The present paper is devoted to study initial-boundary value problems associating an integral condition with Neumann condition for a pseudohyperbolic equation. We establish the existence, uniqueness and continuous dependence upon the data of a classical and weak solutions in a nonclassical function space.

1. Introduction

A great number of physical problems can be modeled by the following partial differential equation:

$$\frac{\partial^2\theta}{\partial t^2} - \alpha\frac{\partial^2\theta}{\partial x^2} - \beta\frac{\partial^3\theta}{\partial t\partial x^2} = h(x,t), \tag{1.1}$$

for $(x,t) \in Q = \Omega \times I$, where $\Omega = (0,l)$, $I = (0,T)$ are bounded intervals in $\mathbb{R}_+$, α and β are positive constants, and $\beta\frac{\partial^3\theta}{\partial t\partial x^2}$ is the weak viscosity.

We consider the function $\theta = \theta(x,t)$ satisfying equation (3.1) in Q, and

$$\theta(x,0) = \Phi(x), \quad \frac{\partial\theta(x,0)}{\partial t} = \Psi(x), \qquad x \in \Omega, \tag{1.2}$$

$$\frac{\partial\theta(0,t)}{\partial x} = \mu(t), \quad \int_\Omega \theta(x,t)dx = m(t), \qquad t \in I, \tag{1.3}$$

where h, Φ, Ψ, μ and m are known functions verifying:

$$h \in C\left(\overline{Q}\right), \ \Phi, \Psi \in C^1\left(\overline{\Omega}\right), \ \mu \text{ and } m \in C^2\left(\overline{I}\right),$$

and the compatibility conditions

$$\Phi'(0) = \mu(0), \ \int_\Omega \Phi(x)dx = m(0), \ \Psi'(0) = \mu'(0), \ \int_\Omega \Psi(x)dx = m'(0).$$

The present work can be regarded as a continuation of results obtained in [7], in so far as in [7], are shown the existence and uniqueness of a strong solution of a similar problem, in which the proof is based on an energy inequality and on the density of the range of the operator generated by the studied problem. However, in the present work, we are interested in classical and weak solutions of the considered problem. To this end, we convert problem (1.1)-(1.3) into an equivalent problem as in [4], while the Neumann and integral conditions are homogeneous:

$$\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial t \partial x^2} = f(x,t), \qquad (x,t) \in Q, \tag{1.4}$$

$$u(x,0) = \varphi(x), \qquad \frac{\partial u(x,0)}{\partial t} = \psi(x), \qquad x \in \Omega, \tag{1.5}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad \int_\Omega u(x,t)dx = 0, \qquad t \in I. \tag{1.6}$$

with

$$\varphi'(0) = 0, \int_\Omega \varphi(x)dx = 0, \ \psi'(0) = 0, \int_\Omega \psi(x)dx = 0. \tag{1.7}$$

Let us mention that this paper can be considered as a continuation of [1 , 3 , 5 , 6 , 8 , 9 , 10, 11, 13], where mixed problems with nonlocal condition(s) for other linear equations were considered.

The plan of the paper is the following:

2. Notations and preliminary results
3. Classical solution
3.1. Uniqueness and continuous dependence of the solution
3.2. Existence of the solution
3. Weak solution
3.1. Weak formulation
3.2. A priori estimates
3.3. Existence of the solution
3.4. Uniqueness and continuous dependence of the solution.

2. Notation and preliminary results

The following notation will be used throughout the paper:

Let H be a Hilbert space with norm $\|.\|_H$. For simplicity, throughout the paper, $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$ and $\|\cdot\|$ its associated norm. By

$L_0^2(\Omega)$, we denote the space of square integrable functions with zero mean value on Ω, namely

$$L_0^2(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega u dx = 0 \right\},$$

which is clearly a Hilbert space for the scalar product $(\cdot,\cdot)$. We denote by $C(Q), C^i(Q)$ $(i = 1,2), L^2(I;H), H^1(I;H)$ the standard function spaces; by $C_0(\Omega)$ the space of continuous functions with compact support in Ω; by $B_2^1(\Omega)$ the space of square summable primitive functions, first introduced by the author [3 , 4 , 8], which can be considered as a completion of space $C_0(\Omega)$ for the scalar product

$$(u,w)_{B_2^1(\Omega)} = \int_\Omega \Im_x u \, \Im_x w \, dx,$$

and the corresponding norm

$$\|u\|_{B_2^1(\Omega)} = \sqrt{(u,u)_{B_2^1(\Omega)}} = \|\Im_x u\|,$$

for every fixed $x \in \Omega$, where $\Im_x u = \int_0^x u(\xi)\, d\xi$. It is easy to get

$$\|u\|^2_{B_2^1(\Omega)} \leq \frac{l^2}{2} \|u\|^2, \tag{2.1}$$

for every $u \in L^2(\Omega)$, from which we deduce the continuity of the imbedding $L^2(\Omega) \to B_2^1(\Omega)$.

To obtain the classical solution, we use Fourier's method. For this, we have need to recall the following properties of Fourier series:

LEMMA 2.1([12]). *If a continuous periodic function $\widetilde{u}(x,.)$ has continuous derivatives up to and including $(k-1)-th$ order, whilst the $k-th$ order derivative is piecewise continuous, then the Fourier coefficients a_n and b_n of $\widetilde{u}(x,.)$ are of order $O\left(\frac{1}{n^{k+1}}\right)$.*

REMARK 2.1. If we have to expand the function $u(x,.)$ given in the interval Ω in a series $\cos\frac{n\pi}{l}x$ then the previous requirements must be fulfilled for the function $\widetilde{u}(x,.)$ obtained for the even extension of $u(x,.)$. In particular, for the continuity of $\widetilde{u}(x,.)$ it is necessary that $u'(0,.) = u'(l,.) = 0$, since otherwise the even extension has a discontinuity at points $x = 0$ and $x = l$. The continuity of $\widetilde{u}(x,.)$ at points $x = 0$ and $x = l$ holds automatically for the even extension. In general, for the continuity of odd derivatives of the extended function must have $u^{(k)}(0,.) = u^{(k)}(l,.) = 0$ $(k = 1,3,5,...,2n+1)$. The even derivatives are automatically continuous.

3. Classical solution

The nearest result to the investigation considered in this subsection can be found in [2], in which is studied a similar problem with $\alpha = 1$, $\beta = 0$ and instead of the Neumann condition it is given the Dirichlet condition. Since the presence of the integral in boundary conditions can greatly complicate the application of standard techniques, the author of [2] converted the posed problem to an equivalent form in which the integral term did not appear, and used the Fourier method to prove the existence and uniqueness of a classical solution. However, in this subsection and in the remainder of the paper we conserve the problem as is posed and we work in the nonclassical function space $B_2^1(\Omega)$. This greatly facilitate the solvability of the considered problem.

We start by stating the definition of classical solution to problem (1.4)-(1.6):

DEFINITION 3.1. A classical solution of problem (1.4)-(1.6) is a function $u : \overline{I} \to L^2(\Omega)$ such that:
i) $u \in C^2(\overline{Q})$ with continuous derivative in $\overline{Q}$ of the form $\frac{\partial^3 u}{\partial t \partial x^2}$,
ii) u satisfying equation (1.4),
iii) u fulfils the initial conditions (1.5), and the boundary conditions (1.6).

3.1. Uniqueness and continuous dependence of the solution

In this section, we first establish some a priori estimates, then we conclude directly the uniqueness and continuous dependence of the solution upon the data.

THEOREM 3.1. *If $u(x,t)$ is a classical solution of problem* (1.4)-(1.6) *and $f \in C(\overline{Q})$, then it satisfies the following a priori estimates:*

$$\|u(.,\tau)\|^2 \leq c_1 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\, dt + \|\varphi\|^2 + \|\psi\|^2_{B_2^1(\Omega)} \right), \tag{3.1}$$

$$\left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{B_2^1(\Omega)} \leq c_2 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\, dt + \|\varphi\|^2 + \|\psi\|^2_{B_2^1(\Omega)} \right), \tag{3.2}$$

where $c_1 = \frac{1}{\alpha} \max\left(1, \alpha, \frac{l^2}{4\beta}\right)$, $c_2 = \max\left(1, \alpha, \frac{l^2}{4\beta}\right)$ *and* $0 \leq \tau \leq T$.

PROOF. Taking the scalar product in $B_2^1(\Omega)$ of equation (1.4) and $\frac{\partial u}{\partial t}$, and integrating the result over $(0,\tau)$, we have

$$\begin{aligned} &\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt - \alpha \int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt \\ &-\beta \int_0^\tau \left(\frac{\partial^3 u(.,t)}{\partial t \partial x^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt = \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt. \end{aligned} \tag{3.3}$$

The standard integration by parts of the left-hand side of (3.3) leads to

$$\alpha \|u(.,\tau)\|^2 + \left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{B_2^1(\Omega)} + 2\beta \int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2 dt \tag{3.4}$$
$$= 2\int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(\Omega)} dt + \alpha \|\varphi\|^2 + \|\psi\|^2_{B_2^1(\Omega)} .$$

In light of the Cauchy inequality and (2.1), the first term on the right-hand side of (3.3) is bounded by

$$\varepsilon \int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\, dt + \frac{l^2}{2\varepsilon} \int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2 dt.$$

We choose $\varepsilon = \frac{l^2}{4\beta}$ so that the second term will be simplified by the third term in the left-hand side. Thus, we have

$$\alpha \|u(.,\tau)\|^2 + \left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{B_2^1(\Omega)} \tag{3.5}$$
$$\leq \frac{l^2}{4\beta} \int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\, dt + \alpha \|\varphi\|^2 + \|\psi\|^2_{B_2^1(\Omega)} .$$

From (3.5), we obtain estimates (3.1) and (3.2).

COROLLARY 3.2. *The classical solution of problem* (1.4)-(1.6), *if there exists, is unique and depends continuously upon* φ, ψ *and* f.

3.2. Existence of the solution

Problem (1.4)-(1.6) being linear, then its solution can be found as the sum of solutions of the following two problems:

$$\frac{\partial^2 \omega}{\partial t^2} - \alpha \frac{\partial^2 \omega}{\partial x^2} - \beta \frac{\partial^3 \omega}{\partial t \partial x^2} = 0, \tag{3.6}$$

$$\omega(x,0) = \varphi(x), \qquad \frac{\partial \omega(x,0)}{\partial t} = \psi(x), \tag{3.7}$$

$$\frac{\partial \omega(0,t)}{\partial x} = 0, \qquad \int_\Omega \omega(x,t)dx = 0, \tag{3.8}$$

and

$$\frac{\partial^2 \varpi}{\partial t^2} - \alpha \frac{\partial^2 \varpi}{\partial x^2} - \beta \frac{\partial^3 \varpi}{\partial t \partial x^2} = f(x,t), \tag{3.9}$$

$$\varpi(x,0)=0, \qquad \frac{\partial \varpi(x,0)}{\partial t}=0, \tag{3.10}$$

$$\frac{\partial \varpi(0,t)}{\partial x}=0, \qquad \int_{\Omega}\varpi(x,t)dx=0. \tag{3.11}$$

Let us start with the first problem. Applying the method of separation of variables, so we shall find a non-trivial solution of the form $\omega(x,t)=X(x)\cdot T(t)$. Substituting it in equation (3.6), we get

$$\frac{X''(x)}{X(x)}=\frac{T''(t)}{\beta T'(t)+\alpha T(t)},$$

therefore, we obtain the equations

$$X''(x)+\lambda X(x)=0, \tag{3.12}$$

$$T''(t)+\beta\lambda T'(t)+\alpha\lambda T(t)=0, \tag{3.13}$$

where λ is a strictly positive constant. The boundary conditions (3.8) give

$$X'(0)=0, \qquad \int_{\Omega}X(x)dx=0. \tag{3.14}$$

According to conditions (3.14), the non-trivial solution of (3.12) is given by

$$X_n(x)=A_n\cos\sqrt{\lambda_n}x, \tag{3.15}$$

with $\lambda_n=\left(\frac{n\pi}{l}\right)^2$. To each of these values of λ_n, we must find the solution of equation (3.13). Being a differential equation with constant coefficients, then its characteristic equation is

$$r^2+\beta\left(\frac{n\pi}{l}\right)^2 r+\left(\frac{n\pi}{l}\right)^2\alpha=0. \tag{3.16}$$

Since the discriminant Δ of (3.16), defined by $\Delta=\left(\frac{n\pi}{l}\right)^2\left[\left(\frac{n\pi}{l}\right)^2\beta^2-4\alpha\right]$, is positive for $n\geq n_0=E\left(\frac{2l\sqrt{\alpha}}{\beta\pi}\right)+1$, thus the equation admits two distinct roots:

$$r_n^{(1)}=\left(-\left(\tfrac{n\pi}{l}\right)^2\beta+\sqrt{\Delta}\right)\Big/2, r_n^{(2)}=\left(-\left(\tfrac{n\pi}{l}\right)^2\beta-\sqrt{\Delta}\right)\Big/2. \tag{3.17}$$

Hence the solution of (3.13) is given by

$$T_n(t)=B_n e^{r_n^{(1)}t}+\widetilde{B}_n e^{r_n^{(2)}t}. \tag{3.18}$$

REMARK 3.1. The roots $r_n^{(1)}$ and $r_n^{(2)}$ are negative. Indeed, suppose that $r_n^{(1)}$ is positive, i.e., $-\left(\frac{n\pi}{l}\right)^2\beta+\sqrt{\Delta}\geq 0$, thus $\Delta\geq\left(\frac{n\pi}{l}\right)^4\beta^2$ or $\left(\frac{n\pi}{l}\right)^4\beta^2-4\alpha\left(\frac{n\pi}{l}\right)^2\geq\left(\frac{n\pi}{l}\right)^4\beta^2$, which is absurd.

Consequently, we have

$$e^{r_n^{(1)}t}<1 \text{ and } e^{r_n^{(2)}t}<1. \tag{3.19}$$

It then follows from (3.15) and (3.18), that

$$\omega(x,t)=\sum_{n\geq n_0}\omega_n(x,t)=\sum_{n\geq n_0}\cos\frac{n\pi}{l}x\left(C_ne^{r_n^{(1)}t}+D_ne^{r_n^{(2)}t}\right),$$

where

$$C_n=\frac{r_n^{(2)}\varphi_n-\psi_n}{r_n^{(2)}-r_n^{(1)}},\ D_n=\frac{\psi_n-r_n^{(1)}\varphi_n}{r_n^{(2)}-r_n^{(1)}},$$

with

$$\varphi_n=\frac{2}{l}\int_\Omega\varphi(x)\cos\frac{n\pi}{l}xdx,\ \psi_n=\frac{2}{l}\int_\Omega\psi(x)\cos\frac{n\pi}{l}xdx.$$

Thus the solution $\omega(x,t)$ can be written as follows

$$\omega(x,t)=\sum_{n\geq n_0}\cos\frac{n\pi}{l}x\left(\frac{r_n^{(2)}e^{r_n^{(1)}t}-r_n^{(1)}e^{r_n^{(2)}t}}{r_n^{(2)}-r_n^{(1)}}\varphi_n+\frac{e^{r_n^{(2)}t}-e^{r_n^{(1)}t}}{r_n^{(2)}-r_n^{(1)}}\psi_n\right). \tag{3.20}$$

We now show that the series (3.20) does in fact give the classical solution to problem (3.6)-(3.8).

THEOREM 3.3. *Suppose that:*
C1: $\varphi,\ \psi\in C^2\left(\overline{\Omega}\right)$ *and their third derivatives are piecewise continuous,*
C2: φ *and* ψ *verify the compatibility conditions* (1.7).
Then, there exists a classical solution of problem (3.6)-(3.8) *which can be represented as a series converging uniformly on* $\overline{Q}$. *This series can be differentiated, term by term, twice with respect to* x *and* t *and on the form* $\frac{\partial^3}{\partial t\partial x^2}$. *The obtained series for the derivatives of the solution converge uniformly in the rectangle* $\overline{Q}$.

PROOF. First, we shall verify the continuity of the function $\omega(x,t)$. For this, it is sufficient to prove the uniform convergence of the series (3.20). From which, we have

$$|\omega_n(x,t)|\leq\frac{\left|r_n^{(2)}e^{r_n^{(1)}t}-r_n^{(1)}e^{r_n^{(2)}t}\right|}{\left|r_n^{(2)}-r_n^{(1)}\right|}|\varphi_n|+\frac{\left|e^{r_n^{(2)}t}-e^{r_n^{(1)}t}\right|}{\left|r_n^{(2)}-r_n^{(1)}\right|}|\psi_n|.$$

In light of (3.19), we get

$$|\omega_n(x,t)| \leq \frac{\left|r_n^{(2)}\right| + \left|r_n^{(1)}\right|}{\left|r_n^{(2)} - r_n^{(1)}\right|} |\varphi_n| + \frac{2}{\left|r_n^{(2)} - r_n^{(1)}\right|} |\psi_n| .$$

According to conditions C1 in Theorem 3.3 and making use Lemma 2.1, we obtain

$$|\varphi_n| = O\left(\tfrac{1}{n^4}\right), \tag{3.21}$$

$$|\psi_n| = O\left(\tfrac{1}{n^4}\right). \tag{3.22}$$

It follows from (3.17) that

$$\left|r_n^{(2)}\right| + \left|r_n^{(1)}\right| = O\left(n^2\right), \tag{3.23}$$

$$\frac{1}{\left|r_n^{(2)} - r_n^{(1)}\right|} = O\left(\tfrac{1}{n^2}\right), \tag{3.24}$$

$$\left|r_n^{(2)} \cdot r_n^{(1)}\right| = O\left(n^2\right), \tag{3.25}$$

$$\left|r_n^{(2)}\right|^2 + \left|r_n^{(1)}\right|^2 = O\left(n^4\right). \tag{3.26}$$

By virtue of (3.21)-(3.24), we conclude that $|\omega_n| = O\left(\frac{1}{n^4}\right)$, then the series (3.20) converge uniformly, i.e., the function $\omega(x,t)$ continuously adjoins its initial condition. We must prove the uniform convergence of the series

$$\frac{\partial \omega(x,t)}{\partial t} = \sum_{n \geq n_0} \cos\frac{n\pi}{l}x \left(\frac{r_n^{(1)} r_n^{(2)} \left(e^{r_n^{(1)}t} - e^{r_n^{(2)}t}\right)}{r_n^{(2)} - r_n^{(1)}} \varphi_n + \frac{r_n^{(2)} e^{r_n^{(2)}t} - r_n^{(1)} e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}} \psi_n \right),$$

$$\frac{\partial \omega(x,t)}{\partial x} = -\sum_{n \geq n_0} \frac{n\pi}{l} \sin\frac{n\pi}{l}x \left(\frac{r_n^{(2)} e^{r_n^{(1)}t} - r_n^{(1)} e^{r_n^{(2)}t}}{r_n^{(2)} - r_n^{(1)}} \varphi_n + \frac{e^{r_n^{(2)}t} - e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}} \psi_n \right),$$

$$\int_0^x \omega(\xi,t)d\xi = \sum_{n \geq n_0} \frac{l}{n\pi} \sin\frac{n\pi}{l}x \left(\frac{r_n^{(2)} e^{r_n^{(1)}t} - r_n^{(1)} e^{r_n^{(2)}t}}{r_n^{(2)} - r_n^{(1)}} \varphi_n + \frac{e^{r_n^{(2)}t} - e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}} \psi_n \right).$$

Similarly, by taking (3.21)-(3.25) into account, we get

$$\left|\tfrac{\partial \omega_n(x,t)}{\partial t}\right| = O\left(\tfrac{1}{n^4}\right), \quad \left|\tfrac{\partial \omega_n(x,t)}{\partial x}\right| = O\left(\tfrac{1}{n^3}\right) \text{ and } \left|\int_0^x \omega_n(\xi,t)d\xi\right| = O\left(\tfrac{1}{n^5}\right).$$

Thus the functions $\frac{\partial\omega(x,t)}{\partial t}$, $\frac{\partial\omega(x,t)}{\partial x}$ and $\int_0^x \omega(\xi,t)d\xi$, adjoin continuously $\frac{\partial\omega(x,0)}{\partial t}$, $\frac{\partial\omega(0,t)}{\partial x}$ and $\int_\Omega \omega(\xi,t)d\xi$, respectively.

We now show that the functions $\frac{\partial^2\omega}{\partial t^2}$, $\frac{\partial^2\omega}{\partial x^2}$, $\frac{\partial^3\omega}{\partial t\partial x^2}$ converge uniformly. To do this, differentiating the series (3.20) with respect to t and x as follows:

$$\frac{\partial^2\omega}{\partial t^2} = \sum_{n\geq n_0} \cos\frac{n\pi}{l}x \left(\frac{r_n^{(1)}r_n^{(2)}\left(r_n^{(1)}e^{r_n^{(1)}t} - r_n^{(2)}e^{r_n^{(2)}t}\right)}{r_n^{(2)} - r_n^{(1)}}\varphi_n + \frac{\left(r_n^{(2)}\right)^2 e^{r_n^{(2)}t} - \left(r_n^{(1)}\right)^2 e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}}\psi_n \right),$$

$$\frac{\partial^2\omega}{\partial x^2} = -\sum_{n\geq n_0} \left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi}{l}x \left(\frac{r_n^{(2)}e^{r_n^{(1)}t} - r_n^{(1)}e^{r_n^{(2)}t}}{r_n^{(2)} - r_n^{(1)}}\varphi_n + \frac{e^{r_n^{(2)}t} - e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}}\psi_n \right),$$

$$\frac{\partial^3\omega}{\partial t\partial x^2} = -\sum_{n\geq n_0} \left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi}{l}x \left(\frac{r_n^{(1)}r_n^{(2)}\left(e^{r_n^{(1)}t} - e^{r_n^{(2)}t}\right)}{r_n^{(2)} - r_n^{(1)}}\varphi_n + \frac{e^{r_n^{(2)}t} - e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}}\psi_n \right).$$

In light of (3.21)-(3.26), we have

$$\left|\frac{\partial^2\omega}{\partial t^2}\right| = O\left(\tfrac{1}{n^2}\right), \left|\frac{\partial^2\omega}{\partial x^2}\right| = O\left(\tfrac{1}{n^2}\right), \left|\frac{\partial^3\omega}{\partial t\partial x^2}\right| = O\left(\tfrac{1}{n^2}\right),$$

from which we get the result. □

Knowing the solution of problem (3.6)-(3.8), we shall find a solution expressed as a Fourier series of problem (3.9)-(3.11) in the form

$$\varpi(x,t) = \sum_{n\geq n_0} \varpi_n(x,t) = \sum_{n\geq n_0} \varpi_n(t)\cos\frac{n\pi}{l}x. \tag{3.27}$$

For this, we determine $f(x,t)$ as follows:

$$f(x,t) = \sum_{n\geq n_0} f_n(t)\cos\frac{n\pi}{l}x \tag{3.28}$$

with $f_n(t) = \frac{2}{l}\int_\Omega f(x,t)\cos\frac{n\pi}{l}xdx$. Substituting (3.27) and (3.28) in (3.9), yields

$$\sum_{n\geq n_0} \cos\frac{n\pi}{l}x \left(\varpi_n''(t) - \beta\left(\frac{n\pi}{l}\right)^2 \varpi_n'(t) - \alpha\left(\frac{n\pi}{l}\right)^2 \varpi_n(t) - f_n(t) \right) = 0.$$

We observe that it is satisfied if

$$\varpi_n''(t) - \beta\left(\frac{n\pi}{l}\right)^2 \varpi_n'(t) - \alpha\left(\frac{n\pi}{l}\right)^2 \varpi_n(t) = f_n(t).$$

Thus the required solution is

$$\varpi(x,t) = \sum_{n\geq n_0} \cos\frac{n\pi}{l}x \, \frac{1}{\left|r_n^{(2)} - r_n^{(1)}\right|} \int_0^t \left(e^{r_n^{(2)}(t-\tau)} - e^{r_n^{(1)}(t-\tau)}\right) f_n(t)d\tau. \tag{3.29}$$

It follows from (3.20) and (3.29) that

$$\begin{aligned} u(x,t) \;=\; & \sum_{n\geq n_0} \cos\frac{n\pi}{l}x \left(\frac{r_n^{(2)} e^{r_n^{(1)}t} - r_n^{(1)} e^{r_n^{(2)}t}}{r_n^{(2)} - r_n^{(1)}}\varphi_n + \frac{e^{r_n^{(2)}t} - e^{r_n^{(1)}t}}{r_n^{(2)} - r_n^{(1)}}\psi_n \right. \\ & \left. + \frac{1}{\left|r_n^{(2)} - r_n^{(1)}\right|} \int_0^t \left(e^{r_n^{(2)}(t-\tau)} - e^{r_n^{(1)}(t-\tau)}\right) f_n(t)d\tau \right). \end{aligned}$$

4. Weak solution

4.1. Weak formulation

In order to define a weak solution of problem (1.4)-(1.6), we take the scalar product in $L^2\left(I, B_2^1(\Omega)\right)$ of equation (1.4) and the test function $v(x,t) \in V = \left\{v/v \in H^1\left(I, L_0^2(\Omega)\right) : v(.,T) = 0\right\}$. This yields

$$\begin{aligned} & \left(\frac{\partial^2 u}{\partial t^2}, v\right)_{L^2\left(I,B_2^1(\Omega)\right)} - \alpha\left(\frac{\partial^2 u}{\partial x^2}, v\right)_{L^2\left(I,B_2^1(\Omega)\right)} \\ & -\beta\left(\frac{\partial^3 u}{\partial t \partial x^2}, v\right)_{L^2\left(I,B_2^1(\Omega)\right)} = (f,v)_{L^2\left(I,B_2^1(\Omega)\right)}. \end{aligned} \tag{4.1}$$

The standard integration by parts gives

$$\left(\frac{\partial^2 u}{\partial t^2}, v\right)_{L^2\left(I,B_2^1(\Omega)\right)} = -\left(\psi, v(.,0)\right)_{B_2^1(\Omega)} - \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{L^2\left(I,B_2^1(\Omega)\right)}, \tag{4.2}$$

$$-\alpha\left(\frac{\partial^2 u}{\partial x^2}, v\right)_{L^2\left(I;B_2^1(\Omega)\right)} = \alpha\left(u,v\right)_{L^2\left(I,L_0^2(\Omega)\right)}, \tag{4.3}$$

$$-\beta\left(\frac{\partial^3 u}{\partial t \partial x^2}, v\right)_{L^2\left(I,B_2^1(\Omega)\right)} = -\beta\left(\varphi, v(.,0)\right) - \beta\left(u, \frac{\partial v}{\partial t}\right)_{L^2\left(I,L_0^2(\Omega)\right)}. \tag{4.4}$$

Inserting (4.2)-(4.4) into (4.1), we get

$$\begin{aligned} & -\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{L^2\left(I, B_2^1(\Omega)\right)} + \alpha\,(u,v)_{L^2\left(I, L_0^2(\Omega)\right)} - \beta\left(u, \frac{\partial v}{\partial t}\right)_{L^2\left(I, L_0^2(\Omega)\right)} \\ = \; & (f,v)_{L^2\left(I, B_2^1(\Omega)\right)} + \beta\,(\varphi, v(.,0)) + (\psi, v(.,0))_{B_2^1(\Omega)}\,. \end{aligned}$$

The above equality can be written as follows:

$$A\,(u,v) = (f,v)_{L^2\left(I, B_2^1(\Omega)\right)} + \beta\,(\varphi, v(.,0)) + (\psi, v(.,0))_{B_2^1(\Omega)}\,, \tag{4.5}$$

where $A\,(u,v)$ is its left-hand side.

DEFINITION 4.1. The function u is called a weak solution of problem (1.4)-(1.6), if the following properties are fulfilled:
i) $u \in L^2\left(I, L_0^2\,(\Omega)\right)$,
ii) $\frac{\partial u}{\partial t} \in L^2\left(I, B_2^1\,(\Omega)\right)$,
iii) u verifies the integral condition (1.6_2),
iv) the integral identity (4.5) holds for all functions $v \in V$.

We observe that due to i), the integral condition (1.6_2) has sense, and by virtue of i) and ii) and (1.7) each term in the integral identity (4.5) is well defined.

4.2. A priori estimates

Suppose that there are sequences of functions $f^{(n)}, \varphi^{(n)}, \psi^{(n)}, n = 1, 2, ...,$ satisfying conditions of Theorem 3.3, such that

$$\begin{cases} f^{(n)} \underset{n\to\infty}{\to} f & \text{in } B_2^1(\Omega), \quad \forall t \in \overline{I} \\ \varphi^{(n)} \underset{n\to\infty}{\to} \varphi & \text{in } L_0^2(\Omega), \\ \psi^{(n)} \underset{n\to\infty}{\to} \psi & \text{in } B_2^1(\Omega), \end{cases} \tag{4.6}$$

and for each $n = 1, 2, ...,$ there is a unique classical solution $u^{(n)}(x,t)$, in the sense of Definition 4.1, of the problem

$$\frac{\partial^2 u^{(n)}}{\partial t^2} - \alpha \frac{\partial^2 u^{(n)}}{\partial x^2} - \beta \frac{\partial^3 u^{(n)}}{\partial t \partial x^2} = f^{(n)}(x,t), \tag{4.7}$$

$$u^{(n)}(x,0) = \varphi^{(n)}(x), \quad \frac{\partial u^{(n)}(x,0)}{\partial t} = \psi^{(n)}(x), \tag{4.8}$$

$$\frac{\partial u^{(n)}(0,t)}{\partial x} = 0, \quad \int_\Omega u^{(n)}(x,t)dx = 0. \tag{4.9}$$

Let us consider problem (4.7)-(4.9) for $n = k$ and $n = m$. Subtracting these equalities, we obtain

$$\frac{\partial^2 u^{(k,m)}}{\partial t^2} - \alpha \frac{\partial^2 u^{(k,m)}}{\partial x^2} - \beta \frac{\partial^3 u^{(k,m)}}{\partial t \partial x^2} = f^{(k,m)}(x,t), \tag{4.10}$$

$$u^{(k,m)}(x,0) = \varphi^{(k,m)}(x), \quad \frac{\partial u^{(k,m)}(x,0)}{\partial t} = \psi^{(k,m)}(x), \tag{4.11}$$

$$\frac{\partial u^{(k,m)}(0,t)}{\partial x} = 0, \quad \int_\Omega u^{(k,m)}(x,t)dx = 0, \tag{4.12}$$

with $u^{(k,m)}(x,t) = u^{(k)}(x,t) - u^{(m)}(x,t)$, $f^{(k,m)}(x,t) = f^{(k)}(x,t) - f^{(m)}(x,t)$, $\varphi^{(k,m)}(x,t) = \varphi^{(k)}(x,t) - \varphi^{(m)}(x,t)$, $\psi^{(k,m)}(x,t) = \psi^{(k)}(x,t) - \psi^{(m)}(x,t)$.

In this subsection, we derive estimates for classical solutions to problem (4.10)-(4.12). The existence of a weak solution is obtained by subtracting a convergent sequence, once a priori estimates are established.

THEOREM 4.1. *For all $k, m \in \mathbb{N}^*$, the following a priori estimates hold:*

(4.13)
$$\left\|u^{(k,m)}\right\|^2_{L^2\left(I,L^2_0(\Omega)\right)} \leq \frac{c_3}{\alpha}\left(\left\|f^{(k,m)}\right\|^2_{L^2\left(I,B^1_2(\Omega)\right)} + \left\|\varphi^{(k,m)}\right\|^2 + \left\|\psi^{(k,m)}\right\|^2_{B^1_2(\Omega)}\right),$$

(4.14)
$$\left\|\frac{\partial u^{(k,m)}}{\partial t}\right\|^2_{L^2\left(I,B^1_2(\Omega)\right)} \leq c_3\left(\left\|f^{(k,m)}\right\|^2_{L^2\left(I,B^1_2(\Omega)\right)} + \left\|\varphi^{(k,m)}\right\|^2 + \left\|\psi^{(k,m)}\right\|^2_{B^1_2(\Omega)}\right),$$

where $c_3 = \max\left(1, \alpha, \frac{l^2}{2\beta}\right) Te^T$.

PROOF. Taking the scalar product in $B^1_2(\Omega)$ of equation (4.10) and $\frac{\partial u^{(k,m)}}{\partial t}$, and integrating the result over $(0,\tau)$ with respect to t, we get

$$\alpha\left\|u^{(k,m)}(.,\tau)\right\|^2 + \left\|\frac{\partial u^{(k,m)}(.,\tau)}{\partial t}\right\|^2_{B^1_2(\Omega)} + 2\beta\int_0^\tau \left\|\frac{\partial u^{(k,m)}(.,t)}{\partial t}\right\|^2 dt$$
$$= 2\int_0^\tau \left(f^{(k,m)}(.,t), \frac{\partial u^{(k,m)}(.,t)}{\partial t}\right)_{B^1_2(\Omega)} dt + \alpha\left\|\varphi^{(k,m)}\right\|^2 + \left\|\psi^{(k,m)}\right\|^2_{B^1_2(\Omega)}. \tag{4.15}$$

Omitting the third term on the left-hand side of (4.15) and estimating the first

integral on the right-hand side by applying the Cauchy inequality, we obtain

$$\begin{aligned}
&\alpha\left\|u^{(k,m)}(.,\tau)\right\|^2+\left\|\frac{\partial u^{(k,m)}(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)}\\
\leq\ &\left(\frac{l^2}{2\beta}\int_0^\tau\left\|f^{(k,m)}(.,t)\right\|^2_{B_2^1(\Omega)}dt+\alpha\left\|\varphi^{(k,m)}\right\|^2+\left\|\psi^{(k,m)}\right\|^2_{B_2^1(\Omega)}\right)\\
&+\int_0^\tau\left\|\frac{\partial u^{(k,m)}(.,t)}{\partial t}\right\|^2_{B_2^1(\Omega)}dt\\
\leq\ &\left(\frac{l^2}{2\beta}\int_0^\tau\left\|f^{(k,m)}(.,t)\right\|^2_{B_2^1(\Omega)}dt+\alpha\left\|\varphi^{(k,m)}\right\|^2+\left\|\psi^{(k,m)}\right\|^2_{B_2^1(\Omega)}\right)\\
&+\int_0^\tau\left(\alpha\left\|u^{(k,m)}(.,t)\right\|^2+\left\|\frac{\partial u^{(k,m)}(.,t)}{\partial t}\right\|^2_{B_2^1(\Omega)}\right)dt.
\end{aligned}$$

Therefore, from Gronwall's Lemma, we come to the conclusion that

$$\begin{aligned}
&\alpha\left\|u^{(k,m)}(.,\tau)\right\|^2+\left\|\frac{\partial u^{(k,m)}(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)}\\
\leq\ &\max\left(1,\alpha,\frac{l^2}{2\beta}\right)Te^T\left(\int_0^T\left\|f^{(k,m)}(.,t)\right\|^2_{B_2^1(\Omega)}dt\right.\\
&\left.+\left\|\varphi^{(k,m)}\right\|^2+\left\|\psi^{(k,m)}\right\|^2_{B_2^1(\Omega)}\right).
\end{aligned}\tag{4.16}$$

Integrating (4.16) with respect to τ over I, we get

$$\begin{aligned}
&\alpha\int_0^T\left\|u^{(k,m)}(.,\tau)\right\|^2dt+\int_0^T\left\|\frac{\partial u^{(k,m)}(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)}dt\\
\leq\ &c_3\left(\left\|f^{(k,m)}\right\|^2_{L^2\left(I,B_2^1(\Omega)\right)}+\left\|\varphi^{(k,m)}\right\|^2+\left\|\psi^{(k,m)}\right\|^2_{B_2^1(\Omega)}\right).
\end{aligned}$$

from which we have estimates (4.13) and (4.14).

REMARK 4.1. Since the right-hand side of inequality (4.16) is taken independent of τ, we can obtain other estimates by taking the least upper bound of the left-hand side of (4.16) with respect to τ from 0 to T. Thus, we have

$$\left\|u^{(k,m)}\right\|^2_{C(I,L^2(\Omega))}\leq\frac{c_4}{\alpha}\left(\left\|f^{(k,m)}\right\|^2_{L^2\left(I,B_2^1(\Omega)\right)}+\left\|\varphi^{(k,m)}\right\|^2+\left\|\psi^{(k,m)}\right\|^2_{B_2^1(\Omega)}\right),$$

$$\left\|\frac{\partial u^{(k,m)}}{\partial t}\right\|^2_{C\left(I,B_2^1(\Omega)\right)}\leq c_4\left(\left\|f^{(k,m)}\right\|^2_{L^2\left(I,B_2^1(\Omega)\right)}+\left\|\varphi^{(k,m)}\right\|^2+\left\|\psi^{(k,m)}\right\|^2_{B_2^1(\Omega)}\right),$$

where $c_4 = \max\left(1, \alpha, \frac{l^2}{2\beta}\right) e^T$.

COROLLARY 4.2. *The sequences* $\left\{u^{(n)}\right\}_n$ *and* $\left\{\frac{\partial u^{(n)}}{\partial t}\right\}_n$ *converge to* $\overline{u}$ *and* $\frac{\partial \overline{u}}{\partial t}$, *respectively, in the following sense:*

$$u^{(n)} \to \overline{u} \qquad in\ L^2\left(I, L_0^2(\Omega)\right), \tag{4.17}$$

$$\frac{\partial u^{(n)}}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \qquad in\ L^2\left(I, B_2^1(\Omega)\right). \tag{4.18}$$

4.3. Existence of the solution

In this section, we shall prove that the limit-function $\overline{u}$ is a solution of problem (1.4)-(1.6) in the sense of Definition 3.2.

THEOREM 4.3. *Suppose that* $\varphi \in L_0^2(\Omega), \psi \in B_2^1(\Omega)$, $f \in L^2\left(I, B_2^1(\Omega)\right)$. *Then problem* (1.4)-(1.6) *possesses a weak solution.*

PROOF. We must show that $\overline{u}$ satisfies identity (4.5). For this, we take the weak formulation of the approximated problem

$$A\left(u^{(n)}, v\right) = \left(f^{(n)}, v\right)_{L^2\left(I, B_2^1(\Omega)\right)} + \beta\left(\varphi^{(n)}, v(.,0)\right) + \left(\psi^{(n)}, v(.,0)\right)_{B_2^1(\Omega)}.$$

Due to the linearity of the problem, the last identity imply that, when $n \to \infty$, the following holds, for all $v \in V$ and $n = 1, 2, \ldots$:

$$\begin{aligned}
&\lim_{n\to\infty} A\left(u^{(n)} - \overline{u}, v\right) + A\left(\overline{u}, v\right) \\
= &\lim_{n\to\infty} \left(f^{(n)} - f, v\right)_{L^2\left(I, B_2^1(\Omega)\right)} + (f, v)_{L^2\left(I, B_2^1(\Omega)\right)} \\
&+\beta \lim_{n\to\infty} \left(\varphi^{(n)} - \varphi, v(.,0)\right) + \beta\left(\varphi, v(.,0)\right) \\
&+ \lim_{n\to\infty} \left(\psi^{(n)} - \psi, v(.,0)\right)_{B_2^1(\Omega)} + \left(\psi, v(.,0)\right)_{B_2^1(\Omega)}.
\end{aligned}$$

According to Schwarz inequality and inequalities (2.1), we get

$$
\begin{aligned}
(4.19)\qquad & A\left(u^{(n)}-\overline{u},v\right)\\
\leq\ & \frac{l}{\sqrt{2}}\left\|\frac{\partial u^{(n)}}{\partial t}-\frac{\partial\overline{u}}{\partial t}\right\|_{L^2\left(I,B_2^1(\Omega)\right)}\left\|\frac{\partial v}{\partial t}\right\|_{L^2\left(I,L_0^2(\Omega)\right)}\\
& +\alpha\left\|u^{(n)}-\overline{u}\right\|_{L^2\left(I,L_0^2(\Omega)\right)}\|v\|_{L^2\left(I,L_0^2(\Omega)\right)}\\
& +\beta\left\|u^{(n)}-\overline{u}\right\|_{L^2(I,L^2(\Omega))}\left\|\frac{\partial v}{\partial t}\right\|_{L^2\left(I,L_0^2(\Omega)\right)}\\
\leq\ & \max\left(\tfrac{l}{\sqrt{2}},\alpha+\beta\right)\left(\left\|u^{(n)}-\overline{u}\right\|_{L^2\left(I,L_0^2(\Omega)\right)}\right.\\
& \left.+\left\|\tfrac{\partial u^{(n)}}{\partial t}-\tfrac{\partial\overline{u}}{\partial t}\right\|_{L^2\left(I,B_2^1(\Omega)\right)}\right)\|v\|_{H^1\left(I,L_0^2(\Omega)\right)}.
\end{aligned}
$$

If we pass to the limit in (4.19) as $n\to\infty$, by taking into account of (4.17)-(4.18), we obtain

$$
(4.20)\qquad \lim_{n\to\infty}A\left(u^{(n)}-\overline{u},v\right)=0.
$$

Analogously, by taking (4.6) into account, we prove that

$$
\begin{aligned}
(4.21)\qquad & \lim_{n\to\infty}\left\{\left(f^{(n)}-f,v\right)_{L^2\left(I,B_2^1(\Omega)\right)}+\beta\left(\varphi^{(n)}-\varphi,v(.,0)\right)\right.\\
& \left.+\left(\psi^{(n)}-\psi,v(.,0)\right)_{B_2^1(\Omega)}\right\}=0.
\end{aligned}
$$

It then follows from (4.20) and (4.21) that the limit function $\overline{u}(x,t)$ satisfies identity (4.5) almost everywhere in Q.

Thus we deduce that the limit-function $\overline{u}(x,t)$ is a weak solution of problem (1.4)-(1.6), in the sense of Definition 4.1.

4.4. Uniqueness and continuous dependence of the solution

THEOREM 4.4. *Let* (f,φ,ψ) *and* $(f^*,\varphi^*,\psi^*)\in L^2\left(I,B_2^1(\Omega)\right)\times L_0^2(\Omega)\times B_2^1(\Omega)$ *and let* u *and* u^* *be two weak solutions of problem* (1.4)-(1.6), *corresponding, respectively, to the above data. Then the following estimates hold:*

$$
\begin{aligned}
(4.22)\qquad \|u-u^*\|^2_{L^2\left(I,L_0^2(\Omega)\right)} \leq\ & c_1T\left(\|f-f^*\|^2_{L^2\left(I,B_2^1(\Omega)\right)}\right.\\
& \left.+\|\varphi-\varphi^*\|^2+\|\psi-\psi^*\|^2_{B_2^1(\Omega)}\right),
\end{aligned}
$$

$$(4.23)\quad \left\| \frac{\partial u}{\partial t} - \frac{\partial u^*}{\partial t} \right\|^2_{L^2(I,B_2^1(\Omega))} \leq c_2 T \left(\|f - f^*\|^2_{L^2(I,B_2^1(\Omega))} + \|\varphi - \varphi^*\|^2 + \|\psi - \psi^*\|^2_{B_2^1(\Omega)} \right),$$

where c_i $(i = 1, 2)$ are positive constants independent of the solutions.

PROOF. We, first, prove that estimations (3.1)-(3.2) remain valid to weak solution u of problem (1.4)-(1.6). To this end, we consider the sequence $u^{(n)}(x,t)$, $n = 1, 2, ...$, of classical solutions of problem (4.7)-(4.9), converging to the weak solution. From (3.1), with u is replaced by $u^{(n)}$, we have

$$(4.24)\quad \left\| u^{(n)}(.,\tau) \right\|^2 \leq c_1 \left(\int_0^\tau \left\| f^{(n)}(.,t) \right\|^2_{B_2^1(\Omega)} dt + \left\| \varphi^{(n)} \right\|^2 + \left\| \psi^{(n)} \right\|^2_{B_2^1(\Omega)} \right).$$

If we pass to the limit in (4.23), by taking (4.6) and (4.17) into account, and we integrate over I with respect to t, it yields

$$(4.25)\quad \|u\|^2_{L^2(I,L^2_0(\Omega))} \leq c_1 T \left(\|f\|^2_{L^2(I,B_2^1(\Omega))} + \|\varphi\|^2 + \|\psi\|^2_{B_2^1(\Omega)} \right).$$

Set $w = u - u^*, g = f - f^*, u^0 = \varphi - \varphi^*$, $u^1 = \psi - \psi^*$. Since problem (1.4)-(1.6) is linear, we apply inequality (4.24) to the solution w, we get (4.21).

Estimate (4.22) is established in the similar way, by taking into (4.18) account, respectively, in place of (4.17).

COROLLARY 4.5. *Under hypotheses of Theorem 4.4, the solution of problem* (1.4)-(1.6) *is unique.*

References

[1] N. E. Benouar, N. I. Yurchuk, Mixed problem with an integral condition for parabolic equations with the Bessel operator, Differentsial'nye Uravneniya **27** (1991), 2094-2098.

[2] S. A. Beïlin, Existence of solutions for one-dimensional wave equations with nonlocal conditions, Electron. J. Differential Equations **2001** (2001), no. 76, 1-8.

[3] A. Bouziani, Mixed problem with boundary integral conditions for a certain parabolic equation, J. Appl. Math. Stochastic Anal. **9** (1996), no. 3, 323-330.

[4] A. Bouziani, Solution forte d'un problème mixte avec une condition non locale pour une classe d'équations hyperboliques, Acad. Roy. Belg. Bull. Cl. Sci. **8** (1997), 53–70.

[5] A. Bouziani, Strong solution for a mixed problem with nonlocal condition for certain pluriparabolic equations, Hiroshima Math. J. **27** (1997), no. 3, 373–390.

[6] A. Bouziani, On the solvability of nonlocal pluriparabolic problems, Electron. J. Differential Equations **2001** (2001), 1-16.

[7] A. Bouziani, Initial-boundary value problem with nonlocal condition for a viscosity equation, Int. J. Math. & Math. Sci. **30** (2002), no. 6, 327-338.

[8] A. Bouziani, On the solvability of parabolic and hyperbolic problems with a boundary integral condition, Internat. J. Math. & Math. Sci. **31** (2002), 201-213.

[9] A. Bouziani, On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in nonclassical function spaces, Internat. J. Math. & Math. Sci. **30** (2002), 435-447.

[10] A. Bouziani, On a class of nonclassical hyperbolic equations with nonlocal conditions. J. Appl. Math. Stochastic Anal. **15** (2002), no. 2, 135–153.

[11] A. Bouziani; N. E. Benouar, Mixed problem with integral conditions for a third order parabolic equation, Kobe J. Math. **15** (1998), no. 1, 47–58.

[12] A. I. Tikhonov and A. A. Samarskii, Equations of Mathematical physics, Edit. Mir, Moscow, 1984.

[13] N. I.Yurchuk, Mixed problem with an integral condition for certain parabolic equations, Differents. Uravn., **22** (1986), 2117-2126.

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