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THE CLASSIFICATION OF ISOPARAMETRIC SURFACES IN S_1^3

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Abstract

A non-degenerate surface in the de Sitter space S_1^3 is called an isoparametric surface, if the trace and the determinant of its shape-operator S are constant. In this paper we classify all isoparametric surfaces in the de Sitter space S_1^3. We show that an isoparametric in S_1^3 with diagonalizable shape-operator S is Lorentz equivalent to an open part of either a 2-sphere, a hyperboloid, a paraboloid or the surface $H^1(r) \times S^1(\sqrt{1+r^2})$, or $H^1(\sqrt{1-r^2}) \times S^1(r)$ in S_1^3. To classify all isoparametric surfaces in S_1^3 with undiagonalizable shape-operator we show that given any solution κ of the equation $\kappa_{uv} = -(1+\lambda^2)e^{\kappa}$ with constant λ we can construct an isoparametric surface in S_1^3 with undiagonalizable S by solving an associated ODE, and that any undiagonalizable isoparametric surfaces can be constructed in this way.

1. Introduction

Isoparametric hypersurfaces in R^{n+1}, H^{n+1} and S^{n+1} have been extensively studied in the last 20 years. It is known that an isoparametric hypersurface in R^{n+1} is either totally umbilical or an open subset of a standard spherical cylinder $S^k(r) \times R^{n-k}$ (cf. Levi-Civita [7] and B. Segre [10]), and an isoparametric hypersurface in the hyperbolic space H^{n+1} is either totally umbilical or an open part of the standard product $S^k(r) \times H^{n-k}(\sqrt{1+r^2})$ (cf. P.J. Ryan [9]). The study of isoparametric hypersurfaces in the sphere S^{n+1} is more complicated and exciting than the cases in R^{n+1} and H^{n+1}, due to the works of E. Cartan ([1]–[4]), H.F. Münzner ([8]) and many others.

In this paper we give a complete classification of isoparametric surfaces in the de Sitter space S_1^3, the unit sphere with respect to the Lorentz product $\langle\ ,\ \rangle$ in the 4-dimensional Lorentz space R_1^4. It is well-known that $(S_1^3, \langle\ ,\ \rangle)$ is a 3-dimensional Lorentz manifold of constant curvature.

A surface $x\colon M \to S_1^3$ is called non-degenerate, if at each point $p \in M$ the tangent plane $dx(T_pM)$ is non-degenerate with respect to the Lorentz product $\langle\ ,\ \rangle$. In this case we can determine a section ξ of the normal bundle with $\langle \xi, \xi\rangle = \pm 1$. Then the shape operator $S\colon T_pM \to T_pM$ is defined by $dx(S(v)) = -d\xi(v)$, which is a self-adjoint operator with respect to $\langle\ ,\ \rangle$. A surface in S_1^3 is called an isoparametric surface if it is non-degenerate and both its mean curvature $\mathrm{Trace}(S)$ and Gauss-Kroncker curvature $\det S$ are constant. Thus for an isoparametric surface in S_1^3 any eigenvalue-function λ of the shape-operator (if any) is constant. For any eigenvalue λ of S we call $x_\lambda := \lambda x + \xi$ the focal surface of x with respect to λ. It is clear that all focal surfaces of an isoparametric surface in S_1^3 are degenerate to one point or a curve in R_1^4.

In this paper R_1^4 is the Lorentz space with inner product

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

where $x = (x_0, x_1, x_2, x_3), y = (y_0, y_1, y_2, y_3)$. The de Sitter space

$$S_1^3 = \left\{x \in R_1^4 \;\middle|\; \langle x, x\rangle = 1\right\}$$

is a Lorentz hypersurface in R_1^4 with constant sectional curvature.

It is easy to check the following examples of isoparametric surfaces in S_1^3:

(i) The 2-spheres given by the intersection of the hyperplane $x_0 = r$ with S_1^3.

(ii) The hyperboloid given by the intersection of the hyperplane $x_1 = r$ for some $r \neq 1$ with S_1^3.

(iii) The paraboloid given by the intersection of the hyperplane $x_0 = x_1 - 1$ with S_1^3.

(iv) The surface $H^1(r) \times S^1(\sqrt{1+r^2})$:

$$\left\{\left(r\cosh\frac{u}{r}, r\sinh\frac{u}{r}, \sqrt{1+r^2}\cos\frac{v}{\sqrt{1+r^2}}, \sqrt{1+r^2}\sin\frac{v}{\sqrt{1+r^2}}\right)\;\middle|\; (u,v) \in R^2\right\}.$$

Its shape-operator S has two different eigenvalues λ_1 and λ_2 with $\lambda_1\lambda_2 = 1$.

(v) The surface $H^1(\sqrt{1-r^2}) \times S^1(r)$ for $0 < r < 1$:

$$\left\{\left(\sqrt{1-r^2}\sinh\frac{u}{\sqrt{1-r^2}}, \sqrt{1-r^2}\cosh\frac{u}{\sqrt{1-r^2}}, r\cos\frac{v}{r}, r\sin\frac{v}{r}\right)\;\middle|\; (u,v) \in R^2\right\}.$$

Its shape-operator S has two different eigenvalues λ_1 and λ_2 with $\lambda_1\lambda_2 = -1$.

The isoparametric surfaces given above have a common property that its shape operator S is diagonalizable.

Our purpose in this paper is to prove the following classification theorems:

CLASSIFICATION THEOREM I. *Any isoparametric surface in S_1^3 with diagonalizable shape-operator is Lorentz equivalent to an open subset of either an 2-sphere, a hyperboloid, a paraboloid, or the surface $H^1(r) \times S^1(\sqrt{1+r^2})$, $H^1(\sqrt{1-r^2}) \times S^1(r)$.*

CLASSIFICATION THEOREM II. *Given any solution κ of the PDE $\kappa_{uv} - (1+\lambda^2)e^{\kappa} = 0$ one can construct an isoparametric surface in S_1^3 with undiagonalizable shape-operator by solving an associated ODE. Moreover, any isoparametric surface with undiagonalizable shape operator can be constructed in this way.*

This paper is organized as follows. In the section 2 we give the structure equations of surfaces in S_1^3. In the section 3 we study isoparametric surfaces in S_1^3 with diagonalizable shape-operator and prove the Classification Theorem I. In the section 4 we study undiagonalizable isoparametric surfaces S_1^3 and prove the Classification Theorem II.

2. Structure equations for surfaces in S_1^3

Let $x\colon M \to S_1^3$ be a non-degenerate surface. We can find a unit normal $\xi\colon M \to R_1^4$ such that $\langle \xi, \xi \rangle = \epsilon = \pm 1$. The shape operator $S\colon T_pM \to T_pM$ of x is defined by

$$d\xi(v) = -dx(S(v)), \quad \forall v \in T_pM. \tag{2.2}$$

The first fundamental form I of x is given by $I(v,w) = \langle dx(v), dx(w) \rangle$. It follows from (2.2) that $I(S(v),w) = I(v,S(w))$ for all $v, w \in T_pM$. Thus S is self-adjoint with respect to I.

Let $\{e_1, e_2\}$ be a local basis for TM. Then $\{e_1(x), e_2(x), \xi, x\}$ is a moving frame of the surface x in R_1^4 along M. We denote by $\{\omega^1, \omega^2\}$ the dual basis of $\{e_1, e_2\}$. Then the structure equations of x read:

$$de_i(x) = \sum_j \omega_i^j e_j(x) + \sum_j h_{ij}\omega^j \xi - \sum_j g_{ij}\omega^j x, \tag{2.3}$$

$$dx = \sum_i \omega^i e_i(x), \quad d\xi = -\sum_{ij} S_j^i \omega^j e_i(x), \tag{2.4}$$

where

$$g_{ij} = \langle e_i(x), e_j(x)\rangle, \quad \langle de_i(x), e_j(x)\rangle = \sum_k \omega_i^k g_{kj}, \tag{2.5}$$

$$S(e_i) = \sum_j S_i^j e_j, \quad h_{ij} = \epsilon \sum_k g_{ik} S_j^k. \tag{2.6}$$

We note that $I = \sum_{ij} g_{ij}\omega^i\omega^j$ and $II = \sum_{ij} h_{ij}\omega^i\omega^j$ are the first and the second fundamental form of x, they are symmetric 2-forms. If we denote

$$d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j = -\frac{1}{2}\sum_{kl} R_{ikl}^j \omega^k \wedge \omega^l, \quad R_{ikl}^j = -R_{ilk}^j; \tag{2.7}$$

$$dh_{ij} - \sum_k h_{ik}\omega_j^k - \sum_{jk} h_{kj}\omega_i^k = \sum_k h_{ij,k}\omega^k. \tag{2.8}$$

Then the Gauss and Codazzi equations for the surface $x\colon M \to S_1^3$ are given by

$$R_{ikl}^j = h_{ik}S_l^j - h_{il}S_k^j + \delta_l^j g_{ik} - \delta_k^j g_{il}, \quad h_{ij,k} = h_{ik,j}. \tag{2.9}$$

These equations are important in the classification of isoparametric surfaces in S_1^3.

The characteristic polynomial of the shape operator S is given by $t^2 - \mathrm{Tr}(S)t + \det S = 0$. By definition, $x\colon M \to S_1^3$ is called isoparametric, if the coefficients of this polynomial are constant. In particular, the minimal polynomial $m(t)$ of the shape operator S of an isoparametric surface has constant coefficients.

An isoparametric surface in S_1^3 is called diagonalizable, if the minimal polynomial of S is given as $m(t) = t - \lambda$ or $m(t) = (t - \lambda_1)(t - \lambda_2)$ with $\lambda_1 \neq \lambda_2$. Otherwise we call the isoparametric surface undiagonalizable. We will study diagonalizable isoparametric surfaces and undiagonalizable isoparametric surfaces in the sections 3 and 4 respectively.

3. Diagonalizable isoparametric surfaces in S_1^3

Let $x\colon M \to S_1^3$ be a diagonalizable isoparametric surface. By definition the minimal polynomial of its shape operator S is either (I) $m(t) = t - \lambda$ or (II) $m(t) = (t - \lambda_1)(t - \lambda_2)$ with $\lambda_1 \neq \lambda_2$.

First we consider the case (I). In this case we have $S = \lambda\,\mathrm{id}$ for some constant λ. From (2.2) we get $\xi + \lambda x = c$ for some non-zero constant vector $c \in R_1^4$. Since $\langle \xi, \xi\rangle = \epsilon$ and $\langle \xi, x\rangle = 0$, we get

$$\langle c, c\rangle = \epsilon + \lambda^2, \quad \langle x, c\rangle = \lambda. \tag{3.1}$$

If $\langle c, c\rangle < 0$, we get from the first equation of (3.1) that $\epsilon = -1$ and $\lambda^2 < 1$. Then we can find a Lorentz transformation $T \in O(3,1)$ such that $cT = \sqrt{1-\lambda^2}(1,0,0,0)$. By consider the surface $\tilde{x} = xT$, if necessary, we may assume that $c = \sqrt{1-\lambda^2}(1,0,0,0)$. From the second equation of (3.1) we get $x_0 = -\frac{\lambda}{\sqrt{1-\lambda^2}} := r$. Thus $x\colon M \to S_1^3$ is an open part of the surface (i) in the Classification Theorem I.

If $\langle c, c\rangle > 0$, we get $\epsilon + \lambda^2 > 0$. By taking a Lorentz transformation, if necessary, we may assume that $c = \sqrt{\epsilon+\lambda^2}(0,1,0,0)$. It follows from the second equation of (3.1) we get $x_1 = \frac{\lambda}{\sqrt{\epsilon+\lambda^2}} := r$. Thus $x\colon M \to S_1^3$ is an open part of either the one sheet hyperboloid ($\epsilon = 1$) or the two sheet hyperboloid ($\epsilon = -1$) given by (ii) in the Classification Theorem I.

If $\langle c, c\rangle = 0$, we get from (3.1) $\epsilon = -1$ and $\lambda^2 = 1$. By changing ξ to $-\xi$, if necessary, we may assume that $\lambda = 1$. By taking a Lorentz transformation, if necessary, we may assume that $c = (-1,1,0,0)$. From the second equation of (3.1) we get $-x_0 + x_1 = 1$. Thus $x\colon M \to S_1^3$ is an open part of the paraboloid given by (iii) in the Classification Theorem I.

Next we consider the case (II). In this case $m(t) = (t-\lambda_1)(t-\lambda_2)$ with $\lambda_1 \neq \lambda_2$. Thus we have the Lorentz orthogonal decomposition

$$TS_1^3 = E_1 \oplus E_2 \oplus R\xi,$$

where E_1 and E_2 are real line bundle consisting of eigenvectors of λ_1 and λ_2, respectively. Since $(S-\lambda_1)^2 \neq 0$ and $(S-\lambda_2)^2 \neq 0$ and S is self-adjoint with respect to the first fundamental form I, around each point there exist non-vanishing vector field V and W such that

$$I((S-\lambda_1)V, (S-\lambda_1)V) \neq 0, \quad I((S-\lambda_2)W, (S-\lambda_2)W) \neq 0, \tag{3.2}$$

We note that $(S-\lambda_1)V$ is a local section of E_2 and $(S-\lambda_2)W$ is a smooth local section of E_1. Since $\langle \xi, \xi\rangle = \epsilon$ and $(TS_1^3, \langle\ ,\ \rangle)$ is Lorentzian, by exchanging λ_1 and λ_2 and normalizing V and W if necessary, we may assume that

$$I((S-\lambda_1)V, (S-\lambda_1)V) = 1, \quad I((S-\lambda_2)W, (S-\lambda_2)W) = -\epsilon. \tag{3.3}$$

By letting $e_1 = (S-\lambda_2)W$ and $e_2 = (S-\lambda_1)V$ we get a moving frame $\{e_1(x), e_2(x), \xi, x\}$ in R_1^4 along M, which satisfies

(3.4)

$$g_{11} = \langle e_1(x), e_1(x)\rangle = -\epsilon, \quad g_{12} = \langle e_1(x), e_2(x)\rangle = 0, \quad g_{22} = \langle e_2(x), e_2(x)\rangle = 1$$

and $S_i^j = \lambda_i \delta_i^j$. It follows from (2.6) that

$$h_{11} = -\lambda_1, \quad h_{12} = h_{21} = 0, \quad h_{22} = \epsilon\lambda_2. \tag{3.5}$$

From (3.4) and the second equation of (2.5) we get

$$\omega_1^1 = \omega_2^2 = 0, \quad \omega_1^2 = \epsilon\omega_2^1. \tag{3.6}$$

By letting $i = j$ in (2.8) and using (3.5) and (3.6) we get $h_{ii,k} = 0$ for all i, k. Since $\{h_{ij,k}\}$ is totally symmetric and $\lambda_1 \neq \lambda_2$, by letting $i = 1$ and $j = 2$ in (2.8) we get $\omega_2^1 = \omega_1^2 = 0$, which implies that

$$R^j_{ikl} = 0, \quad [e_1, e_2] = 0. \tag{3.7}$$

It follows from (2.9) that $\epsilon\lambda_1\lambda_2 + 1 = -R^1_{212} = 0$. Since $[e_1, e_2] = 0$, we can introduce coordinate (u, v) such that $e_1 = \frac{\partial}{\partial u}$ and $e_2 = \frac{\partial}{\partial v}$. Thus the structure equations (2.3) and (2.4) read:

$$\frac{\partial^2 x}{\partial u^2} = -\lambda_1\xi + \epsilon x, \quad \frac{\partial^2 x}{\partial u \partial v} = 0, \quad \frac{\partial^2 x}{\partial v^2} = \epsilon\lambda_2\xi - x; \tag{3.7}$$

$$\frac{\partial \xi}{\partial u} = -\lambda_1 \frac{\partial x}{\partial u}, \quad \frac{\partial \xi}{\partial v} = -\lambda_2 \frac{\partial x}{\partial v}. \tag{3.8}$$

We note that any surface $x(u, v)$ in S^3_1 satisfying these structure equations is unique up to a Lorentz transformation.

In the case that $\epsilon = -1$ we have $\lambda_1\lambda_2 = 1$. By changing ξ to $-\xi$, if necessary, we may assume that λ_1 and λ_2 are positive. By exchanging u and v in (3.7) and (3.8), if necessary, we may assume that $\lambda_1 > 1$. We write $\lambda_1 = \frac{\sqrt{1+r^2}}{r}$ and $\lambda_2 = \frac{r}{\sqrt{1+r^2}}$. It is easy to check that the surface

$$\left(r\cosh\frac{u}{r}, r\sinh\frac{u}{r}, \sqrt{1+r^2}\cos\frac{v}{\sqrt{1+r^2}}, \sqrt{1+r^2}\sin\frac{v}{\sqrt{1+r^2}}\right), \quad (u, v) \in R^2$$

satisfies the structure equation (3.7) and (3.8) with $\epsilon = -1$. Thus $x\colon M \to S^3_1$ is Lorentz equivalent to the surface $H^1(r) \times S^1(\sqrt{1+r^2})$ given in the Classification Theorem I.

In the case that $\epsilon = 1$ we have $\lambda_1\lambda_2 = -1$. We may assume that $\lambda_1 = -\frac{r}{\sqrt{1-r^2}}$ and $\lambda_2 = \frac{\sqrt{1-r^2}}{r}$ with $0 < r < 1$. It is easy to check that the surface

$$\left(\sqrt{1-r^2}\sinh\frac{u}{\sqrt{1-r^2}}, \sqrt{1-r^2}\cosh\frac{u}{\sqrt{1-r^2}}, r\cos\frac{v}{r}, r\sin\frac{v}{r}\right), \quad (u, v) \in R^2$$

satisfying the structure equation (3.7) and (3.8) with $\epsilon = 1$. Thus $x\colon M \to S^3_1$ is Lorentz equivalent to the surface $H^1(\sqrt{1-r^2}) \times S^1(r)$ given in the Classification Theorem I.

Thus we complete the proof of the Classification Theorem I.

4. Undiagonalizable isoparametric surfaces in S_1^3

In this section we classify all undiagonalizable isoparametric surfaces in S_1^3.

LEMMA 4.1. *If the shape operator S of an isoparamteric surface $x\colon M \to S_1^3$ is undiagonalizable, then its minimal polynomial is given by $m(t) = (t-\lambda)^2$ for some $\lambda \in R$.*

PROOF. Since the characteristic polynomial of S has constant coefficients, only possible minimal polynomial for the undiagonalizable S is either (i) $m(t) = (t-\lambda)^2$ for some $\lambda \in R$, or (ii) $m(t) = (t-\lambda)(t-\bar{\lambda})$ for some $\lambda \in C$ with $\lambda \neq \bar{\lambda}$. Our purpose is to prove the case (ii) cannot occur. For this purpose we extend R_1^4 to the complex space C^4 with the hermitian inner product

$$\langle z, w\rangle = -z_0\overline{w}_0 + z_1\overline{w}_1 + z_2\overline{w}_2 + z_3\overline{w}_3, \quad \forall z, w \in C^4. \tag{4.1}$$

The shape operator can be extended to the real operator $S\colon TM \otimes C \to TM \otimes C$ by

$$S(v + iw) = Sv + iSw, \quad v, w \in TM.$$

It is easy to check

$$\langle Sz, w\rangle = \langle z, Sw\rangle, \quad \forall z, w \in TM \otimes C, \tag{4.2}$$

which implies $\langle (S-\lambda)z, w\rangle = \langle z, (S-\overline{\lambda})w\rangle$. Let V be a local non-vanishing vector field in $TM \otimes C$ such that $Z = (S-\overline{\lambda})V \neq 0$. Then we have $S(Z) = \lambda Z$ and $\langle Z, Z\rangle = 0$. We write $Z = v + iw$ for some real vector field v and w. Since $\lambda \neq \overline{\lambda}$ and $\langle Z, Z\rangle = 0$, we know that v and w is linearly independent, which satisfies $\langle v, v\rangle + \langle w, w\rangle = 0$. We claim

$$\langle Z, \overline{Z}\rangle = \langle v, v\rangle - \langle w, w\rangle + 2i\langle v, w\rangle \neq 0.$$

Otherwise we get $\langle v, v\rangle = \langle v, w\rangle = \langle w, w\rangle = 0$, which implies that $\langle w_0 v - v_0 w, w_0 v - v_0 w\rangle = 0$. Since the first coordinate of the vector $w_0 v - v_0 w$ vanishes, we get $w_0 v - v_0 w = 0$, a contradiction to the fact that v and w are linearly independent. We write $\langle Z, \overline{Z}\rangle = r^2 e^{2i\theta}$ with some positive function r and real function θ and define $W = \frac{\sqrt{2}}{r} e^{-i\theta} Z$, then we have

$$\langle W, W\rangle = 0, \quad \langle W, \overline{W}\rangle = 2. \tag{4.3}$$

We write $W = e_1 + ie_2$. Then by (4.3) we know that $\{e_1, e_2\}$ is a local basis for TM, which satisfies

$$\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1, \quad \langle e_1, e_2\rangle = 0. \tag{4.4}$$

Let ξ be the unit normal of $x\colon M \to S_1^3$. Because of the signature of S_1^3 we have $\langle \xi, \xi \rangle = 1$. Since $S(W) = \lambda W = (a + ib)W$ with $b \neq 0$, we get

$$S(e_1) = ae_1 - be_2, \quad S(e_2) = be_1 + ae_2. \tag{4.5}$$

It follows from (2.6), (4.3) and (4.4) that

$$h_{11} = a,\ h_{22} = -a, \quad h_{12} = h_{21} = b, \quad \omega_1^1 = \omega_2^2 = 0, \quad \omega_1^2 = \omega_2^1. \tag{4.6}$$

By letting $i = 1$ and $j = 2$ in (2.8) and using (4.4) and (4.5) we get $h_{12,k} = 0$ for all k. Since $\{h_{ij,k}\}$ is totally symmetric and $b \neq 0$, by letting $i = j$ in (2.8) we get

$$-2b\omega_1^2 = \sum_k h_{11,1}\omega^1, \quad -2b\omega_2^1 = h_{22,2}\omega^2,$$

which implies that $\omega_1^2 = \omega_2^1 = 0$ and $R^j_{ikl} = 0$. In particular, we get $0 = R^1_{212} = b^2 + a^2 + 1$, a contradiction. Thus the minimal polynomial of S cannot take the form $m(t) = (t - \lambda)(t - \bar{\lambda})$ with $\lambda \neq \bar{\lambda}$. The only possible minimal polynomial for undiagonalizable shape operator S is given by $m(t) = (t - \lambda)^2$.

Now let $x\colon M \to S_1^3$ be an isoparametric surface whose shape operator S has minimal polynomial $m(t) = (t - \lambda)^2$ with $\lambda \in R$. Since $S - \lambda \neq 0$, we can find a smooth local vector field V with $(S - \lambda)V \neq 0$. Thus $\{V, (S - \lambda)V\}$ is a local basis for TM. Since $\langle\ ,\ \rangle$ is non-degenerate on TM, we get $\langle V, (S - \lambda)V \rangle \neq 0$ (otherwise we will have $\langle (S - \lambda)V, W \rangle = 0$ for all $W \in TM$). We can normalize V such that $\langle (S - \lambda)V, V \rangle = \varepsilon = \pm 1$. We define

$$W = V - \frac{\varepsilon}{2}\langle V, V \rangle (S - \lambda)V.$$

Then we have

$$\langle W, W \rangle = 0, \quad \langle (S - \lambda)W, W \rangle = \varepsilon.$$

It is easy to check that such W is uniquely determined up to sign. We define $e_1 = W$ and $e_2 = \varepsilon(S - \lambda)W$. Then we have

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 0, \quad \langle e_1, e_2 \rangle = 1,\ S(e_1) = \lambda e_1 + \varepsilon e_2, \quad S(e_2) = \lambda e_2. \tag{4.7}$$

By the signature of $(S_1^3, \langle\ ,\ \rangle)$ we know that the unit normal ξ satisfies $\langle \xi, \xi \rangle = 1$. From (4.7) we get

$$h_{11} = \varepsilon, \quad h_{12} = h_{21} = \lambda, \quad h_{22} = 0. \tag{4.8}$$

By letting $i = 1$ and $j = 2$ in (2.8), we get $h_{12,k} = 0$ for all k. Since $h_{ij,k}$ are totally symmetric, we get from (2.8) with $i = j = 1$ that $\omega_1^1 = -\omega_2^2 = \mu\omega^1$ for some smooth function μ. Using the formula $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$ we get $d\omega^1 = 0$. Thus we can find coordinate (u, v) and a smooth function κ such that

$$\omega^1 = du, \quad \omega^2 = e^{\kappa}\, dv. \tag{4.9}$$

Using the formula $d\omega^2 = \omega^2 \wedge \omega_2^2$ we get $\mu = \kappa_u$. Thus the structure equations of $x\colon M \to S_1^3$ read:

$$x_{uu} = \kappa_u x_u + \varepsilon\xi, \quad x_{uv} = \lambda e^{\kappa}\xi - e^{\kappa}x, \quad x_{vv} = \kappa_v x_v; \tag{4.10}$$

$$\xi_u = -\lambda x_u - \varepsilon e^{-\kappa} x_v, \quad \xi_v = -\lambda x_v. \tag{4.11}$$

The integrability condition is given by

$$\kappa_{uv} = -(\lambda^2 + 1)e^{\kappa}. \tag{4.12}$$

Thus we get the following classification theorem:

THEOREM 4.2. *An undiagonalizable isoparametric surface in S_1^3 determines a solution κ of the PDE* (4.12). *Conversely, given any solution κ of* (4.12) *we can construct an unique undiagonalizable isoparametric surface $x\colon R^2 \to S_1^3$ with λ as principal curvature by solving PDE* (4.11) *with the initial condition*

$$\begin{aligned} &x(0,0) = (0,0,0,1),\\ &x_u(0,0) = \frac{1}{\sqrt{2}}(1,1,0,0), \quad x_v(0,0) = \frac{1}{\sqrt{2}}(-1,1,0,0),\\ &\xi(0,0) = (0,0,1,0). \end{aligned}$$

In the rest of this section we show that for any given solution κ of PDE (4.12) one can get a explicit expression for the associated isoparametric surface $x(u, v)$ by solving an ODE of four order. To explain our construction, let κ be a solution of (4.12). Then

$$\kappa_{uu} - \frac{1}{2}k_u^2 := \tau(u) \tag{4.13}$$

is a function depending only on u. Now let $F(u) \in R_1^4$ be the solution of the ODE

$$F^{(4)}(u) + (2\tau(u) + 2\lambda\varepsilon)F''(u) + \tau'(u)F'(u) - F(u) = 0 \tag{4.14}$$

with the initial conditions that $\{F(0), F'(0), F''(0), F'''(0)\}$ is a basis in R_1^4 satisfying the following product table:

$$\langle F(0), F(0)\rangle = \langle F''(0), F''(0)\rangle = -\langle F'(0), F'''(0)\rangle = 1+\lambda^2, \tag{4.15}$$

$$\langle F'''(0), F'''(0)\rangle = (1+\lambda^2)(2\tau(0)+2\lambda\varepsilon), \tag{4.16}$$

$$\langle F^{(i)}(0), F^{(j)}(0)\rangle = 0 \tag{4.17}$$

for other $0 \le i, j \le 3$. By letting

$$x(u,v) = \frac{1}{1+\lambda^2}(\lambda F(u) + \varepsilon\kappa_u F'(u) + \varepsilon F''(u)), \tag{4.18}$$

we claim that

(i) $x(u,v)$ is an undiagonalizable isoparametric surface in S_1^3;

(ii) all undiagonalizable isoparametric surfaces in S_1^3 can be obtained in this way.

To prove (i) we define

$$X(u) = \begin{pmatrix} F(u) \\ F'(u) \\ F''(u) \\ F'''(u) \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \tau'(u) & -2(\tau(u)+\lambda\varepsilon) & 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2(\tau(u)+\lambda\varepsilon) \end{pmatrix}, \quad I_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $F(u)$ is the solution of ODE (4.14) with the initial conditions (4.15), (4.16) and (4.17). Now let $Y(u) = X(u)I_1X(u)^t - (1+\lambda^2)A$. Since

$$X'(u) = TX(u), \quad A'(u) = TA + AT^t,$$

we get an ODE $Y'(u) = TY(u) + Y(u)T^t$ with initial condition $Y(0) = 0$. Then we have the only solution $Y(u) = 0$, which implies that

$$\langle F(u), F(u)\rangle = \langle F''(u), F''(u)\rangle = -\langle F'(u), F'''(u)\rangle = 1+\lambda^2, \tag{4.19}$$

$$\langle F'''(u), F'''(u)\rangle = (1+\lambda^2)(2\tau(u)+2\lambda\varepsilon), \tag{4.20}$$

$$\langle F^{(i)}(u), F^{(j)}(u)\rangle = 0 \tag{4.21}$$

for other $0 \leq i, j \leq 3$. By defining $\xi(u,v) = F(u) - \lambda x(u,v)$, we can easily verify that $x(u,v)$ is a surface satisfying the PDE (4.10) and (4.11) with the unit normal $\xi(u,v)$, which implies that $x(u,v)$ is an undiagonalizable isoparametric surface in S_1^3.

To prove (ii) we note that if $x(u,v)$ is an isoparametric surface in S_1^3 with the unit normal $\xi(u,v)$, then by (4.11) we know that its focal surface $F(u) = \lambda x(u,v) + \xi(u,v)$ is a curve. Using (4.10) and (4.11) we can verify that $F(u)$ satisfies the ODE (4.14) and the surface $x(u,v)$ is given by (4.18).

Thus we complete the proof of the Classification Theorem II.

REMARK. As Professor Takeshi Sasaki mentions to us, the PDE (4.12) can be easily rewritten as

$$\kappa_{xy} = \exp(2\kappa)$$

and since this is a Liouville equation, we have solutions of type

$$\kappa = \frac{1}{2\log(X(x)'Y(y)')} - \log(X(x) + Y(y)).$$

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