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A NOTE ON ROSEMAN MOVES

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Abstract

D. Roseman introduced seven types of local deformations of surface diagrams as a generalisation of Reidemeister moves. The present paper proves that there are six types of local deformations, which can describe each of Roseman's moves.

1. Introduction

A *surface-knot* is an embedded oriented closed connected surface in $\mathbf{R}^4$. Let $\pi\colon \mathbf{R}^4 \to \mathbf{R}^3$ be the projection defined by $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. A projection of a surface-knot in $\mathbf{R}^3$ is *generic* if each point is either a regular point, a double point, a triple point or a branch point (see [7]). (The set of double points forms double curves.) A generic projection of a surface-knot F with crossing information is called a *surface diagram* (see [1]) and is denoted by D_F. Let $|D_F|$ denote the generic surface without the crossing information on the crossing set. If a generic surface in $\mathbf{R}^3$ is a projection of a surface-knot, then the generic surface is said to be *liftable* ([3], [2], [11]). Two surface-knots F_1 and F_2 are *equivalent* if there is an isotopy deformation between them. For equivalent surface-knots F_1 and F_2, their surface diagrams D_{F_1} and D_{F_2} are said to be *equivalent*. A surface-knot is *trivial* if it is equivalent to the boundary of the standard handlebody in $\mathbf{R}^3 \times \{0\} \subset \mathbf{R}^4$. D. Roseman [10] introduced seven types of local deformations for surface-knots (see Figures 1 and 2) as a generalisation of Reidemeister moves [9] in the theory of classical knots. We call these moves *Roseman moves*. The following holds.

LEMMA 1.1 (D. Roseman). *Let F_1 and F_2 be equivalent surface-knots and let D_{F_1} and D_{F_2} be their surface diagrams respectively. Then there exists a finite sequence of seven types of local deformations such that it deforms D_{F_1} into D_{F_2}.*

Note that for each picture of Figure 1 and Figure 2, the move from the left side to the right side is denoted by R_i^+, otherwise by R_i^- for each $i = 1, \ldots, 6$. We do not distinguish two directions in R_7 moves. If we ignore the crossing

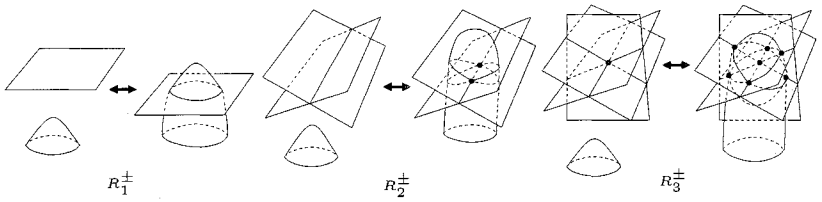


Fig. 1.

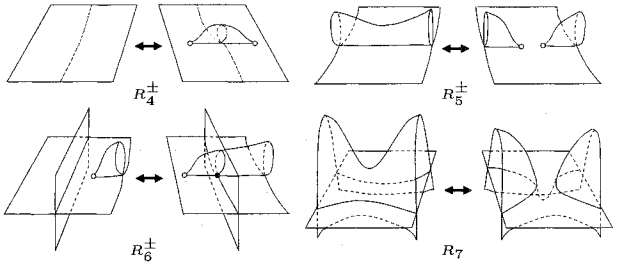
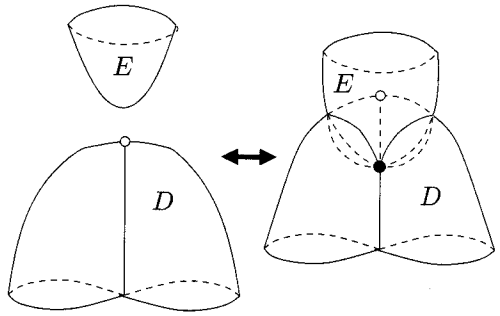


Fig. 2.

information of each of local moves $M^{\pm}$, then we denote the local deformation by $|M|^{\pm}$. Let F be a surface-knot and let D_F be a surface diagram of F. Let Δ be a closed 3-ball in $\mathbf{R}^3$ with $D_F \cap \mathrm{Int}(\Delta) \neq \emptyset$. Then we call $\Delta \cap D_F$ a *sub-diagram* of D_F and denote it by Δ_F. We will describe a local move called a *finger move* as follows. The initial diagram is $E \cup D$ consisting of a properly embedded disc E in Δ and a sub-diagram D in Δ with $E \cap D = \emptyset$. Then choose two points $y_0 \in E$ and $y_1 \in D$ so that there exists a simple arc γ from y_0 to y_1 in Δ. Take a small disc neighbourhood U_0 of y_0 in E and push U_0 at y_0 to y_1 along the arc γ. We call E a *moving disc* and we call D a *target diagram.* Note that this move may create extra intersection between E and D. If a finger move is a projected isotopy deformation in $\mathbf{R}^4$, then we will also call the isotopy deformation a *finger move.* On the other hand, for generic surfaces in 3-manifolds, T. Homma and T. Nagase and others studied generic homotopy between generic surfaces ([4], [5], [8]). In their papers they introduce four types of projected finger moves, $|R_4|^{\pm}$ and $|R_7|$ moves. One of four finger moves is formed by a disc E and the Whitney umbrella D, where D can be viewed as a neighbourhood of a branch point in D_F so that it contains the branch point and a double arc from the branch point (see Figure 3). Take a point $y_0 \in E$ and take a point y_1 in the interior of the double arc of D. Also take a simple arc γ from y_0 to y_1 via the branch point. Then apply a finger move along γ so that it creates a triple point and a pair of double loops based at the triple point. These deformations can be expressed as projected local isotopy deformations of F in $\mathbf{R}^4$ (Lemma 2.1). We will denote



$X_5^{\pm}$

Fig. 3.

the move and its reversed deformation by $X_5^{\pm}$ and they are depicted in Figure 3. Let D_F be a surface diagram of a surface-knot F. Let $M^{\pm}$ represent Roseman moves or $X_5^{\pm}$ moves. Our main results are the following.

THEOREM 1.1. *The following holds.*

(1) *A Roseman move $R_5^{\pm}$ is decomposed into a finite sequence of $R_4^{\pm}$ and R_7 moves.*

(2) *A Roseman move $R_6^{\pm}$ is decomposed into a finite sequence of $X_5^{\pm}$ and R_7 moves.*

COROLLARY 1.1. *Let F_1 and F_2 be equivalent surface-knots and let D_{F_1} and D_{F_2} be their surface diagrams. Then D_{F_1} is deformed into D_{F_2} by a finite sequence of six types of Roseman moves $R_1^{\pm}$, $R_2^{\pm}$, $R_3^{\pm}$, $R_4^{\pm}$, $R_6^{\pm}$ and R_7, and also by a finite sequence of six types of local deformations, $R_1^{\pm}$, $R_2^{\pm}$, $R_3^{\pm}$, $R_4^{\pm}$, $X_5^{\pm}$ and R_7.*

PROOF. It is a direct consequence of Theorem 1.1 and Lemma 1.1. □

We will prove Theorem 1.1 in Section 4. For a topological space X, for a subset $A \subset X$, $\mathrm{Int}(A)$ and $\mathrm{Cl}(A)$ denote the interior and the closure of A in X respectively.

2. Local deformations

In this section we will describe properties of a finger move. We have the following lemma.

LEMMA 2.1. *Let F be a surface-knot and let D_F be a surface diagram of F. Let Δ_F be a sub-diagram of D_F in a 3-ball Δ. Assume that Δ_F contains a proper*

disc E and a proper sub-diagram D such that $E \cap D = \emptyset$. Then there is a finger move from E into D in $\mathrm{Int}(\Delta)$ *such that the finger can be lifted as the highest one or lowest one in the cylinder* $\pi^{-1}(\Delta)$.

PROOF. If X is a subset of D_F, then let $\widetilde{X}$ denote a subset of F such that $\pi(\widetilde{X}) = X$. We construct a finger move in Δ, which can be lifted to the highest or lowest in $\pi^{-1}(\Delta)$. Let y_0 be a point of E and let y_1 be a point of D. Let $\widetilde{y}_0$ be the point of F such that $\pi(\widetilde{y}_0) = y_0$. Take a small disc neighbourhood U_0 of y_0 in E. Take a simple arc $\gamma\colon I \to \mathrm{Int}(\Delta)$ from y_0 to y_1. Let $N(\gamma)$ be a tubular neighbourhood of γ in $\mathbf{R}^3$ so that there exists a diffeomorphism $\xi\colon I \times B^2 \to N(\gamma)$. Push the point y_0 along γ to y_1 in $N(\gamma)$ so that U_0 is modified into a disc U_0', and U_0' meets D in Δ_F. Let $\widetilde{\gamma}$ denote a lift of γ from $\widetilde{y}_0$ so that the height of $\widetilde{\gamma}$ is the same as the height of $\widetilde{y}_0$. We can deform $\widetilde{E}$ along $\widetilde{\gamma}$ so that the deformation covers the finger move on U_0. Let B^2 denote the unit disc in $\mathbf{R}^2$: $B^2 = \{x \in \mathbf{R}^2\colon |x| < 1\}$, and let $B^2(0.5)$ be a disc with radius 0.5: $B^2(0.5) = \{x \in \mathbf{R}^2\colon |x| < 0.5\}$. There is a diffeomorphism $\psi\colon B^2 \to U_0'$. Assume that U_0' intersects D transversely. Then there exists $t_0 = \min\{t \in I\colon \gamma(t) \in D\}$. Set $t_1 = (1/2)t_0$. Let $N(\gamma([0,t_1])$ be the restricted $N(\gamma)$ over $[0,t_1]$. We can assume that $\psi(B^2 \setminus B^2(0.5)) \subset N([0,t_1])$ and $\psi(B^2 \setminus B^2(0.5)) \cap D = \emptyset$. The diffeomorphism ψ can be extended to $\widehat{\psi}\colon \mathbf{R}^2 \to U$, where U is an open disc in E such that $\mathrm{Cl}(U_0) \subset U$. Also $\widehat{\psi}$ has an embedded lift $\widetilde{\psi}\colon \mathbf{R}^2 \to \widetilde{U}$, where $\pi(\widetilde{U}) = U$. We can define a smooth function $\varphi_+\colon \mathbf{R}^2 \to \mathbf{R}$ so that $\varphi_+(x) = 1$ if $x \notin B^2$, $0 < \varphi_+(x) < 1$ if $x \in \mathrm{Int}(B^2 \setminus B^2(0.5))$ and $\varphi_+(x) = 0$ if $x \in B^2(0.5)$. Define $\varphi_-(x) = 1 - \varphi_+(x)$. Let $p_4\colon \mathbf{R}^4 \to \mathbf{R}$ be the projection defined by $p_4(x_1,x_2,x_3,x_4) = x_4$. A height function $h\colon \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$h(x) = \varphi_+(x)p_4 \circ \widetilde{\psi}(x) + \varphi_-(x)M_1, \tag{1}$$

where M_1 is a number greater than $\max\{p_4(\widetilde{y})\colon \widetilde{y} \in \widetilde{D}\}$. Now we define $\widetilde{U_0}'$ in $\mathbf{R}^4$ by $(\psi(x),h(x))$, $x \in B^2$. Denoting the lift of U_0 by $\widetilde{U}_0$, there is an isotopy from $\widetilde{U}_0$ to $\widetilde{U}_0'$ and this gives an isotopy deformation of the surface-knot in $\mathbf{R}^4$ so that $\widetilde{U}_0'$ is the highest in $\pi^{-1}(\Delta)$. Similarly, we can lift the disc U_0' using a number M_0 less than $\min\{p_4(\widetilde{y})\colon \widetilde{y} \in \widetilde{D}\}$ so that the disc $\widetilde{U}_0'$ is lower than D in Δ. This gives another isotopy deformation when the disc $\widetilde{U}_0'$ is the lowest in $\pi^{-1}(\Delta)$. □

The following holds.

PROPOSITION 2.1. *Let D be a surface diagram in $\mathbf{R}^3$. Let $|D| = |D_0|, \ldots, |D_n|$ be a sequence of generic surfaces in $\mathbf{R}^3$ such that for each i $(i = 1, \ldots, n)$, $|D_i|$ is obtained from $|D_{i-1}|$ by an $|R_1|^{\pm}$ move, ..., an $|R_4|^{\pm}$ move or an $|X_5|^{\pm}$ move. Then there is a sequence $D = D_0, \ldots, D_n$ of surface diagrams such that*

for each i $(i = 1, \ldots, n)$, D_i is $|D_i|$ with some crossing information, and that D_i is obtained from D_{i-1} by an $R_1^{\pm}$ move, ..., an $R_4^{\pm}$ move or an $X_5^{\pm}$ move.

PROOF. Note that each of local moves $R_i^{\pm}$, $(i = 1, 2, 3)$ and $X_5^{\pm}$ is a finger move in which the finger can be the highest or the lowest by Lemma 2.1. For an $|R_4|^{\pm}$ move, it is liftable and the lift $R_4^{\pm}$ is compatible with a finger move. For each i $(i = 1, \ldots, n)$, $|D_i|$ is obtained from $|D_{i-1}|$ by a projected finger move or an $|R_4|^{\pm}$ move. Thus there is a desired sequence $D = D_0, \ldots, D_n$. □

3. Cancelling pairs of double arcs

Let F be a surface-knot and let D_F be a surface diagram of F. Let $\widetilde{E}$ be an embedded disc in F. Denote $\pi(\widetilde{E})$ by $E \subset D_F$. Assume that the restriction of double arcs of D_F to E is a pair of disjoint proper arcs in E, say γ_0 and γ_1. The pair is called a *cancelling pair of double arcs* if the orientations of γ_0 and γ_1 are opposite and E is the upper sheet or the lower sheet for both double arcs.

LEMMA 3.1. *Let F be a surface-knot and let D_F be a surface diagram of F. Let Δ be a closed 3-ball with $\Delta \cap D_F \neq \emptyset$. Assume that Δ contains a properly embedded disc E and a sub-diagram D such that $\eta_1 = E \cap D$ is a proper double arc in E. Let $y_0 \in E$ and $y_1 \in D$ with $y_0, y_1 \notin \eta_1$, and let γ be a simple arc from y_0 to y_1 in $\mathrm{Int}(\Delta)$. Assume that there exists a finger move from y_0 to y_1 along γ so that the modified E intersects D transversely. Then the finger move can be chosen so that it creates a double arc η_2 and $\{\eta_1, \eta_2\}$ is a cancelling pair in Δ.*

PROOF. Take a small disc neighbourhood U_0 of y_0 in E. Assume that the finger move is done in a tubular neighbourhood $N(\gamma)$ of γ. Denote the deformed U_0 by U_0'. We use the same notations used in the proof of Lemma 2.1. Let B^2 be the unit open disc and there exists a diffeomorphism $\psi\colon B^2 \to U_0'$. We can assume that $\psi(B^2 \setminus B^2(0.5)) \cap D = \emptyset$ and $\psi(B^2(0.5)) \cap D \neq \emptyset$. Considering $\psi^{-1}(U_0' \cap D)$, it is a set of circles in B^2, which are pre-images of double circles. There exists an outer most circle C_1 in $\psi^{-1}(U_0' \cap D)$ in B^2. We can take a double point $z_0 \in (E \setminus U_0) \cap D$ and $z_1 \in \psi(C_1)$. Take an arc δ on U_0' from z_0 to z_1 so that $\delta \cap D = \{z_0, z_1\}$. Since the finger can be the highest or the lowest by Lemma 2.1, we can choose a suitable height of the disc $\psi(B^2(0.5))$ so that the pair of double arcs around z_0 and z_1 is a cancelling pair of double arcs. □

4. Proof of Theorem 1.1

Some special moves can be defined that decompose each of $|R_5^{\pm}|$ and $|R_6^{\pm}|$ moves into a finite sequence of elementary moves (see [4], [5] for details). We will show that those deformations can be realised as local isotopy deformations

of sub-diagrams of surface diagrams. Note that the R_7 move can be applied if there is an embedded disc G in $\mathbf{R}^3$ such that $G \cap D_F = \partial G$ and ∂G is formed by a pair of arcs δ_1, $\delta_2 \subset D_F$ such that $\delta_1 \cap \delta_2 = \partial\delta_1 = \partial\delta_2$, and points of $\partial\delta_1 = \partial\delta_2$ are in the cancelling pair of double arcs. We call the disc G a *descendent disc* (see [6], [4], [5]).

PROOF OF THEOREM 1.1. We call the singular disc produced by the R_4^+ move (see Figure 2), a *bug* [12]. The diagram has a pair of branch points and a double segments bounded by those branch points. The initial diagram in the R_5^+ move, which has a properly embedded simple double arc depicted in Figure 2, is called a *pipe disc* [12].

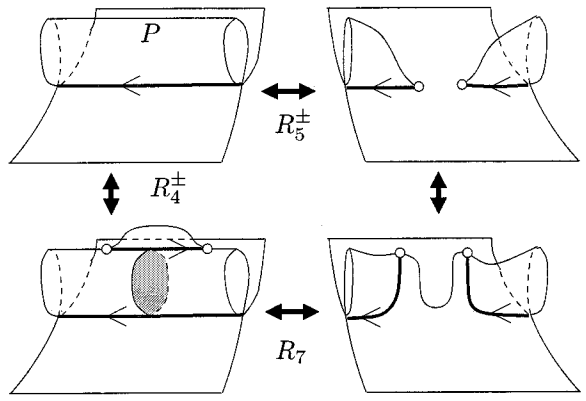


Fig. 4.

(1) We will describe $R_5^{\pm}$ moves by a finite sequence of $R_4^{\pm}$ and R_7 moves. Let P be a pipe disc. Add a bug on the pipe disc so that the double segment of the bug and the double arc of the pipe disc form a cancelling pair of double arcs. Then obviously there exists a descendent disc such that R_7 can be applied to obtain a pair of branch points with separated double arcs (see Figure 4). This deformation is R_5^+ move and the reversed deformation is R_5^- move.
(2) We will describe the R_6^+ move with $X_5^{\pm}$ and R_7 moves. We can view the initial diagram of the R_6^+ move as a sub-diagram consisting of a disc E and a sub-diagram D around a branch point such that $E \cap D$ is a proper double arc on E. Take a point $y_0 \in E \setminus D$ and take a point y_1 on the double arc joining the branch point $b \in D$. Take a simple arc γ from y_0 to y_1 passing through the branch point b along the double curve. Apply the finger move along γ from y_0 to y_1 so that the centre of the finger part creates a triple point with the diagram D. This modification is X_5^+ move and the finger can be the highest or the lowest so that we can create a cancelling pair of double edges by Lemma 3.1. Also we can find

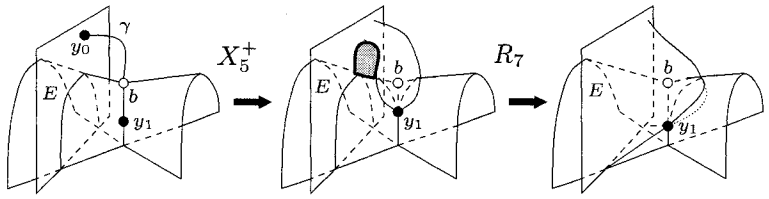


Fig. 5.

a descendent disc. Thus the R_7 move can be applied to obtain a diagram which is equivalent to the diagram obtained by R_6^+ move and the reversed deformation is R_6^- move. □

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