



Forbidden moves for virtual links

Okabayashi, Takayoshi

(Citation)

Kobe Journal of Mathematics, 22(1-2):49-63

(Issue Date)

2005-12

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0034434>

(URL)

<https://hdl.handle.net/20.500.14094/E0034434>



FORBIDDEN MOVES FOR VIRTUAL LINKS

Takayoshi OKABAYASHI

(Received January 14, 2005)

Abstract

We show that two oriented virtual 2-component links are deformed into each other by generalized Reidemeister moves and forbidden moves if and only if they have the same linking and virtual linking numbers. As an application, we show that a special value of the Sawollek polynomial of a 2-component virtual link is determined by the virtual linking number.

1. Introduction

L.H. Kauffman [3] introduced virtual knot theory. In this note, we consider a virtual link, which is an equivalence class of oriented virtual link diagrams. A *virtual link diagram* that is presented by an oriented link diagram which has not only a *classical crossing* (positive or negative crossing) but also a virtual crossing as shown in Fig. 1.

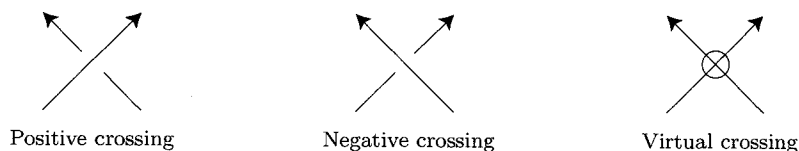


Fig. 1. Three types of crossings.

Two virtual link diagrams are *equivalent* if there exists a finite sequence of *classical Reidemeister moves* A1, A2, A3 or *virtual Reidemeister moves* B1, B2, B3, C as shown in Fig. 2. By a *generalized Reidemeister move* we mean either a classical or virtual Reidemeister move. A *virtual link* is an equivalence class of virtual link diagrams.

The D- and F-moves as shown in Fig. 3 are called the *forbidden moves*. In fact, T. Kanenobu [2] and S. Nelson [5] have shown independently that any virtual knot diagram can be deformed into the trivial knot if we allow forbidden moves.

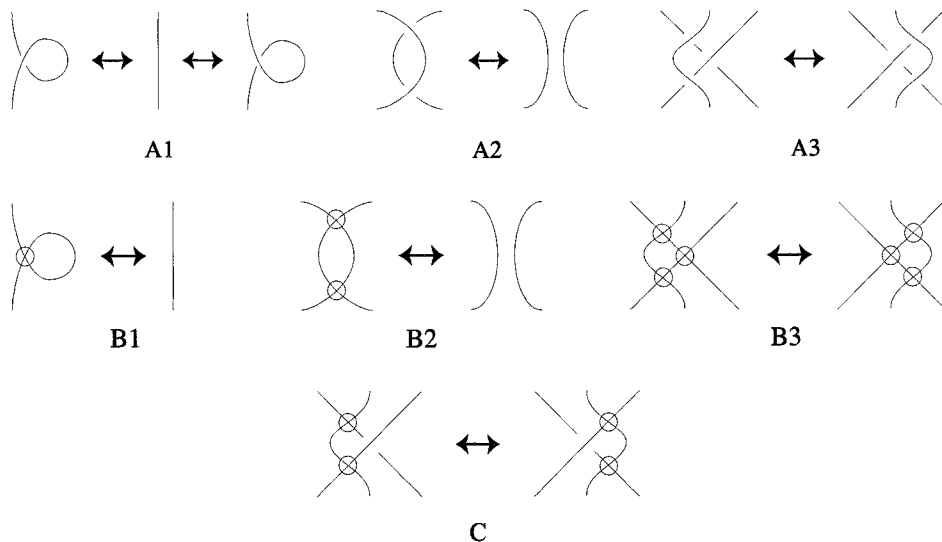


Fig. 2. Classical Reidemeister moves and virtual Reidemeister moves.

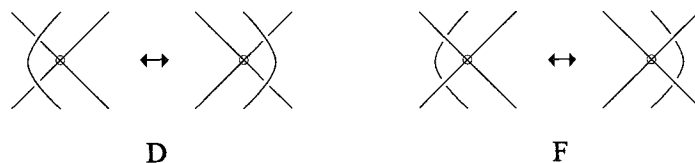


Fig. 3. Forbidden moves.

In this note, we generalize their result to a virtual link; allowing forbidden moves we show that any virtual oriented 2-component link can be deformed into a certain standard virtual 2-braid link (Theorem 6), and thus two virtual 2-component links with the same linking and virtual linking numbers can be deformed into each other if we allow forbidden moves (Corollary 7). This can be generalized to a virtual link with three components or more (Theorem 8).

As an application, we consider a special value of the Sawollek polynomial $Z_L(x, y)$ [6] of a virtual 2-component link L . T. Kishino [4] has shown that for a virtual knot K , the Sawollek polynomial at $x = 1$ always vanishes: $Z_K(1, y) = 0$. Showing that a forbidden move preserves the Sawollek polynomial at $x = 1$ (Theorem 10), we have: for a 2-component virtual link L with virtual linking number $n/2$, $Z_L(1, y) = (1 - (-y)^n)(1 - (-y)^{-n})$ (Theorem 12). Notice the above-mentioned result of Kanenobu and Nelson implies Kishino's result. In fact, the Sawollek polynomial of a classical link is zero.

This note is organized as follows: In the section 2, we define the linking number and the virtual linking number for an oriented virtual link, and define a virtual 2-braid link. In the section 3, we prepare some lemmas to prove our main theorem (Theorem 6). In the section 4, we prove our main theorem for a 2-component virtual link. In the section 5, we apply our main theorem to the Sawollek polynomial.

2. Linking numbers and a virtual 2-braid

Let $L = \bigcup_{i=1}^r K_i$ be an oriented r -component virtual link presented by a diagram D . For a classical crossing c , we define the sign, $\text{sgn}(c)$, in a usual way, that is, $+1$ or -1 according as c is positive or negative. For a virtual crossing v between different components K_i and K_j , we define the ij -sign of v , $\text{sgn}_{ij}(v)$, as shown in Fig. 4.



Fig. 4. The ij -sign for a virtual crossing.

Then we define the *linking number* of K_i and K_j , $\text{lk}(K_i, K_j)$, by

$$\text{lk}(K_i, K_j) = \frac{1}{2} \sum_{c \in C} \text{sgn}(c),$$

where C is the set of classical crossings of D between K_i and K_j . Similarly, we define the *virtual linking number* of K_i and K_j , $\text{vlk}(K_i, K_j)$, by

$$\text{vlk}(K_i, K_j) = \frac{1}{2} \sum_{v \in V} \text{sgn}_{ij}(v),$$

where V is the set of virtual crossings between K_i and K_j . Clearly, $\text{vlk}(K_i, K_j) = -\text{vlk}(K_j, K_i)$. It is easy to see the following.

PROPOSITION 1. *The linking number $\text{lk}(K_i, K_j)$ and the virtual linking number $\text{vlk}(K_i, K_j)$ are preserved by a generalized Reidemeister move, and so they are invariants of an oriented virtual link. Furthermore, they are preserved by a forbidden move.*

Let us consider a 2-component virtual link $K_1 \cup K_2$. There are six types of crossings between different components as shown in Fig. 5. Let n_i be the number

of crossings of Type i , $1 \leq i \leq 6$. Then we have:

$$\begin{aligned} \text{lk}(K_1, K_2) &= \frac{1}{2}(n_1 + n_2 - n_3 - n_4); \\ \text{vlk}(K_1, K_2) &= \frac{1}{2}(n_5 - n_6). \end{aligned}$$

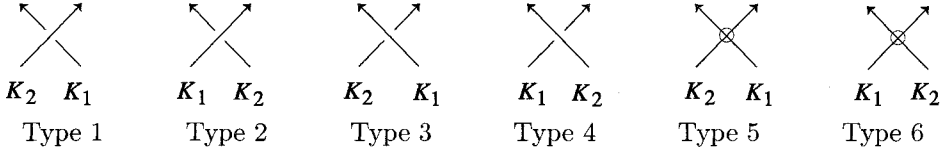


Fig. 5. Types of crossings between different components.

Furthermore, we have:

PROPOSITION 2.

$$\begin{aligned} \text{vlk}(K_1, K_2) &= \frac{1}{2}(n_5 - n_6) = \frac{1}{2}(-n_1 + n_2 - n_3 + n_4); \\ \text{lk}(K_1, K_2) + \text{vlk}(K_1, K_2) &= n_2 - n_3. \end{aligned}$$

PROOF. Let $L' = K'_1 \cup K'_2$ be a virtual link obtained from L by changing all classical crossings into virtual crossings, and let n'_i be the number of Type i crossings of L' . Then $n'_5 = n_1 + n_3 + n_5$, $n'_6 = n_2 + n_4 + n_6$, and so $\text{vlk}(K'_1, K'_2) = (1/2)((n_1 + n_3 + n_5) - (n_2 + n_4 + n_6))$, which is zero since L' is a trivial link. This gives the first formula, which further implies the second one. $\square$

We call each of the four virtual links as shown in Fig. 6 the *virtual Hopf link*. Let the left component be K_1 and the right one K_2 . Then from left to right, $\text{lk}(K_1, K_2) = 1/2, 1/2, -1/2, -1/2$; $\text{vlk}(K_1, K_2) = -1/2, 1/2, -1/2, 1/2$, which shows that they are mutually inequivalent. Also, since the linking number of a classical link is an integer, a virtual Hopf link is not a classical link.



Fig. 6. Virtual Hopf links.

More generally, we denote the 2-braid virtual link as shown in Fig. 7 by $L(P_1, P_2, \dots, P_n)$, where P_k is one of the *oriented virtual Hopf tangles* H_r , $1 \leq r \leq 6$, as shown in Fig. 8. Let the left component be K_1 and the right one K_2 .

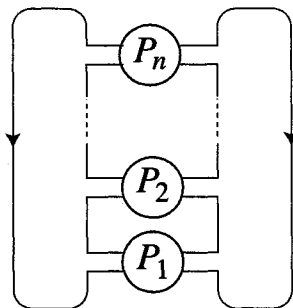
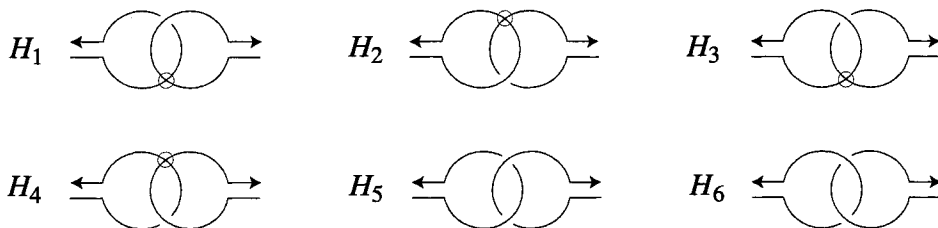

 Fig. 7. The virtual 2-braid $L(P_1, P_2, \dots, P_n)$.


Fig. 8. Oriented virtual Hopf tangles.

Further, we will denote by $L(m_1; m_2; m_3; m_4; m_5; m_6)$ the 2-braid virtual link

$$L(\underbrace{H_1, \dots, H_1}_{m_1}, \underbrace{H_2, \dots, H_2}_{m_2}, \dots, \underbrace{H_6, \dots, H_6}_{m_6}).$$

3. Lemmas

We define the S1-, S2-, and S3-moves on virtual link diagrams as shown in Fig. 9; in [2] S1- and S2-moves are called F' - and F'_2 -moves, respectively.

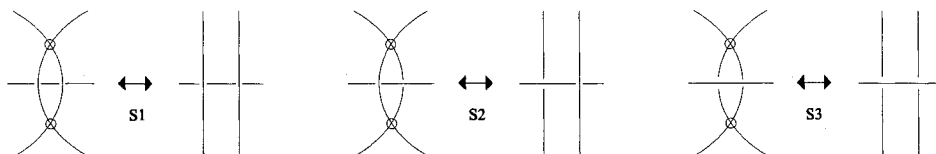


Fig. 9. The S1-, S2-, and S3-moves.

LEMMA 3. *Each of the S1-, S2-, and S3-moves is realized by generalized Reidemeister moves and forbidden moves.*

PROOF. In [2, Lemma], it is proved that each of the S1- and S2-moves are equivalent to the D-move. It is easy to see that the S3-move is realized by the D-move and the B2-move. $\square$

Kanenobu uses the following lemma to prove that any virtual knot can be deformed into the trivial knot allowing forbidden moves; see the proof of Theorem 1 and Example in [2].

LEMMA 4. *We can deform any self-classical crossing into a virtual crossing by using generalized Reidemeister moves and forbidden moves.*

PROOF. We show the deformations in Fig. 10, where the deformation indicated by an arrow $\rightarrow$ uses generalized Reidemeister moves and forbidden moves and the deformation indicated by an arrow with a tilde $\leadsto$ uses only generalized Reidemeister moves and no forbidden move. (In the following, we will use this notation.) Pick a self-classical crossing P and slide it along the arc until it arrives at the starting point. When it meets a particular crossing, we deform the diagram near it by using one of the S1-, S2-, S3-moves to pass through it. Then we obtain the second diagram, where near the dotted curves new virtual crossings appear. Then by using the B1-, B2-, B3, and C-moves, it becomes the third diagram, which further becomes the fourth one in a similar way. Using the A1-move, we obtain the last diagram. $\square$

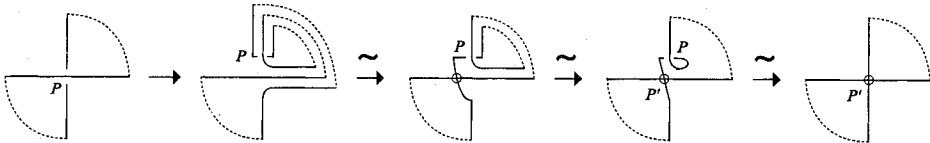


Fig. 10. Proof of Lemma 4.

LEMMA 5. *The three virtual links that are identical except the 2 crossings of the diagram as shown in Fig. 11 can be deformed one another by using generalized Reidemeister moves and forbidden moves.*

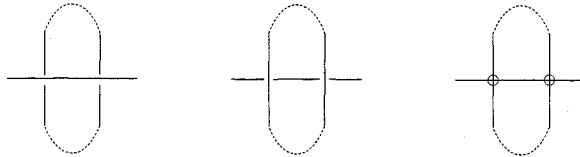


Fig. 11. Three virtual links of Lemma 5.

PROOF. We can prove in a similar way to the proof of Lemma 4. See Fig. 12, which shows the deformation between the second and third links of Fig. 11. $\square$

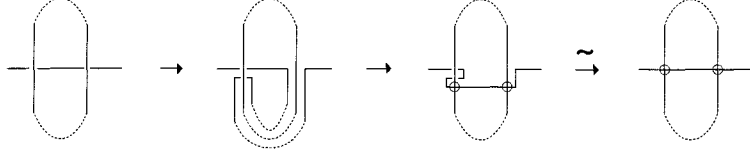


Fig. 12. Proof of Lemma 5.

4. 2-component virtual link

In this section, we show our main theorem:

THEOREM 6. *Let $L = K_1 \cup K_2$ be a 2-component virtual link. If $\text{lk}(K_1, K_2) = \lambda$ and $\text{v lk}(K_1, K_2) = \mu$, then L can be deformed into the following 2-braid virtual link by generalized Reidemeister moves and forbidden moves:*

$L(-2\mu; 0; 0; 0; \lambda + \mu; 0)$	<i>if</i> $0 \leq -\mu < \lambda$;
$L(0; 2\mu; 0; 0; \lambda - \mu; 0)$	<i>if</i> $0 \leq \mu < \lambda$;
$L(0; 0; -2\mu; 0; 0; -\lambda + \mu)$	<i>if</i> $0 \leq -\mu < -\lambda$;
$L(0; 0; 0; 2\mu; 0; -\lambda - \mu)$	<i>if</i> $0 \leq \mu < -\lambda$;
$L(0; \lambda + \mu; 0; - \lambda + \mu; 0; 0)$	<i>if</i> $0 < \lambda \leq \mu$;
$L(\lambda - \mu; 0; - \lambda - \mu; 0; 0; 0)$	<i>if</i> $0 < \lambda \leq -\mu$.

In particular, we have:

COROLLARY 7. *Two 2-component virtual links $L = K_1 \cup K_2$ and $L' = K'_1 \cup K'_2$ have the same linking numbers and virtual linking numbers: $\text{lk}(K_1, K_2) = \text{lk}(K'_1, K'_2)$, $\text{v lk}(K_1, K_2) = \text{v lk}(K'_1, K'_2)$ if and only if L can be deformed into L' by generalized Reidemeister moves and forbidden moves.*

PROOF OF THEOREM 6. Let n_r be the number of classical crossings between K_1 and K_2 of Type r , $1 \leq r \leq 4$, and $n = \sum_{r=1}^4 n_r$. Then $2\lambda = n_1 + n_2 - n_3 - n_4$ and $2\mu = -n_1 + n_2 - n_3 + n_4$ by Proposition 2.

Step 1. Let L_1 be a virtual link obtained from L by changing every self-classical crossing of L to a virtual crossing. Then by Lemma 4 L can be deformed into L_1 by using generalized Reidemeister moves and forbidden moves.

Step 2. Let L_2 be a virtual link obtained from L_1 by the move B2 near each classical crossing between different components as shown in Fig. 13; L_2 is

the result of fusions for a virtual link L'_2 with n_r sets of virtual Hopf links H_r , $1 \leq r \leq 4$, where L'_2 is a virtual link obtained from L by changing all classical crossings to virtual crossings. Thus L'_2 is a trivial 2-component link.

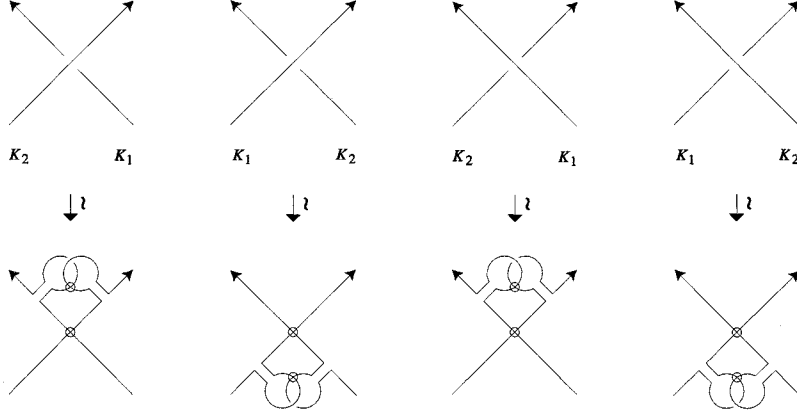


Fig. 13. Changing crossings by fusions of virtual Hopf links.

Step 3. Pull all virtual Hopf links outside L'_2 using the deformation as shown in Fig. 14, which is performed by the generalized Reidemeister moves. Next, slide each root of the fusion-bands with virtual Hopf links along the links L'_2 so that the roots on each component are close each other; see Fig. 15, where P_1 and P_2 are virtual Hopf tangles. Further using the deformation given in Fig. 14 we pull the virtual Hopf tangles so that they align as shown in Fig. 16, where the tangle T_3 contains only virtual crossings; we denote this resultant link by L_3 . In particular, notice that the crossings between the boundary of fusion bands are virtual crossings.

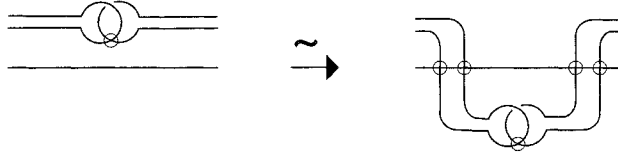


Fig. 14. Pulling a virtual Hopf tangle.

Step 4. Using the B1-, B2-, B3-moves, we can separate the arcs of K_1 and K_2 in the tangle T_3 , that is, we can deform L_3 into L_4 as shown in Fig. 17, where the tangle T_4 contains only arcs belong to K_1 and T'_4 contains only arcs belong to K_2 .

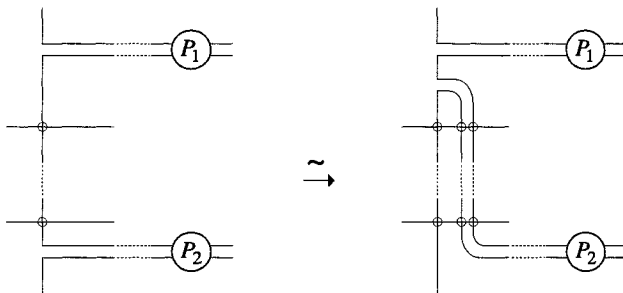
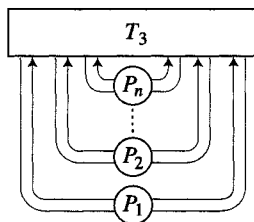
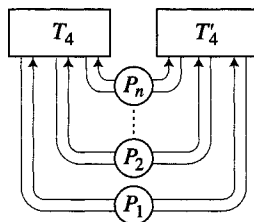


Fig. 15. Sliding the root of a fusion-band.


 Fig. 16. The virtual link L_3 .

 Fig. 17. The virtual link L_4 .

Step 5. We can exchange fusion bands as shown in Fig. 18 using generalized Reidemeister moves and forbidden moves, where P_1 and P_2 are one of the virtual Hopf tangles H_r , $1 \leq r \leq 4$; see Fig. 19 for the proof. The deformations from the first diagram to the second one, from the 4th diagram to the 5th one in Fig. 19 use Lemma 5. Thus after exchanging the fusion bands appropriately, we can deform L_4 into a 2-braid virtual link $L(P_1, P_2, \dots, P_n)$ as shown in Fig. 7 using the B1-, B2-, B3-moves.

Step 6. Consider the case $0 \leq -\mu < \lambda$. Then $0 \leq -(-n_1 + n_2 - n_3 + n_4) < n_1 + n_2 - n_3 - n_4$, and so $0 < n_2 - n_3 < n_1 - n_4$. We can change the order of the tangles $P_1, P_2, \dots, P_n$ by exchanging the fusion bands as in Step 5. Thus we

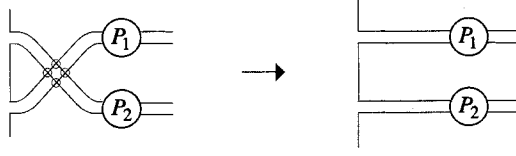


Fig. 18. Exchanging fusion bands.

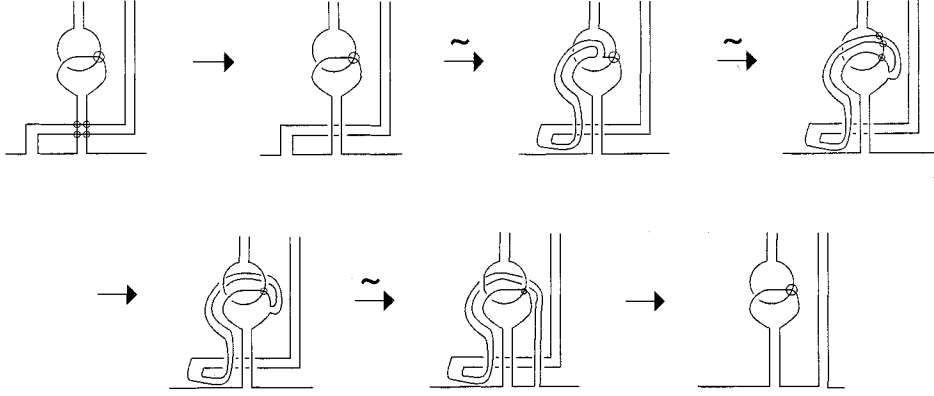


Fig. 19. Proof for exchanging fusion bands.

deform $L(P_1, P_2, \dots, P_n)$ into

$$L(\underbrace{H_1, H_4, H_1, H_4, \dots, H_1, H_4}_{n_4 \text{ pairs}}, \underbrace{H_2, H_3, H_2, H_3, \dots, H_2, H_3}_{n_3 \text{ pairs}}, \\ \underbrace{H_1, H_2, H_1, H_2, \dots, H_1, H_2}_{(n_2 - n_3) \text{ pairs}}, \underbrace{H_1, H_1, \dots, H_1}_{n_1 - n_2 + n_3 - n_4}),$$

which is equivalent to $L(-2\mu; 0; 0; 0; \lambda + \mu; 0)$ by using the A2-, B2-moves. We can prove for other cases similarly. This completes the proof. $\square$

More generally, we have

THEOREM 8. *Two r -component virtual links $L = K_1 \cup K_2 \cup \dots \cup K_r$ and $L' = K'_1 \cup K'_2 \cup \dots \cup K'_r$, $r \leq 2$, satisfy $\text{lk}(K_i, K_j) = \text{lk}(K'_i, K'_j)$, $\text{v} \text{lk}(K_i, K_j) = \text{v} \text{lk}(K'_i, K'_j)$ for each pair (i, j) , $1 \leq i < j \leq r$, if and only if L can be deformed into L' by generalized Reidemeister moves and forbidden moves.*

The proof is similar to that of Theorem 6; we deform each sublink $K_i \cup K_j$ of a virtual link L into the result of fusion for a trivial 2-component link with a certain 2-braid virtual link as given in Theorem 6.

5. Sawollek polynomials

First we review the definition of Sawollek polynomial. Let D be a virtual link diagram with n classical crossings $c_1, c_2, \dots, c_n$, $n \geq 1$. Define the $2n \times 2n$ matrix M as a block matrix $M = \text{diag}(M_1, \dots, M_n)$, where M_i is either

$$\begin{pmatrix} 1-x & -y \\ -xy^{-1} & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -x^{-1}y \\ -y^{-1} & 1-x^{-1} \end{pmatrix}$$

according as c_i is positive or negative. Next, define the $2n \times 2n$ matrix $P = (p_{(i,a),(j,b)})$ with rows and columns enumerated $(1,l), (1,r), (2,l), (2,r), \dots, (n,l), (n,r)$ as follows: We label the four vertices $D \cap \partial\Delta_i$ as in Fig. 20, where Δ_i is a disk neighborhood of the classical crossing c_i . Deleting all open neighborhoods $D \cap \text{Int } \Delta_1, \dots, D \cap \text{Int } \Delta_n$ near the classical crossings, we obtain a finite number of arcs together with virtual crossings. If the vertices i_a^- and j_b^+ , $1 \leq i, j \leq n$, $a, b = l, r$, are the end points of a common such arc, then $p_{(i,a),(j,b)} = 1$; otherwise $p_{(i,a),(j,b)} = 0$.

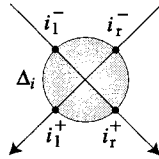


Fig. 20. Labelling the four vertices at the crossing c_i .

The Sawollek polynomial is defined by

$$Z_D(x, y) = (-1)^{w(D)} \det(M - P),$$

where $w(D)$ denotes the writhe of D , that is, the sum of the signs of classical crossings. Further, for a virtual link L presented by a virtual diagram D , we define the *normalized Sawollek polynomial* $\tilde{Z}_L(x, y)$ by

$$\tilde{Z}_L(x, y) = \begin{cases} x^{-N} Z_D(x, y) & \text{if } Z_D(x, y) \neq 0; \\ 0 & \text{if } Z_D(x, y) = 0, \end{cases}$$

where N is the lowest power of x in $Z_D(x, y)$. Then $Z_D(x, y)$ is an invariant of virtual links up to multiplication by powers of $x^{\pm 1}$ [6, Theorem 1], and so the normalized Sawollek polynomial is an invariant of virtual links. Several properties of the Sawollek polynomial are known; in particular, we have the following [6, Theorem 6 c)]:

PROPOSITION 9. *The Sawollek polynomial at $x = 1$, $Z_D(1, y)$, does not depend on the over-under information of the diagram's classical crossings.*

Moreover, we have

THEOREM 10. *A forbidden move preserves the Sawollek polynomial at $x = 1$, $Z_D(1, y)$.*

PROOF. By virtue of Proposition 9, we have only to prove that a D-move preserves $Z_D(1, y)$. Let D_1, D_2 be two virtual link diagrams that are identical except near a point where they are as in Fig. 21. We show that $Z_{D_1}(1, y) = Z_{D_2}(1, y)$. Other types of oriented D-moves are realized by this move and A2-moves; Fig. 22 shows one of other types of oriented D-moves are realized by the D-move given in Fig. 21 and A2-moves.

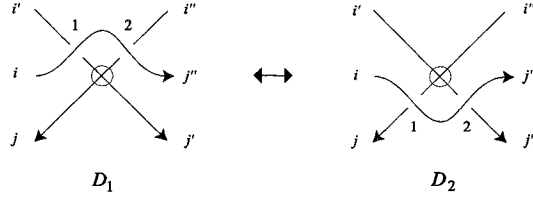


Fig. 21. Oriented D-move.

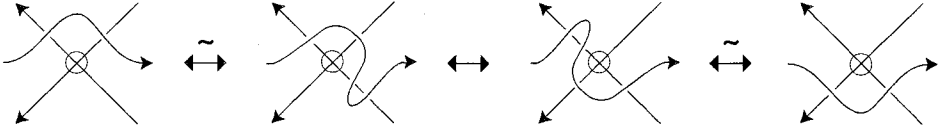


Fig. 22. Another oriented D-move.

In Fig. 21, each number k means the classical crossing c_k , and so, for D_1 , the following 6 pairs of vertices are connected by arcs:

$$\begin{aligned} &1_l^- \text{ and } i_a^+, \quad 1_r^- \text{ and } i_b'^+, \quad 2_r^- \text{ and } i_c''^+, \\ &j_e'^- \text{ and } 1_l^+, \quad j_d^- \text{ and } 2_l^+, \quad j_f''^- \text{ and } 2_r^+; \end{aligned}$$

and for D_2 :

$$\begin{aligned} &1_l^- \text{ and } i_a^+, \quad 1_r^- \text{ and } i_c''^+, \quad 2_r^- \text{ and } i_b'^+, \\ &j_d^- \text{ and } 1_l^+, \quad j_e'^- \text{ and } 2_l^+, \quad j_f''^- \text{ and } 2_r^+, \end{aligned}$$

forbidden moves. By Proposition 9 and Theorem 10, the Sawollek polynomials at $x = 1$ of L and L_n are equal. Thus by Lemma 11, we obtain the result. $\square$

Acknowledgements. The author would like to thank Akio Kawauchi, Taizo Kanenobu, Teruhisa Kadokami and Yasutaka Nakanishi for their useful advice and encouragement, and Toshimasa Kishino for informing him about his result [4].

References

- [1] M. Goussarov, M. Polyak and O. Viro, Finite type invariants of classical and virtual knots, *Topology* **39** (2000), 1045–1068.
- [2] T. Kanenobu, Forbidden moves unknot a virtual knot, *J. Knot Theory Ramifications* **10** (2001), 89–96.
- [3] L.H. Kauffman, Virtual knot theory, *Europ. J. Combinatorics* **20** (1999), 663–690.
- [4] T. Kishino, On prime virtual knots with up to 6 crossings (6 kouten ika no so na musubime ni tsuite) (in Japanese), Master's Thesis, Osaka City University (2002).
- [5] S. Nelson, Unknotting virtual knots with Gauss diagram forbidden moves, *J. Knot Theory Ramifications* **10** (2001), 931–935.
- [6] J. Sawollek, On Alexander-Conway polynomials for virtual knots and links, preprint (revised 2001).

Department of Mathematics
 Osaka City University
 Sugimoto, Sumiyoshi-ku, Osaka 558-8585
 Japan
 E-mail: taka@tshome.org