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PDF issue: 2026-11-24

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(Citation)
Kobe Journal of Mathematics,24(1):21-39

(Issue Date)
2007-06

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0034445

(URL)
https://hdl.handle.net/20.500.14094/E0034445

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KOBE J. MATH.,
24 (2007) 21–39

BOUNDARY SLOPES OF 2-BRIDGE LINKS DETERMINE THE CROSSING NUMBER

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(Received March 30, 2006)
(Revised June 15, 2006)

Abstract

A *diagonal* surface in a link exterior M is a properly embedded, incompressible, boundary incompressible surface which furthermore has the same number of boundary components and the same slope on each component of ∂M. We derive a formula for the boundary slope of a diagonal surface in the exterior of a 2-bridge link which is analogous to the formula for the boundary slope of a 2-bridge knot found by Hatcher and Thurston. Using this formula we show that the *diameter* of a 2-bridge link, that is, the difference between the smallest and largest finite slopes of diagonal surfaces, is equal to the crossing number.

1. Introduction

Let $\mathrm{cr}(K)$ denote the minimal crossing number of a knot K in the 3-sphere, and let $D(K)$ be the diameter of the set of finite boundary slopes of the knot. It was conjectured by Ichihara that

$$2\,\mathrm{cr}(K) \geq D(K) \tag{1}$$

for all knots K. This conjecture has been proven for 2-bridge knots by Mattman, Maybrun, and Robinson [12] and for Montesinos knots with three or more tangles by Ichihara and Mizushima [10]. Moreover, for alternating knots, the difference between the boundary slopes of the two checkerboard surfaces (in the reduced alternating diagram) is always twice the crossing number. Hence,

$$2\,\mathrm{cr}(K) = D(K) \tag{2}$$

for all alternating Montesinos knots. Neither [12] nor [10], however, discuss possible extensions of statements (1) or (2) to link exteriors. In this paper we do this by considering a restricted set of essential surfaces in the link exterior which we

call "diagonal" surfaces. Our main result, Theorem 2, provides a formula for the boundary slope of a diagonal surface of a 2-bridge link L which is analogous to the formula given by Hatcher and Thurston for the boundary slope of a 2-bridge knot. As an application of this formula, we prove Theorem 11, that

$$\mathrm{cr}(L) = D_\Delta(L)$$

where L is a 2-bridge link and $D_\Delta(L)$ is the diameter of the finite slopes of diagonal surfaces. In addition, if L is a non-split, n component, alternating link, and if both checkerboard surfaces are diagonal, then we show in Proposition 12 that $\frac{2}{n}\,\mathrm{cr}(L) \leq D_\Delta(L)$. This together with Theorem 11 suggests that

$$\frac{2}{n}\,\mathrm{cr}(L) \geq D_\Delta(L),$$

with equality in the case of alternating links, is a possible generalization of Ichihara's conjecture to non-split links.

The paper will proceed as follows. We begin by reexamining the beautiful relationship between boundary slopes of 2-bridge knots or links and minimal edge paths in diagrams of curve systems on the 4-punctured sphere developed by Hatcher and Thurston [7] and by Floyd and Hatcher [3]. In Section 2 we review the salient features of this theory and use results from our paper [9] in order to derive a formula for the boundary slope of a diagonal surface. We apply this formula to prove Theorem 11 in Section 3. Finally, in Section 4, we discuss extensions of these ideas to n component, non-split links.

2. Boundary Slopes

We begin with some basic terminology. An *essential* surface S in a compact, orientable 3-manifold with boundary is a properly embedded surface which is both incompressible and boundary incompressible. If the 3-manifold is the exterior of a link of n components, then we can choose a preferred basis $\{\mu_i, \lambda_i\}$ for each boundary torus T_i^2, $1 \leq i \leq n$. The intersection of S with T_i^2 is a collection of k_i simple, closed, nontrivial, parallel curves which determine an isotopy class represented by $\mu_i^{p_i}\lambda_i^{q_i}$ for some co-prime integers p_i, q_i. The *boundary slope* of S on component i is defined to be the ratio p_i/q_i. One can also consider the $2n$-tuple $(k_1p_1, k_1q_1, k_2p_2, k_2q_2, \ldots, k_np_n, k_nq_n)$ which encodes the boundary slopes and the number of sheets of S on each boundary component.

According to Hatcher [8], a knot can have only a finite number of boundary slopes. However for a link there may be infinitely many for each component. Therefore, in order to define a diameter we restrict our attention to a special subset of surfaces. Define a *diagonal surface* to be an essential surface whose

associated $2n$-tuple has the form $(kp, kq, kp, kq, \ldots, kp, kq)$. That is, the boundary slope and number of sheets of S on each component is the same. To each diagonal surface we can assign the single slope p/q. If L is a non-split link, then Floyd and Oertel [4] prove that there are a finite number of branched surfaces in the link exterior that carry all essential surfaces. As pointed out in [8], if S_1 and S_2 are two essential surfaces carried by the same branched surface, then the intersection number $\partial S_1 \cdot \partial S_2$ is equal to zero. If S_1 and S_2 are diagonal with $2n$-tuples $(kp, kq, kp, kq, \ldots, kp, kq)$ and $(jr, js, jr, js, \ldots, jr, js)$, then

$$\partial S_1 \cdot \partial S_2 = kj(pr - qs) = 0.$$

Hence, S_1 and S_2 have the same slope and so there can only be a finite number of diagonal boundary slopes for any non-split link. It follows that the diameter $D_\Delta(L)$ of any non-split link is finite.

We assume the reader is familiar with the fact that corresponding to each reduced rational number p/q with $0 \leq p \leq q$, is a 2-bridge knot if q is odd or 2-bridge link if q is even. However, we warn the reader that what some authors call the 2-bridge knot or link $L_{p/q}$, others call the mirror image of $L_{p/q}$. In this paper we follow the convention used in [7] which is opposite that used in [3], [9], and [11]. In the papers by Hatcher and Thurston [7] and Floyd and Hatcher [3], the set of essential surfaces in the exterior of a 2-bridge knot or link $L_{p/q}$ are completely described and classified. These papers develop a beautiful correspondence between the essential surfaces in $L_{p/q}$ and certain paths in diagrams of curve systems on the 4-punctured sphere. For all essential surfaces in knot exteriors, and for diagonal surfaces in link exteriors, this diagram is called D_1 and is shown in Figure 1.

The diagram D_1 is a tessellation of $\mathbb{H}^2$ by ideal triangles. The rationals, together with $\frac{1}{0}$, are arranged around the unit circle as shown, and two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are connected by a geodesic if and only if $ad - bc = \pm 1$. The group of orientation preserving symmetries of D_1 is $\mathrm{PSL}_2\mathbb{Z}$. Let $G \subset \mathrm{PSL}_2\mathbb{Z}$ be the subgroup of Möbius transformations given by $z \to \frac{az+b}{cz+d}$ where c is even. It follows that the ideal triangle $\left\{\frac{1}{0}, \frac{0}{1}, \frac{1}{1}\right\}$ is a fundamental domain for the action of G and the G-images of the *fundamental* ideal quadrilateral $Q = \left\{\frac{1}{0}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$ tessellate $\mathbb{H}^2$. There are two distinct orbits of edges which are labeled A and C.

An edge path in D_1 is *minimal* if it never contains two consecutive edges that lie in the same triangle. According to [7] and [3], each minimal edge path in D_1 from $\frac{1}{0}$ to $\frac{p}{q}$ determines a diagonal surface in $L_{p/q}$. (A similar correspondence exists for non-diagonal surfaces but involving paths in a more complicated diagram D_t.) For a particular fraction $\frac{p}{q}$ there can only be a finite number of minimal edge paths connecting it to $\frac{1}{0}$. This follows from the fact that these minimal paths are all contained in a unique minimal chain of quadrilaterals consisting of

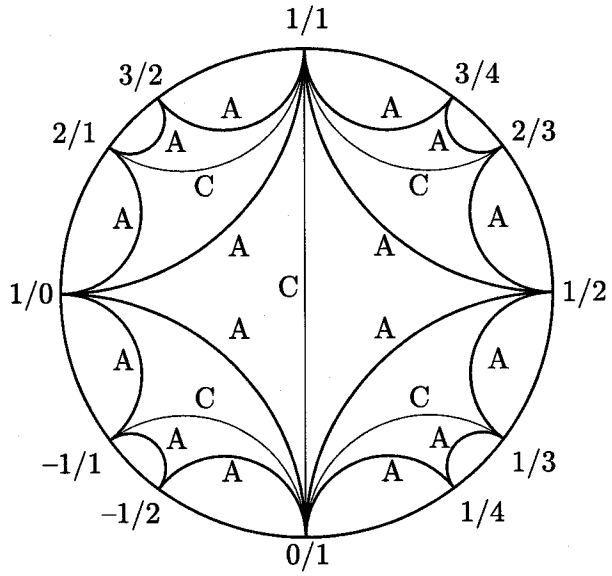

Figure 1. The Diagram D_1.

Q and a finite number of its translates under G.

In order to determine the slope of a diagonal surface we must first describe several important features of edge paths in D_1. Each edge path from $\frac{1}{0}$ to $\frac{p}{q}$ corresponds to a continued fraction expansion[1]

$$\frac{p}{q} = r + [b_1, b_2, \ldots, b_k] = r + \cfrac{1}{b_1 - \cfrac{1}{b_2 - \ddots - \cfrac{1}{b_k}}}$$

where the partial sums

$$\frac{p_i}{q_i} = r + [b_1, b_2, \ldots, b_i]$$

are the consecutive vertices on the path. At the vertex p_i/q_i the path turns left with b_{i+1} triangles on the left if $b_{i+1} > 0$ or to the right with $-b_{i+1}$ triangles on the right if $b_{i+1} < 0$. For example, the path γ shown in Figure 5 corresponds to the expansion

$$\frac{13}{34} = 0 + [2, -1, 1, -1, 1, -2].$$

[1]We follow the notational convention of [7].

Because b_i can be interpreted in terms of the amount of turning at vertex p_{i-1}/q_{i-1}, we call the b_i's the *turning numbers* of the path. For any path γ let n_γ^+ and n_γ^- be the number of positive and negative turning numbers respectively. Minimality of a path can now be stated in terms of the turning numbers: a path is minimal if and only if all the turning numbers are 2 or more in absolute value.

We may recursively generate D_1 by starting with the initial pair $\frac{1}{0}$ and $\frac{0}{1}$ and then introducing *mediants*. We first introduce the mediant $\frac{1+0}{0+1} = \frac{1}{1}$, obtaining the sequence $\{\frac{1}{0}, \frac{1}{1}, \frac{0}{1}\}$. We now insert mediants again between each consecutive pair of fractions to obtain $\{\frac{1}{0}, \frac{2}{1}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1}\}$ and so on. This process keeps the sequence in decreasing order and also preserves the fact that the determinant $p_iq_{i+1} - p_{i+1}q_i$ of consecutive fractions is always $+1$. Viewed this way we see that every vertex in D_1 has two *parents*: the fractions that gave birth to it when taking mediants. Both parents of a link (a fraction with even denominator, and thus corresponding to a 2-bridge link) are knots, while the parents of a knot (a fraction with odd denominator) are a mixture of a knot and a link. Furthermore, the numerators of the parents of a link must have opposite parity. It is not hard to see that each vertex in a minimal path from $\frac{1}{0}$ to $\frac{p}{q}$ must be a parent of the next vertex.

A minimal path is called *even* if all of the turning numbers are even. Note that an even path starting at $1/0$, can never traverse the diagonal (C-type edge) of any quadrilateral. Thus, the vertices along an even path must alternate between knots and links. Notice that there exists a unique even path from $1/0$ to $0/1$ and also from $1/0$ to $1/1$ while there exist two even paths from $1/0$ to $1/2$. This pattern persists in general. That is, each knot p/q has a unique even path, $e(p/q)$, connecting it to $1/0$ while each link p/q has exactly two such even paths. In the case of a link, we denote the even path which arrives via the parent with even numerator $e^0(p/q)$ and the one which arrives via the parent with odd numerator $e^1(p/q)$. These facts follow from an inductive argument as follows. Suppose p/q is a knot. Exactly one of its parents, say a/c, is a link. By the first remark above, any even path to p/q must arrive through a/c. Since a/c is closer to $1/0$ than p/q, we may assume inductively that there exist two even paths from $1/0$ to a/c. Each of these may be extended to a path to p/q by adjoining the edge $\{a/c, p/q\}$. The final turning numbers of these two paths differ by one, hence exactly one of them is even. If, on the other hand, p/q is a link, then both its parents are knots. Again by our inductive hypothesis, each parent is connected to $1/0$ by a unique even path. Now both of these paths extend to even paths to p/q because the turning number at a knot vertex along any path which consists of A-type edges only is always even.

Given p and q define $\epsilon_i(p/q)$ as

$$\epsilon_i(p/q) = (-1)^{\lfloor ip/q \rfloor} \quad \text{for} \quad 0 < i < q,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. These numbers play several important roles in relation to the 2-bridge knot or link $L_{p/q}$. For example, they can be used to express the single relation in a certain 2-generator presentation of the fundamental group of the complement. Or, in the case of a link, the sum of all the ϵ_i's where i is odd is the linking number of the two components (assuming a certain orientation convention). It is convenient to introduce notation for the sum of the even and the sum of the odd ϵ_i's. Let

$$\sigma_0(p/q) = \sum_{i=1}^{\lfloor (q-1)/2 \rfloor} \epsilon_{2i}(p/q) \quad \text{and} \quad \sigma_1(p/q) = \sum_{i=0}^{\lfloor (q-2)/2 \rfloor} \epsilon_{2i+1}(p/q).$$

If $\gamma = \{1/0, p_1/q_1, \ldots, p_n/q_n\}$ is any oriented edge-path in D_1 from $1/0$ to p_n/q_n, we define $m(\gamma)$ to be

$$m(\gamma) = \sum_{i=1}^{n-1} \det \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix}.$$

More generally, if γ is any oriented edge-path in D_1 we define $m(\gamma)$ to be the sum of the *determinants* of its edges, excluding any edge containing $\frac{1}{0}$.

We may now state the following result which gives three different formulations for the boundary slope of a 2-bridge knot corresponding to a specific minimal path in D_1. The first of these is Proposition 2 of [7]. Later in this section we will show that the second two formulations follow from the first.

THEOREM 1. *If γ is a minimal path in D_1 from $\frac{1}{0}$ to the 2-bridge knot $\frac{p}{q}$, then each of the following gives the boundary slope of the corresponding essential surface.*

1. $2\left[(n^+_\gamma - n^-_\gamma) - \left(n^+_{e(p/q)} - n^-_{e(p/q)}\right)\right]$
2. $-2[m(\gamma) - m(e(p/q))]$
3. $-2[m(\gamma) - 2\sigma_0(p/q)]$

It is worth noting that the first formula for the boundary slope given above can be thought of as a slope of $2(n^+_\gamma - n^-_\gamma)$ with respect to a non-preferred longitude, which is then rewritten in terms of the preferred longitude by subtracting the "correction" term $2\left(n^+_{e(p/q)} - n^-_{e(p/q)}\right)$. Similarly, $m(e(p/q))$ and $2\sigma_0(p/q)$ provide the correction in the second and third formulas.

In [3], Floyd and Hatcher extended the work of Hatcher and Thurston, classifying all essential surfaces in 2-bridge link exteriors. Lash [11] then developed an algorithm to compute the boundary slopes of these surfaces. Lash's algorithm was not published, but is described in both [6] and [9]. The following theorem provides the analog of Theorem 1 for diagonal surfaces in 2-bridge link exteriors.

The third of these formulations is easily derived from [9] which simplifies and extends the work of [11].

THEOREM 2. *If γ is a minimal path in D_1 from $\frac{1}{0}$ to the 2-bridge link $\frac{p}{q}$, then each of the following gives the boundary slope of the corresponding diagonal surface.*

1. $(n_\gamma^+ - n_\gamma^-) - \frac{1}{2}\left[\left(n^+_{e^0(p/q)} - n^-_{e^0(p/q)}\right) + \left(n^+_{e^1(p/q)} - n^-_{e^1(p/q)}\right)\right]$
2. $-\left[m(\gamma) - \frac{1}{2}\left(m(e^0(p/q)) + m(e^1(p/q))\right)\right]$
3. $-[m(\gamma) - \sigma_0(p/q)]$

The similarity between Theorem 1 and Theorem 2 can be further enhanced by thinking of the "correction" terms as being obtained by averaging over all even paths. For example, there are two even paths to a link and so $(m(e^0)+m(e^1))/2$ is the average value of $m(\gamma)$ averaged over all even paths γ. Since there is only one even path to a knot, $m(e)$ is again the average value of m averaged over all even paths.

Continuing to compare Theorems 1 and 2, we also see that it is necessary to multiply by a factor of 2 when going from links to knots. This makes sense because when the two components of a link are "joined" to form a single knot, and boundary curves on each component are connected to form a single boundary curve, the numbers of longitudes and meridians comprising each of the original boundary curves must be combined. Since diagonal surfaces have the same data on each component this combination amounts to multiplication by 2.

The remainder of this section will be devoted to proving Theorems 1 and 2. A number of lemmas are required to relate the quantities $n_\gamma^+ - n_\gamma^-, m(\gamma)$ and $\sigma_0(p/q)$ for different paths γ. We begin with a result describing how $m(\gamma)$ changes when γ undergoes a simple change.

Let $\mathcal{C}$ be the minimal chain of quadrilaterals from $\frac{1}{0}$ to $\frac{p}{q}$. Suppose that γ is any path in $\mathcal{C}$ from $\frac{1}{0}$ to $\frac{p}{q}$ and that T is a triangle in $\mathcal{C}$ having one or two edges in γ. If we remove from γ the edges of T and then replace them with the edges of T that are not in γ we obtain a new path in $\mathcal{C}$ from from $\frac{1}{0}$ to $\frac{p}{q}$. We call this a *triangle move.* We may further refine our definition to *left* and *right* triangle moves depending on whether T lies to the left or right of the original path.

LEMMA 3. *Changing γ by a left triangle move increases $m(\gamma)$ by 1.*

PROOF. Every quadrilateral in $\mathcal{C}$ is the image of the fundamental quadrilateral by an element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where c is even and $ad - bc = 1$. Thus, the vertices of the quadrilateral are (in counter-clockwise order)

$$\left\{\frac{a}{c}, \frac{b}{d}, \frac{a+2b}{c+2d}, \frac{a+b}{c+d}\right\}.$$

Furthermore, the diagonal connects $\frac{b}{d}$ to $\frac{a+b}{c+d}$. Let T_1 be the triangle with vertices $\left\{\frac{a}{c}, \frac{b}{d}, \frac{a+b}{c+d}\right\}$ and T_2 the triangle with vertices $\left\{\frac{b}{d}, \frac{a+2b}{c+2d}, \frac{a+b}{c+d}\right\}$. If we orient the boundary of each triangle counter-clockwise, then it is a simple matter to check that $m(\partial T_i) = -1$ for $i = 1, 2$. Now suppose that γ' is obtained from γ by a left triangle move across the triangle T. If $m(\gamma) = x + y$ where y is the contribution to $m(\gamma)$ due to the edges of T in γ, then the edges of T in γ' contribute $y + 1$ to $m(\gamma')$. Hence

$$m(\gamma') = x + y + 1 = m(\gamma) + 1.$$

Therefore, a left triangle move always increases m by 1. □

There are two paths in $\mathcal{C}$ from $\frac{1}{0}$ to $\frac{p}{q}$ which we will call the *upper* and *lower* paths. Topologically, $\mathcal{C}$ is a disk. The lower path follows the perimeter of $\mathcal{C}$ from $\frac{1}{0}$ to $\frac{p}{q}$ in the counter-clockwise direction while the upper path follows the perimeter in the clockwise direction. Except when $q = 1$, neither path can contain three edges in a row from a single quadrilateral since $\mathcal{C}$ is minimal. However, it is possible that two edges in a row are from the same quadrilateral. In this case, if the vertex common to the two edges has an even denominator, then the path is not minimal. If we replace each such occurrence with the diagonal of that quadrilateral, then the path will be minimal. Call these two paths the *lower minimal path* γ_ℓ and the *upper minimal path* γ_u. (If $q = 1$, then $p/q = 0/1$, or $1/1$ and $\gamma_\ell = \gamma_u$.)

LEMMA 4. *The determinant of every edge in both the lower path and the lower minimal path of* $\mathcal{C}$ *(except for the first edge, which contains* $\frac{1}{0}$*) is* -1*. The determinant of every edge in both the upper path and the upper minimal path of* $\mathcal{C}$ *(except for the first edge, which contains* $\frac{1}{0}$*) is* $+1$*.*

PROOF. We show first that as we traverse the perimeter of $\mathcal{C}$ in the counter-clockwise direction the determinant of each edge, other than the edge $\left\{\frac{1}{0}, \frac{0}{1}\right\}$, is -1. If $\mathcal{C}$ consists of a single quadrilateral then this is easy to check. Proceeding by induction, imagine that the last quadrilateral of the chain has been attached to all the previous ones along the edge $\left\{\frac{a}{c}, \frac{b}{d}\right\}$, where the vertex $\frac{a}{c}$ is reached before $\frac{b}{d}$ as one travels counter-clockwise from $\frac{1}{0}$. Thus, by our inductive hypothesis, $ad - bc = -1$. If c is even, then the perimeter of $\mathcal{C}$ has

been changed by replacing the edge $\{\frac{a}{c}, \frac{b}{d}\}$ with the sequence of three edges $\{\{\frac{a}{c}, \frac{a+b}{c+d}\}, \{\frac{a+b}{c+d}, \frac{a+2b}{c+2d}\}, \{\frac{a+2b}{c+2d}, \frac{b}{d}\}\}$. Each of these three new edges has determinant -1. If instead d is even, the edge $\{\frac{a}{c}, \frac{b}{d}\}$ is replaced with the sequence $\{\{\frac{a}{c}, \frac{2a+b}{2c+d}\}, \{\frac{2a+b}{2c+d}, \frac{a+b}{c+d}\}, \{\frac{a+b}{c+d}, \frac{b}{d}\}\}$. Once again, the determinants of these new edges are all -1. Since reversing the direction of an edge negates its determinant, we see that every edge of the upper path has a determinant of $+1$.

If the lower path is not minimal, then we may change the lower path to the lower minimal path by left triangle moves where, moreover, each triangle move replaces two edges with one edge. Such a move increases m by 1 and hence the new edge still has a determinant of -1. A similar argument applies to the upper minimal path. □

LEMMA 5. *Let γ be any path from $1/0$ to p/q in $\mathcal{C}$. Then*

$$-m(\gamma) = n_\gamma^+ - n_\gamma^-.$$

PROOF. Our strategy is to first prove the result for a specific path γ and then show that it remains true as γ is changed to any other path by triangle moves. Let γ be the lower (not necessarily minimal) path in $\mathcal{C}$. It follows that $r = 0$ and all the turning numbers $b_1, b_2, \ldots, b_n$ are positive. Thus $n_\gamma^+ - n_\gamma^- = n - 0 = n$. But, from Lemma 4, $m(\gamma) = -n$. Thus, $-m(\gamma) = n_\gamma^+ - n_\gamma^-$.

Now suppose that γ is any path in $\mathcal{C}$ and we change γ by a left triangle move. By Lemma 3 this will increase m by one. We wish to show that $n_\gamma^+ - n_\gamma^-$ decreases by one. Consider first the case where one edge of γ is replaced by two edges. The new path has one more vertex and one more turning number. It is not hard to see that the new turning is negative while all the other turning numbers keep the same sign. Hence, n_γ^- increases by one and the difference $n_\gamma^+ - n_\gamma^-$ decreases by one.

If the left triangle move exchanges two edges for one, then we may treat it as a right triangle move that exchanges one edge for two. The proof is now nearly the same as before except that the new turning number contributes to n_γ^+ instead of n_γ^-. Therefore such a right triangle move increases $n_\gamma^+ - n_\gamma^-$ by one while m decreases by one. □

We now turn our attention to the ϵ_i's in order to relate σ_0 to m and $n^+ - n^-$.

LEMMA 6. *If $0 < i < q - 1$, then*

$$\epsilon_i(p/q) = (-1)^{p+1}\epsilon_{q-i}(p/q).$$

PROOF. There is a beautiful, and quite useful, geometric interpretation of the ϵ_i's. Figure 2 shows a line of slope p/q extending from $(0,0)$ to (q,p). It cuts

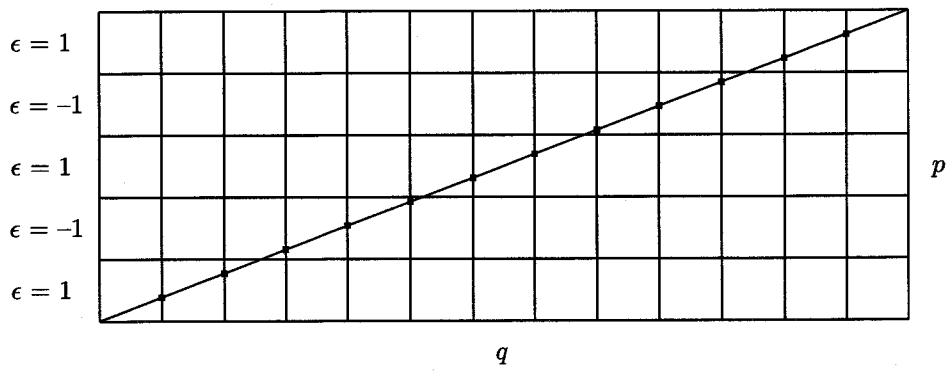

Figure 2. The heights of the dots on the line of slope p/q give the ϵ_i's.

the line $x = i$ at a point P_i with height ip/q. Thus $\lfloor ip/q \rfloor$ is the height of the integer lattice point just beneath P_i. Each time the line passes through another horizontal line in the lattice, the signs of the ϵ_i's change. The result now follows if we consider rotating this figure 180 degrees around its center. □

The following result follows directly from Lemma 6 and also can be visualized nicely in Figure 2.

LEMMA 7. *If q is odd, then*

$$\sigma_0(p/q) = (-1)^{p+1}\sigma_1(p/q).$$

If $p/q = (a+b)/(c+d)$ is the mediant of a/c and b/d, then we would like to relate the sum of the even or odd ϵ_i's for p/q to the corresponding sums for its parents a/c and b/d. The next lemma provides the first step in this direction.

LEMMA 8. *Let a, b, c, and d be positive integers such that $\gcd(a,c) = \gcd(b,d) = 1$, $0 < a/c < b/d$ and $ad - bc = -1$. Let $p/q = (a+b)/(c+d)$ be the mediant of a/c and b/d. Then*

$$\epsilon_i(a/c) = \epsilon_i(p/q) \quad \textit{for} \quad 0 < i < c, \textit{ and}$$
$$\epsilon_i(b/d) = \epsilon_i(p/q) \quad \textit{for} \quad 0 < i < d.$$

PROOF. We give a geometric proof based on Figure 3. Consider the parallelogram P which is the image of the unit square, $[0,1] \times [0,1]$ under the linear transformation T given by

$$T = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

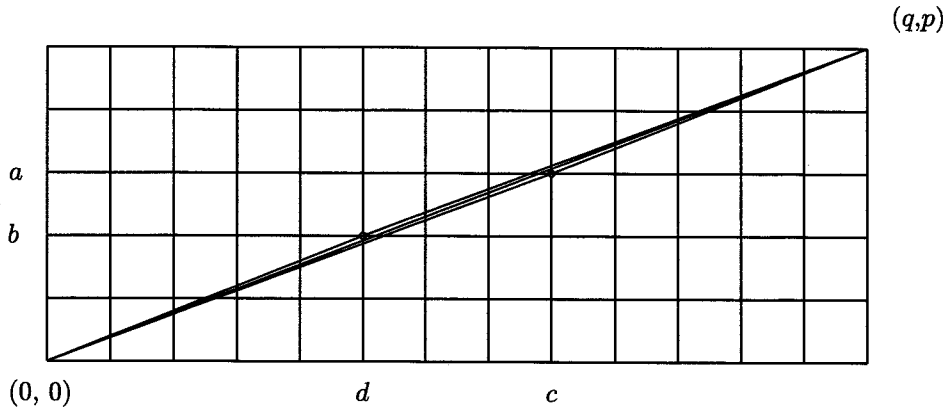

Figure 3. The parallelogram contains no lattice points in its interior.

No integer lattice point lies in the interior of P since T takes the interior of the unit square to the interior of P. For $0 < i < c$, the points $(i, ia/c)$ and $(i, ip/q)$ lie in P and hence cannot have an integer lattice point between them. Thus $\lfloor ia/c \rfloor = \lfloor ip/q \rfloor$ and so $\epsilon_i(a/c) = \epsilon_i(p/q)$. A similar argument shows that $\epsilon_i(b/d) = \epsilon_i(p/q)$ if $0 < i < d$.

(The fact that the parallelogram P has an area of 1 unit is the basis of a neat parlor trick! See page 96 of [5].) □

Using Lemma 8 we may now express $\sigma_i(p/q)$ in terms of its parents. The following formulae can all be discovered by examining Figure 3.

LEMMA 9. *Assume the hypotheses of Lemma* 8.
If q is even, then

$$\sigma_0(p/q) = \sigma_0(a/c) + \sigma_0(b/d), \quad \textit{and}$$
$$\sigma_1(p/q) = \sigma_1(a/c) + \sigma_1(b/d) + (-1)^a.$$

If q is odd, then

$$\sigma_0(p/q) = \begin{cases} \sigma_0(a/c) + (-1)^{p+1}\sigma_1(b/d) & \textit{if } c \textit{ is odd,} \\ \sigma_0(b/d) + (-1)^{p+1}\sigma_1(a/c) & \textit{if } d \textit{ is odd.} \end{cases}$$

PROOF. Suppose q is even and therefore both c and d are odd. Using Lemmas 6 and 8 we obtain

$$\sigma_0(p/q) = \sum_{i=1}^{(q-2)/2} \epsilon_{2i}(p/q)$$

$$
\begin{aligned}
&= \sum_{i=1}^{(c-1)/2} \epsilon_{2i}(p/q) + \sum_{i=(c+1)/2}^{(q-2)/2} \epsilon_{2i}(p/q) \\
&= \sum_{i=1}^{(c-1)/2} \epsilon_{2i}(a/c) + \sum_{i=(c+1)/2}^{(q-2)/2} \epsilon_{q-2i}(p/q) \\
&= \sigma_0(a/c) + \sum_{i=1}^{(d-1)/2} \epsilon_{2i}(p/q) \\
&= \sigma_0(a/c) + \sum_{i=1}^{(d-1)/2} \epsilon_{2i}(b/d) \\
&= \sigma_0(a/c) + \sigma_0(b/d).
\end{aligned}
$$

If we consider the sum of the odd ϵ_i's instead, we obtain

$$
\begin{aligned}
\sigma_1(p/q) &= \sum_{i=0}^{(q-2)/2} \epsilon_{2i+1}(p/q) \\
&= \sum_{i=0}^{(c-3)/2} \epsilon_{2i+1}(p/q) + \epsilon_c(p/q) + \sum_{i=(c+1)/2}^{(q-2)/2} \epsilon_{2i+1}(p/q) \\
&= \sum_{i=0}^{(c-3)/2} \epsilon_{2i+1}(a/c) + (-1)^{\lfloor cp/q \rfloor} + \sum_{i=(c+1)/2}^{(q-2)/2} \epsilon_{q-2i-1}(p/q) \\
&= \sigma_1(a/c) + (-1)^a + \sum_{i=0}^{(d-3)/2} \epsilon_{2i+1}(p/q) \\
&= \sigma_1(a/c) + (-1)^a + \sum_{i=0}^{(d-3)/2} \epsilon_{2i+1}(b/d) \\
&= \sigma_1(a/c) + (-1)^a + \sigma_1(b/d).
\end{aligned}
$$

The cases when q is odd are similar and are left to the reader. □

PROPOSITION 10. *If q is even, then*

$$
\begin{aligned}
m(e^0(p/q)) &= \sigma_0(p/q) - \sigma_1(p/q), \quad \textit{and} \\
m(e^1(p/q)) &= \sigma_0(p/q) + \sigma_1(p/q).
\end{aligned}
$$

If q is odd, then

$$
m(e(p/q)) = 2\sigma_0(p/q).
$$

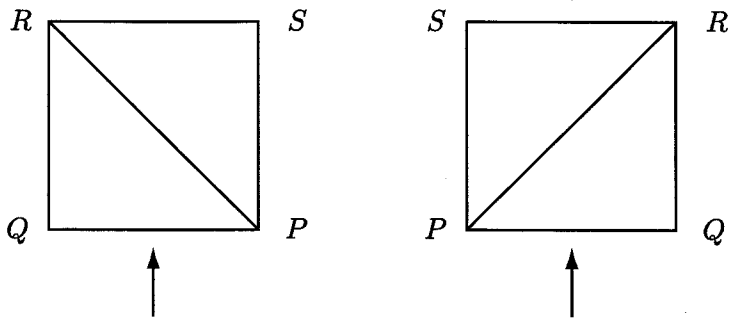

Figure 4. The last quadrilateral in the minimal chain from $1/0$ to either R or S.

PROOF. We use an induction on the number of quadrilaterals in $\mathcal{C}$. If $\mathcal{C}$ is a single quadrilateral, then the proposition is easily verified.

To prove the inductive step, we must consider two cases depending on how the last quadrilateral is attached to the chain. The two cases are shown in Figure 4. Here P, Q, R and S represent reduced fractions with R the mediant of P and Q, and S the mediant of P and R. The denominators of Q and S are even and these vertices correspond to links. The opposite is true of P and R. Finally, the numerators of P and R have opposite parity. In both cases the arrow is used to indicate that the quadrilateral is attached to the previous quadrilateral in the chain along the edge PQ.

Consider the case shown on the left side of Figure 4. We first show that the result holds for the vertex R and then use this to prove the result for the vertex S. Throughout the proof we shall use the notation $[n(A)]$ to denote the numerator of vertex A modulo 2. From Lemma 9 we have

$$2\sigma_0(R) = 2\sigma_0(P) + (-1)^{n(R)+1}2\sigma_1(Q).$$

Using our inductive hypothesis for vertex P and the fact the $n(R)$ and $n(P)$ have opposite parity we can rewrite this equation as

$$2\sigma_0(R) = m(e(P)) + (-1)^{n(P)}2\sigma_1(Q). \tag{3}$$

The unique even path $e(P)$ to P is the extension of an even path to one of its parents and there are two cases to consider depending on whether P is a parent of Q or vice versa. Suppose first that P is a parent of Q. The two even paths to Q arrive through its parents. Thus by definition we have that $e^{[n(P)]}(Q)$ is obtained from $e(P)$ by adjoining the edge PQ. Since the determinant of edge PQ is -1 we have

$$m(e^{[n(P)]}(Q)) = m(e(P)) - 1. \tag{4}$$

There are two even paths to Q and the even path to R is the extension of one of them. It cannot be the extension of $e^{[n(P)]}(Q)$ since in this case the last turning number would be -1. Hence it must be the extension of $e^{[n(R)]}(Q)$. (Here we use the fact that P and R have numerators of opposite parity). Thus

$$m(e(R)) = m(e^{[n(R)]}(Q)) + 1 \tag{5}$$

since the determinant of QR is $+1$. If instead, Q is the parent of P then it turns out that Equations 4 and 5 are still true, the proof of which is an exercise that we leave to the reader.

Thus, in any case, we see that $m(e(P)) = m(e^{[n(P)]}(Q))+1$. Substituting this into Equation 3, using the inductive hypothesis for vertex Q twice, and finally using Equation 5 we obtain:

$$\begin{aligned} 2\sigma_0(R) &= m(e^{[n(P)]}(Q)) + 1 + (-1)^{n(P)} 2\sigma_1(Q) \\ &= \sigma_0(Q) - (-1)^{n(P)}\sigma_1(Q) + 1 + (-1)^{n(P)} 2\sigma_1(Q) \\ &= \sigma_0(Q) + (-1)^{n(P)}\sigma_1(Q) + 1 \\ &= m(e^{[n(P)+1]}(Q)) + 1 \\ &= m(e(R)). \end{aligned}$$

Hence the result is true for vertex R. Now consider vertex S in the left-hand side of Figure 4. From Lemma 9 we have:

$$\begin{aligned} \sigma_0(S) &= \sigma_0(P) + \sigma_0(R) \quad \text{and} \\ \sigma_1(S) &= \sigma_1(P) + \sigma_1(R) + (-1)^{n(P)}. \end{aligned}$$

Adding these two equations and using Lemma 7, the inductive hypothesis, and the result we have already proven for vertex R, we obtain

$$\begin{aligned} \sigma_0(S) + \sigma_1(S) &= \begin{cases} 2\sigma_0(R) + 1 & \text{if } n(P) \text{ is even;} \\ 2\sigma_0(P) - 1 & \text{if } n(P) \text{ is odd.} \end{cases} \\ &= \begin{cases} m(e(R)) + 1 & \text{if } n(P) \text{ is even;} \\ m(e(P)) - 1 & \text{if } n(P) \text{ is odd.} \end{cases} \\ &= m(e^1(S)), \end{aligned}$$

since the determinant of RS is $+1$ and the determinant of PS is -1.

If instead we subtract, we obtain

$$\sigma_0(S) - \sigma_1(S) = \begin{cases} 2\sigma_0(P) - 1 & \text{if } n(P) \text{ is even;} \\ 2\sigma_0(R) + 1 & \text{if } n(P) \text{ is odd.} \end{cases}$$

$$= \begin{cases} m(e(P)) - 1 & \text{if } n(P) \text{ is even;} \\ m(e(R)) + 1 & \text{if } n(P) \text{ is odd.} \end{cases}$$
$$= m(e^0(S)).$$

The case for the right-hand side of Figure 4 is similar. □

We may now prove Theorems 1 and 2.

PROOF OF THEOREM 1. Part 1 of Theorem 1 is Proposition 2 of [7]. Part 2 now follows from Lemma 5 and part 3 follows from Proposition 10. □

PROOF OF THEOREM 2. Part 3 of Theorem 2 may be derived from [9] as follows. Lemma 3 of [9] states that the boundary slope of a diagonal surface corresponding to a minimal edge-path γ with no C-type edges is $m(\gamma)$. To express this slope with respect to a preferred longitude we must subtract $\sigma_0(p/q)$ as described at the end of Section 3 in [9]. Finally, as mentioned already, the definition of $L_{p/q}$ used in [9] is the mirror image of what is used here. Thus, the boundary slope is $-[m(\gamma) - \sigma_0(p/q)]$.

If γ contains C-type edges, then Theorem 6 of [9] applies. For diagonal surfaces, the theorem gives a boundary slope of $x - P + N$. Here $x = m(\gamma')$ where γ' is a path with no C-type edges obtained from γ by P left triangle moves and N right triangle moves. Hence, by Lemma 3, $x - P + N = m(\gamma)$. Again, we must subtract $\sigma_0(p/q)$ and negate the result.

Part 2 of Theorem 2 now follows from Proposition 10. Finally, part 1 follows from Lemma 5. □

3. Diameter and Crossing Number

If L is a link, let $D_\Delta(L)$ be the *diameter* given by the difference between the maximum and minimum (finite) slopes of diagonal surfaces in L. If in fact L is a knot this reduces to the usual notion of diameter. Finally, let $\mathrm{cr}(L)$ denote the crossing number of L. The results of the previous section now allow us to relate the diameter of either a 2-bridge knot or link to its crossing number. We do this in the following theorem, which in the case of knots was proven in [12] using different techniques.

THEOREM 11. *If L is a 2-bridge knot or link with n components, then*

$$D_\Delta(L) = \frac{2}{n}\,\mathrm{cr}(L).$$

PROOF. Suppose $L = L_{p/q}$ is any 2-bridge knot or link and $\mathcal{C}$ is the minimal chain of quadrilaterals from $\frac{1}{0}$ to $\frac{p}{q}$. Let γ_ℓ and γ_u be the lower and upper minimal

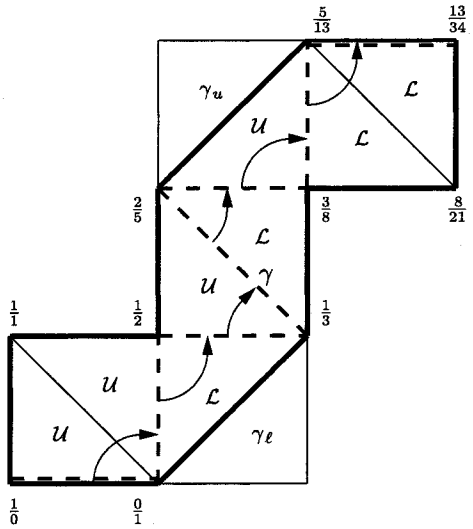

Figure 5. The path γ has turning numbers $2, -1, 1, -1, 1, -2$.

paths in $\mathcal{C}$ respectively. From Lemma 3 we see that $m(\gamma_\ell)$ and $m(\gamma_u)$ provide the extreme values of m since left triangle moves increase m by one. From Theorems 1 and 2 it follows that the diameter is

$$D_\Delta(L) = \frac{2}{n}\left[m(\gamma_u) - m(\gamma_\ell)\right].$$

It remains to show that $m(\gamma_u) - m(\gamma_\ell)$ equals the crossing number.

We illustrate the idea of the proof of this fact with the example $\frac{p}{q} = \frac{13}{34}$ shown in Figure 5.

From Lemma 4 we have that $m(\gamma_u) = i - 1$ and $m(\gamma_\ell) = -j + 1$ where i and j are the number of edges respectively in the two paths. Thus $m(\gamma_u) - m(\gamma_\ell) = i + j - 2$.

The area between γ_u and γ_ℓ is made up of triangles which, except for the first and last triangle, have one edge on one path and the opposite vertex on the other path. Label the first triangle $\mathcal{U}$, the last triangle $\mathcal{L}$ and all the intermediate triangles either $\mathcal{U}$ or $\mathcal{L}$ depending on whether or not they contain an edge on the upper or lower path. This labeling determines a unique path γ which keeps all the triangles labeled $\mathcal{U}$ on its left and all the triangles labeled $\mathcal{L}$ on its right. The path γ is shown dashed in Figure 5. Note that, as in this example, γ may not be minimal. The turning numbers for γ alternate in sign and the sum of the absolute values of these turning numbers is clearly equal to the number of triangles that have been labeled. Since each triangle contains one edge of γ_u or γ_ℓ, except for the first and last, this number equals $i + j - 2$. Finally, it is well

known that γ corresponds to the continued fraction which gives the standard alternating 4-plat diagram of the link. (See for example, Proposition 12.13 of [1].) Thus, the sum of the absolute values of the turning numbers is exactly the crossing number of L. □

4. Final Remarks

In the introduction it was mentioned that the difference between the slopes of the two checkerboard surfaces in the reduced alternating diagram of an alternating knot is equal to twice the crossing number. That is, for an alternating knot K we have $D_\Delta(K) \geq 2\,\mathrm{cr}(K)$. However, this inequality does not immediately generalize to one for alternating 2-component links. While [2] guarantees that checkerboard surfaces in reduced, alternating diagrams of non-split links are essential, they may not be diagonal. For example, one of the checkerboard surfaces for the Whitehead link has slopes of -4 and -2 on the two components. On the other hand, there are infinitely many examples of alternating links for which both checkerboard surfaces are diagonal. (For example, the three-component pretzel link (3,2,3,2,3,2) has diagonal checkerboard surfaces with slopes -2 and 8, and the three-component Montesinos link $K(1/2, 13/17, 1/2, 3/5, 1/10, 3/5)$ has diagonal checkerboard surfaces with slopes -10 and 10). For alternating links where both checkerboard surfaces are diagonal we have the following result.

PROPOSITION 12. *Let L be a non-split alternating link of n components, and assume that both checkerboard surfaces in a reduced alternating diagram of L are diagonal. Then*

$$D_\Delta(L) \geq \frac{2}{n}\,\mathrm{cr}(L).$$

The proof of this proposition makes use of the following lemma.

LEMMA 13. *Let L be a non-split alternating link of n components. Let S and T be the two checkerboard surfaces in a reduced alternating diagram of L. If s_i and t_i are the boundary slopes of S and T respectively on the the i-th component, then*

$$\left|\sum_{i=1}^{n}(s_i - t_i)\right| = 2\,\mathrm{cr}(L).$$

PROOF. By [2] both S and T are essential surfaces. The surface S is a collection of non-nested planar disks connected to each other by twisted bands as shown in Figure 6. By examining any such disk of S, it is easy to see that all

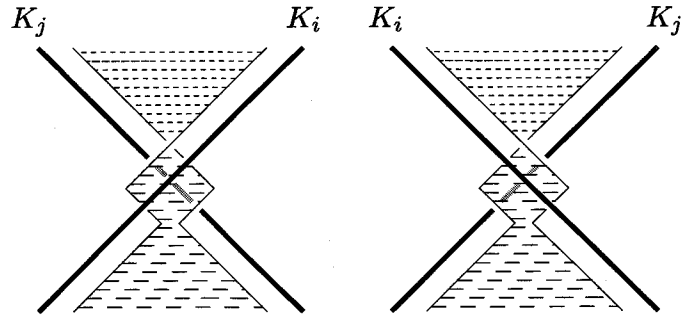

Figure 6. Right and left twisted bands in a checkerboard surface.

of the twisted bands are the same handedness because L is alternating. Furthermore, since the disks of T are the complementary planar regions of the disks of S, it follows that the bands of T have the opposite handedness. Without loss of generality, assume that S has only right twisted bands while T has only left twisted bands. From Figure 6 the contribution to the boundary slope on component i is easily computed. The following table summarizes these contributions.

band type	crossing type	contribution
right twisted	$i \neq j$	$+1$
	$i = j$, positive	$+2$
	$i = j$, negative	0
left twisted	$i \neq j$	-1
	$i = j$, positive	0
	$i = j$, negative	-2

For $1 \leq i \leq n$, let α_i be the number of crossings in the reduced diagram where component i passes over a different component. For self-crossings, let P_i and N_i be the number of positive and negative self-crossings respectively for component i. Using the table we see that $s_i = \alpha_i + 2P_i$ and $t_i = -\alpha_i - 2N_i$. Therefore,

$$\sum_{i=1}^{n}(s_i - t_i) = 2\sum_{i=1}^{n}(\alpha_i + P_i + N_i) = 2\,\mathrm{cr}(L). \qquad \square$$

The proof of Proposition 12 is now simple. Since both S and T are diagonal, $s_i = s_j = s$ and $t_i = t_j = t$ for all $1 \leq i, j \leq n$. Therefore, by Lemma 13 we have $n|s - t| = 2\,\mathrm{cr}(L)$ and the result follows.

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