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ON ABSTRACT FIRST ORDER DIFFERENTIAL EQUATION WITH A NONLOCAL INTEGRAL CONDITION

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Abstract

An abstract first order linear differential equation with unbounded operator generating an analytic C_0 semigroup is considered in a Banach space. For the determination of the initial state, a special nonlocal condition involving an integral of $u(t)$ and an integral of its derivative $u'(t)$ is given. We obtain necessary and sufficient conditions for the unique solvability and well-posedness. We also study the Fredholm solvability. Finally, we obtain a result of the stability of the solution with respect to small perturbation.

1. Introduction

Let E denote a Banach space with norm $\|\,.\,\|$; $\mathcal{L}(E)$ is the Banach algebra of bounded linear operators, mapping E into E, endowed with the usual operator norm; I is the identity operator in $\mathcal{L}(E)$; A is the generator of an analytic C_0 semigroup $U(t)$, acting in the space E; $D(A)$ is the domain of definition of an operator A; $|||\,.\,|||$ is the graph norm on $D(A)$, i.e., $|||\Psi||| = \|\Psi\| + \|A\Psi\|$ for $\Psi \in D(A)$; $\mathbf{D}(A)$ is the Banach space $D(A)$ with the graph norm; $\rho(A)$, $\sigma(A)$ are the resolvent set and the spectrum of an operator A.

A function taking real or complex values, depending on whether the space E is real or complex, will be called a scalar function. A vector function is a function with values in some Banach space. By an operator function we mean a function with values in the Banach algebra $\mathcal{L}(E)$. In the Banach space E we consider the differential equation

$$\frac{du(t)}{dt} = Au(t), \quad 0 \leq t \leq T \tag{1}$$

where A is a closed operator from E into E with dense domain $D(A)$ that generates an analytic C_0 semi group $U(t)$, $0 \leq t \leq T$.

DEFINITION 1. The vector function $u(t) = U(t)f$, $0 \leq t \leq T$, corresponding to some element $f \in E$ is called a *generalized solution* of equation (1). If, in addition, $f \in D(A)$, then the solution $u(t) = U(t)f$ is said to be *classical.*

REMARK 1. In the case, when $f \in D(A)$ obviously, f coincides with the initial state $u(0)$ of the corresponding solution $u(t)$.

Suppose that the initial state f is unknown, and consider the additional relation

$$\int_0^T w_1(t)u(t)\,dt + \int_0^T w_2(t)\frac{du(t)}{dt}\,dt = g \tag{2}$$

with a given element $g \in E$. Here $w_1(t)$ is a scalar measurable function of bounded variation on the segment $[0,T]$, and $w_2(t) \in C^1[0,T]$. Throughout this paper we asume that $w_2(0) = 0$.

Then the integral occurring in (2) is well-defined in the sense of Bochner for any function $u(t) = U(t)f$.

DEFINITION 2. A generalized solution of the problem (1), (2) is defined to be a function $u(t) = U(t)f$, $0 \leq t \leq T$, corresponding to some element $f \in E$ and the reducing relation (2) to a valid identity. If, in addition, $f \in D(A)$, then the corresponding solution $u(t) = U(t)f$ of the problem (1), (2) is called a *classical solution.*

The present paper is devoted to the study of an abstract first order differential equation with an integral boundary condition. The study of this type of boundary value problems goes back to [29]. Such problems belong to the class of inverse problems which have been studied intensively in recent times in connection with their great practical significance [1], [9], [13], [14], [16], [18], [21], [22], [23]. The connection between inverse problems with final and integral condition was shown in [28]. The inverse problems for partial differential equations were considered in very diverse formulations by many authors see [9], [16], [18], [21], [22], [23]. Results for abstract inverse problems were first obtained in [13], [24]. More general nonlocal conditions for different types of partial differential equations were considered later in [4], [5], [6], [7], [8], [10], [11], [17], [19], [25], [26], [27]. In this paper we obtain necessary and sufficient conditions for the unique solvability and well-posedness of the problem (1), (2). We also study the Fredholm solvability. Finally, we obtain a result of the stability of the solution with respect to a small perturbation.

2. Operator equation

It is clear that the solution of the equation (1) is representable in the form $u(t) = U(t)f$. Therefore, the function $u(t) = U(t)f$ satisfies the condition (2) if and only if f satisfies the equation

$$\int_0^T w_1(t)U(t)f\,dt + \int_0^T w_2(t)\frac{dU(t)f}{dt}\,dt = g.$$

Using integration by parts in the sense of Stieltjes and taking into account the assumption on the function $w_2(t)$, we obtain

$$w_2(T)U(T)f + \int_0^T (w_1(t) - w_2'(t))U(t)f\,dt = g. \tag{3}$$

So, for $f \in E$, we have the operator equation $Bf = g$, where

$$Bf = w_2(T)U(T)f + \int_0^T (w_1(t) - w_2'(t))U(t)f\,dt.$$

LEMMA 1. *The operator B maps E into $D(A)$.*

PROOF. If $f \in D(A)$, then $u(t) = U(t)f$ is a classical solution of the equation (1) and $Af \in E$, $U(t)Af = AU(t)f = \frac{d}{dt}(U(t)f)$. Therefore,

$$\begin{aligned} BAf &= w_2(T)U(T)Af + \int_0^T (w_1(t) - w_2'(t))U(t)Af\,dt \\ &= w_2(T)AU(T)f + \int_0^T (w_1(t) - w_2'(t))AU(t)f\,dt. \end{aligned}$$

Since A is a closed operator, it follows that

$$\begin{aligned} &w_2(T)AU(T)f + \int_0^T (w_1(t) - w_2'(t))AU(t)f\,dt \\ &= A\left(w_2(T)U(T)f + \int_0^T (w_1(t) - w_2'(t))U(t)f\,dt\right) \\ &= ABf. \end{aligned}$$

Hence $Bf \in D(A)$, and

$$\begin{aligned} ABf &= w_2(T)AU(T)f + \int_0^T (w_1(t) - w_2'(t))\frac{d}{dt}(U(t)f)\,dt \\ &= w_2(T)AU(T)f + \int_0^T (w_1(t) - w_2'(t))\,d(U(t)f). \end{aligned}$$

Using integration by part in the sense of Stieltjes, we obtain

$$\begin{aligned} & w_2(T)AU(T)f + \int_0^T (w_1(t) - w_2'(t))\, d(U(t)f) \\ & = w_2(T)AU(T)f + (w_1(T) - w_2'(T))U(T)f - (w_1(0) - w_2'(0))f \\ & \quad - \int_0^T U(t)f\, d(w_1(t) - w_2'(t)). \end{aligned}$$

But if f is an arbitrary element from E, there exists a sequence $\{f_n\} \subset D(A)$ converging to f. Since B is a bounded operator, then $Bf_n \to Bf$ and

$$\begin{aligned} ABf_n &= w_2(T)AU(T)f_n + (w_1(T) - w_2'(T))U(T)f_n - (w_1(0) - w_2'(0))f_n \\ &\quad - \int_0^T U(t)f_n\, d(w_1(t) - w_2'(t)) \\ &\to w_2(T)AU(T)f + (w_1(T) - w_2'(T))U(T)f - (w_1(0) - w_2'(0))f \\ &\quad - \int_0^T U(t)f\, d(w_1(t) - w_2'(t)), \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since A is a closed operator, it follows that $Bf \in D(A)$ and

$$\begin{aligned} ABf &= w_2(T)AU(T)f + (w_1(T) - w_2'(T))U(T)f - (w_1(0) - w_2'(0))f \\ &\quad - \int_0^T U(t)f\, d(w_1(t) - w_2'(t)). \end{aligned} \tag{4}$$

So, for any $f \in E$ we conclude that $Bf \in D(A)$. □

REMARK 2. If $g \in E \setminus D(A)$, then the problem (1), (2) is unsolvable (in the sense of Definition 2).

Now both sides of the equation (3) belong to $D(A)$, anda consequently the operator $(\lambda I - A)$, where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), can be applied to them. Assume additionally that $\lambda \in \rho(A)$, this implies that $(\lambda I - A)^{-1}$ exists. Then from (4) we obtain the following equation for the element f

$$Uf = Jf - Kf + \lambda Lf = \Psi. \tag{5}$$

Here J, K, L are bounded linear operators defined by the formulas

$$Jf = (w_1(0) - w_2'(0))I - w_2(T)AU(T), \tag{6}$$

$$Kf = (w_1(T) - w_2'(T))U(T) + \int_0^T U(t)\, d(-(w_1(t) - w_2'(t)), \tag{7}$$

$$L = w_2(T)U(T) + \int_0^T (w_1(t) - w_2'(t))U(t)\, dt. \tag{8}$$

(The integrals of operator-valued functions are defined in the strong topology of $\mathcal{L}(E)$.) The element $\Psi \in E$ in the equation (5) has the form

$$\Psi = (\lambda I - A)g. \tag{9}$$

THEOREM 1. *The function $u(t) = U(t)f$ is a solution of the problem* (1), (2) *if and only if the element f satisfies the equation* (5), *i.e., the equation $Uf = \Psi$, where $U = J - K + \lambda L$. Here J, K, L are the operators defined by the formulas* (6), (7), (8), *respectively, $\lambda \in \rho(A)$, and the element Ψ is defined by* (9). *In addition, the problem* (1), (2) *is uniquely solvable for any g in $D(A)$ if and only if $U^{-1} \in \mathcal{L}(E)$.*

REMARK 3. There is an obvious arbitrariness in the definition of the operator U and the element Ψ, connected with the choice of λ in $\rho(A)$. Since this choice does not play a crucial role in the reasoning below, we shall consider λ fixed henceforth, and the operator U and the element Ψ, thus, uniquely determined.

PROOF. The equivalence of the problem (1), (2) and the equation (5) follows from the derivation of this equation. Let us show that for the unique solvability of the problem (1), (2) it is necessary and sufficient that $U^{-1} \in \mathcal{L}(E)$, the sufficiency is obvious. Indeed, if $U^{-1} \in \mathcal{L}(E)$, then the function $u(t) = U(t)f$, where $f = U^{-1}\Psi$, and Ψ is defined by (9), will be the unique solution of the problem (1), (2).

Necessity: Suppose that the problem (1), (2) is uniquely solvable for any $g \in D(A)$. Then for any Ψ defined by (9) there exists a unique solution of (5). But if g runs through $D(A)$, then the element Ψ runs through the whole of E. Therefore, (5) has a unique solution f for any Ψ in E, i.e., the inverse operator $U^{-1}\colon E \to E$ exists, since the operator U is bounded, by Banach's theorem $U^{-1} \in \mathcal{L}(E)$, which was to be proved. □

3. Well-posedness of problem (1), (2)

The operator equation (5) is the main instrument in investigating the problem (1), (2). In particular, with its help the following theorem is obtained.

THEOREM 2. *Suppose that the problem* (1), (2) *is uniquely solvable for any $g \in D(A)$. Then its solution $u(t) = U(t)f$, corresponding to the element g satisfies the estimation*

$$\|u(t)\| \leq C|||g||| \tag{10}$$

with a constant $C > 0$ independent of g.

PROOF. Since the problem (1), (2) is uniquely solvable, by Theorem 1 its solution is representable in the form $u(t) = U(t)f$, where $f = U^{-1}\Psi$, $U^{-1} \in \mathcal{L}(E)$, and the element Ψ is given by the formula (9). Therefore, $\|f\| \leq \|U^{-1}\| \, \|\Psi\|$. From this, taking the explicit form of Ψ and $u(t)$ into account, the estimate (10) follows immediately. □

We shall say that the problem (1), (2) is well posed, if it has a unique solution $u(t) = U(t)f$, where $f \in E$ for any $g \in D(A)$. By Theorem 2, this solution satisfies the estimation (10) with a constant $C > 0$ independent of g.

Now we formulate the main definition.

DEFINITION 3. Such a constant C will be called a *well-posedness constant* of the problem (1), (2).

In other words, the problem of finding f (the vector function $u(t) = U(t)f$) for given g, well posed in the sense of Definition 3, is well posed in the sense of Hadamard on the pair of Banach spaces $(E, \mathbf{D}(A))$. The basic, although somewhat formal, well-posedness criterion is given by Theorem 1: The problem (1), (2) is well posed if and only if $U^{-1} \in \mathcal{L}(E)$.

In the sequel we constantly use this fact, without special mention.

4. Fredholm property of problem (1), (2)

Let us consider how the uniqueness of solution and the well-posedness of the problem (1), (2) are interrelated. We start with the following definition.

DEFINITION 4. The kernel of the problem (1), (2) is the set of element $u(t) = U(t)f$, $f \in E$ that are solutions of homogeneous condition (2) (i.e., when $g = 0$). If this set consists of the zero element alone, then the kernel of the problem (1), (2) is said to be *trivial.*

It is easy to see that the kernel of the problem (1), (2) coincides with the kernel of operator U defined in Theorem 1, and so it is a closed linear subspace of the space E. Further, if $u_1(t) = U(t)f_1$, $u_2(t) = U(t)f_2$ are solutions of the problem (1), (2) corresponding to one and the same g, then their difference $u(t) = u_1(t) - u_2(t)$ belongs to the kernel of this problem. Thus, for the problem (1), (2) to be a uniquely solvable it is necessary and sufficient that its kernel is trivial; in particular, the kernel of a well-posed problem is, clearly, trivial. It turns out that the converse is also possible, when the triviality of the kernel implies the well-posedness of the problem (1), (2).

THEOREM 3. *Let the semigroup $U(t)$ be compact for $t > 0$, and let the function $w_1(t) - w_2'(t)$ be continuous on the right at $t = 0$ and $J^{-1} \in \mathcal{L}(E)$. Then*

1. *the kernel of the problem* (1), (2) *is of finite dimension;*
2. *if the kernel of the problem* (1), (2) *is trivial, then this problem is well posed.*

The proof of Theorem 3 is based on the following lemma.

LEMMA 2. *Let the semigroup $U(t)$ be compact for $t > 0$, and let the function $w_1(t) - w_2'(t)$ be continuous on the right at $t = 0$. Let $J^{-1} \in \mathcal{L}(E)$. Then the operators K and L defined by the formulas* (6) *and* (7) *are compact.*

PROOF. First, we show the compactness of L. Let $\varepsilon > 0$, and

$$L_\varepsilon = w_2(T)U(T)f + \int_\varepsilon^T (w_1(t) - w_2'(t))U(t)f\,dt.$$

Since $U(t)$ is a compact operator for every $t > 0$, L_ε is compact. But the estimate

$$\begin{aligned} \|Lf - L_\varepsilon f\| &= \left\| \int_0^\varepsilon (w_1(t) - w_2'(t))U(t)f\,dt \right\| \\ &\le \varepsilon \left(\sup_{0\le t\le\varepsilon} \|U(t)f\| \sup_{0\le t\le T} |w_1(t) - w_2'(t)| \right) \\ &\le \varepsilon \left(\sup_{0\le t\le\varepsilon} \|U(t)\| \sup_{0\le t\le T} |w_1(t) - w_2'(t)| \right) \|f\| \end{aligned}$$

holds for any $f \in E$; moreover,

$$\varepsilon \left(\sup_{0\le t\le\varepsilon} \|U(t)\| \sup_{0\le t\le T} |w_1(t) - w_2'(t)| \right) \|f\| \to 0 \quad \text{as} \quad \varepsilon \to 0^+$$

and therefore L is compact as a uniform limit of compacts operators. The compactness of the operator

$$K = (w_1(T) - w_2'(T))U(T) + \int_0^T U(t)\,d(-(w_1(t) - w_2'(t)))$$

is proved analogously. The lemma is proved. □

PROOF OF THEOREM 3. Let us associate the operator $U = J - K + \lambda L$ with the problem (1), (2). Since $J^{-1} \in \mathcal{L}(E)$ (by assumption) and the operators K and L are compact, the well-known Fredholm theorems are valid for U. In

particular, the operator U has a finite-dimensional kernel, the triviality of which is equivalent to $U^{-1} \in \mathcal{L}(E)$. But this is exactly the claim of our theorem, since the kernel of U coincides with the kernel of the problem (1), (2). □

REMARK 4. In essence, in Theorem 3 it is shown that the operator U corresponding to the problem (1), (2) is Fredholm of index zero. In such situations it is customary to say that the problem (1), (2) itself is *Fredholm of index zero.*

5. Well-posedness conditions

In this section we give examples of conditions guaranteeing the well-posedness of the problem (1), (2).

THEOREM 4. *Let $w_1(t)$, $w_2(t)$ be functions of bounded variation on $[0,T]$ such that $w_2(t) \in C^1[0,T]$; $w_2(0) = w_2(T) = 0$; $w_1(t) - w_2'(t)$ nonnegative non increasing; right continuous at $t = 0$ such that $w_1(0) - w_2'(0) > 0$ and let the semigroup $U(t)$ generated by the operator A satisfy the estimate $\|U(t)\| \leq M\exp(-\beta t)$ with constants $M \geq 1$, $\beta > 0$. Then the problem* (1), (2) *is well posed (in the sense of* Definition 3).

PROOF. Consider the operator equation (5) in the given case it is of the form $((w_1(0) - w_2'(0))I - K)f = \Psi$ (where the term λL in (5) may be dropped, since the spectrum $\sigma(A)$ is contained in the half-plane $\{\lambda\colon \operatorname{Re}\lambda \leq -\beta < 0\}$). It suffices to show that $\|K\|_1 < w_1(0) - w_2'(0)$ in some norm $\|\,.\,\|_1$ equivalent to $\|\,.\,\|$. Indeed, then $((w_1(0) - w_2'(0))I - K)^{-1} \in \mathcal{L}(E)$, and by Theorem 1, the problem (1), (2) is well posed. We defined the equivalent norm $\|\,.\,\|_1$ in the space E by the formula $\|h\|_1 = \sup_{t\geq 0}\|U(t)\exp(\beta t)\|$ $(h \in E)$. It is known that $\|U(t)\|_1 \leq \exp(-\beta t)$ (see [12]).

Thus, for this equivalent norm, we have

$$\begin{aligned}\|K\|_1 &= \left\|(w_1(T) - w_2'(T))U(T) + \int_0^T U(t)\,d(-(w_1(t) - w_2'(t))\right\|_1 \\ &\leq \|(w_1(T) - w_2'(T))U(T)\|_1 + \left\|\int_0^T U(t)\,d(-(w_1(t) - w_2'(t))\right\|_1 \\ &\leq (w_1(T) - w_2'(T))\exp(-\beta T) + \int_0^T \exp(-\beta t)\,d(-(w_1(t) - w_2'(t))).\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}\|K\|_1 &\le (w_1(T) - w_2'(T))\exp(-\beta T) - (w_1(T) - w_2'(T))\exp(-\beta T)\\ &\quad + (w_1(0) - w_2'(0)) - \beta\int_0^T \exp(-\beta t)(w_1(t) - w_2'(t))\,dt\\ &= (w_1(0) - w_2'(0)) - \beta\int_0^T \exp(-\beta t)(w_1(t) - w_2'(t))\,dt.\end{aligned}$$

Since $\beta\int_0^T \exp(-\beta t)(w_1(t) - w_2'(t))\,dt$ is positive under the assumptions of the theorem, it follows that

$$(w_1(0) - w_2'(0)) - \beta\int_0^T \exp(-\beta t)(w_1(t) - w_2'(t))\,dt < w_1(0) - w_2'(0).$$

Consequently the problem (1), (2) is well posed, and the theorem is proved. □

REMARK 5. In the case when $T = +\infty$ and $w_2(t) \equiv 0$ this result coincides with [29].

Up to the end of this section we assume that E is a Banach lattice with cone E_+ (see [2]). If $L \in \mathcal{L}(E)$ is a positive operator, i.e., $L(E_+) \subset E_+$, we shall write $L \ge 0$. A semigroup $U(t)$ is said to be positive if $U(t) \ge 0$ for all $t \ge 0$.

THEOREM 5. *Let $w_1(t)$, $w_2(t)$ be functions of bounded variation on $[0,T]$ such that $w_2(t) \in C^1[0,T]$; $w_2(0) = 0$, $w_2(T) > 0$; $w_1(t) - w_2'(t)$ nonnegative non increasing; right continuous at $t = 0$ such that $w_1(0) - w_2'(0) > 0$. Let $J^{-1} \in \mathcal{L}(E)$, $J^{-1} \ge 0$. Let the semigroup $U(t)$ generated by the operator A be positive and compact for $t > 0$. Finaly, assume that the spectrum of A lies in the half-plane $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda < 0\}$. Then the problem* (1), (2) *is well posed (in the sense of* Definition 3).

REMARK 6. Since the semi group $U(t)$ is positive, the requirement on the spectrum $\sigma(A)$ means that $s(A) < 0$, where $s(A) \equiv \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\}$ is the spectral boundary of A. This implies that, in particular, that the resolvent $(\lambda I - A)^{-1}$ exists for $\lambda = 0$ and is positive operator. Then the term λL in (5) may be dropped. Hence the equation (5) becomes $U = J - K$, where

$$J = (w_1(0) - w_2'(0))I - w_2(T)AU(T),$$

$$K = (w_1(T) - w_2'(T))U(T) + \int_0^T U(t)\,d(-(w_1(t) - w_2'(t))).$$

PROOF OF THEOREM 5. First, we prove a uniqueness theorem for the problem (1), (2), and then, combining it with Theorem 3, we obtain an assertion on

the well-posedness of the problem (1), (2). We fix the element $u(t) = U(t)f$ in the kernel of the problem (1), (2) and prove that $u(t) = 0$, i.e., $f = 0$. Using the fact that the space E is a lattice, we present f in the form $f = f^+ - f^-$, where $f^+ = \sup(f, 0)$, $f^- = \inf(-f, 0)$. Clearly, $f^+ \geq 0$, $f^- \geq 0$ and $\inf(f^+, f^-) = 0$. Further, it is obvious that the function $u(t) = U(t)f$ satisfies the condition

$$\begin{aligned} &w_2(T)U(T)f^+ + \int_0^T (w_1(t) - w_2'(t))U(t)f^+ \, dt \\ &= w_2(T)U(T)f^- + \int_0^T (w_1(t) - w_2'(t))U(t)f^- \, dt. \end{aligned} \tag{11}$$

Let φ denote the element of E equal to either of the relations in (11). So, applying the operator A to (11) we obtain

$$A\varphi + Jf^+ = Kf^+, \quad A\varphi + Jf^- = Kf^-, \tag{12}$$

where the operators J, K are defined in (6), (7) respectively. Since the semigroup $U(t)$ is positive, K is positive operator, then, $A\varphi + Jf^+ \geq 0$, $A\varphi + Jf^- \geq 0$, and, consequently,

$$-A\varphi \leq \inf(Jf^+, Jf^-). \tag{13}$$

Put $x \equiv \inf(Jf^+, Jf^-)$. Then $x \geq 0$, since $J \equiv (w_1(0) - w_2'(0))I - w_2(T)AU(T) \geq 0$. On the other hand, since $J^{-1} \geq 0$ (by assumption), the inequalities $Jf^+ \geq x$, $Jf^- \geq x$ imply $J^{-1}x \leq \inf(f^+, f^-) = 0$. Thus $x \leq 0$ and, automatically, $x = 0$. Hence from (13), we conclude that $A\varphi \geq 0$. From Remark 7 we deduce that $\varphi \leq 0$. At the same time, by the definition of φ (see the formula (11)) we have

$$\varphi = w_2(T)U(T)f^+ + \int_0^T (w_1(t) - w_2'(t))U(t)f^+ \, dt \geq 0.$$

Thus, we have shown that $\varphi = 0$, i.e., $w_2(T)U(T)f^+ = 0$ and $\int_0^T (w_1(t) - w_2'(t))U(t)f^+ \, dt = 0$, hence $U(T)f^+ = 0$. From this and (12), it follows that $Kf^+ = 0$, and so $Jf^+ = 0$, which implies that $f^+ = 0$, since J is invertible. Similarly, $f^- = 0$, and so $f = f^+ - f = 0$. Now using the compacteness of $U(t)$ and Theorem 3 we obtain the proof. Consequently the problem (1), (2) is well posed, and the theorem is proved. □

6. Perturbed problem

The remaining part of the paper will be devoted to the question of small perturbations of the problem (1), (2). We consider the case when the equation

(1) is perturbed. Thus, suppose that along with the equation (1) there is the equation

$$\frac{du(t)}{dt} = (A+B)u(t), \quad 0 \le t \le T, \tag{14}$$

in this case, we have a problem (14), (2) instead of the problem (1), (2). Here $B \in \mathcal{L}(E)$. We shall also write $\tilde{A} = A+B$. Since B is a bounded linear operator, the operator $\tilde{A}$ with $D(\tilde{A}) = D(A)$ is the generator of some analytic C_0 semigroup $\tilde{U}(t)$ (see [20]).

THEOREM 6. *Let the problem* (1), (2) *be well posed. Then there exists a number* $\delta > 0$, *depend only on* A *and* $w_1(t)$, $w_2(t)$ *such that if* $\|B\| \le \delta$ *for* $0 \le t \le T$ *and* $w_2(T) = 0$, *then the problem* (14), (2) *is also well posed. For any* $\varepsilon > 0$ *the indicated* δ *can be chosen such that*

$$\|\tilde{u}(t) - u(t)\| \le \varepsilon |||g|||, \tag{15}$$

where $\tilde{u}(t) = \tilde{U}(t)\tilde{f}$, $u(t) = U(t)f$ *are solutions of the problems* (1), (2); (14), (2), *respectively, corresponding to given* $g \in D(A)$.

REMARK 7. The estimate (15) means that $\|\tilde{u}(t) - u(t)\| \to 0$ as $\|B\| \to 0$. Thus, the solutions of well-posed problem (1), (2) are stable under small perturbations of (1).

PROOF. Without loss of generality we may suppose that $\delta \le 1$. We choose constants M and α ($M \ge 1$, $\alpha \in \mathbb{R}$) such that the estimate $\|U(t)\| \le M \exp(\alpha t)$ hold for the semigroup $U(t)$. In this case the following relations hold for the perturbed semigroup $\tilde{U}(t)$ [20]:

$$\|\tilde{U}(t)\| \le M \exp((\alpha + M\|B\|)t),$$
$$\|\tilde{U}(t) - U(t)\| \le M \exp(\alpha t)[\exp(M\|B\|t) - 1].$$

Since $\|B\| \le \delta \le 1$, for $0 \le t \le T$ we have

$$\|\tilde{U}(t)\| \le M_1, \quad \|\tilde{U}(t) - U(t)\| \le M_2(\exp(MT\delta) - 1). \tag{16}$$

(The symbols $M_1, M_2, \ldots$ denote constants dependent only on the data of the problem (1), (2), i.e., on A, $w_1(t)$, $w_2(t)$.)

We consider the equation (5) for the problems (1), (2) and (14), (2). Let $U = J - K + \lambda L$, and $\tilde{U} = \tilde{J} - \tilde{K} + \lambda \tilde{L}$ be the corresponding operators. (The operators $\tilde{J}$, $\tilde{K}$, $\tilde{L}$ are defined by the formulas (6), (7), (8) with the replacement of $U(t)$ by $\tilde{U}(t)$). The number λ in the operators U and $\tilde{U}$ is chosen the same.

Since $s(A) \leq \alpha$ and $s(A+B) \leq \alpha + M\|B\| \leq \alpha + M$, we may assume that, for example, $\lambda = \alpha + M + 1$. Using the explicit form of the operators U and $\tilde{U}$, the estimate (16), it is not difficult to show that

$$\|\tilde{U} - U\| \leq M_3((\exp(MT\delta) - 1)).$$

Consequently, $\|\tilde{U} - U\|$ can be made as small as desired, provided that δ is sufficiently small. By the hypothesis of the theorem, the problem (1), (2) is well posed, and thus, $U^{-1} \in \mathcal{L}(E)$. Choosing δ such that $\|\tilde{U} - U\| \leq \|U^{-1}\|^{-1}$, we get that $\tilde{U}^{-1} \in \mathcal{L}(E)$. But this is equivalent to well-posedness of the problem (14), (2).

To prove the estimate (15), notice that $f = U^{-1}\Psi$, $\tilde{f} = \tilde{U}^{-1}\tilde{\Psi}$, where Ψ, and $\tilde{\Psi}$ are defined by the formula (9) (with the obvious proviso in the case of the problem (14), (2)). Using the explicit form of $\tilde{\Psi}$, Ψ and relation $\|B\| \leq \delta$, we get

$$\|\tilde{\Psi} - \Psi\| \leq \delta\|g\|,$$

and, consequently

$$\|\tilde{\Psi}\| \leq \|\Psi\| + \|\tilde{\Psi} - \Psi\| \leq M_4|||g|||.$$

But then

$$\begin{aligned}\|\tilde{f} - f\| &\leq \|\tilde{U}^{-1} - U^{-1}\|\,\|\tilde{\Psi}\| + \|U^{-1}\|\,\|\tilde{\Psi} - \Psi\| \\ &\leq M_4\|\tilde{U}^{-1} - U^{-1}\|\,|||g||| + \delta\|U^{-1}\|\,\|g\| \\ &\leq M_5[\|\tilde{U}^{-1} - U^{-1}\| + \delta]|||g|||.\end{aligned}$$

Since

$$\|\tilde{U} - U\| \to 0 \quad \text{as} \quad \delta \to 0$$

and the operation of passage to the inverse operator is continuous in the norm of $\mathcal{L}(E)$,

$$\|\tilde{U}^{-1} - U^{-1}\| \to 0 \quad \text{as} \quad \delta \to 0.$$

Now the estimate (15) is obvious. Indeed,

$$\begin{aligned}\|\tilde{u}(t) - u(t)\| &= \|\tilde{U}(t)\tilde{f} - U(t)f + \tilde{U}(t)f - \tilde{U}(t)f\| \\ &= \|\tilde{U}(t)(\tilde{f} - f) + (\tilde{U}(t) - U(t))f\| \\ &\leq \|\tilde{U}(t)\|\,\|\tilde{f} - f\| + \|\tilde{U}(t) - U(t)\|\,\|f\| \\ &\leq M_6[\|\tilde{U}^{-1} - U^{-1}\| + \delta]|||g||| + CM_2(\exp(MT\delta) - 1)|||g|||.\end{aligned}$$

The theorem is proved. □

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