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FIBERED LINKS OF GENUS ZERO WHOSE MONODROMY IS THE IDENTITY

By Taizo KANENOBU

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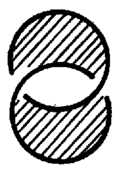
A link L in a 3-manifold M is fibered if $M-L$ is the total space of a fiber bundle over S^1 whose fiber F is well-behaved near L. This means the closure of F is a Seifert surface for L. Thus $M-L$ is homeomorphic to $F \times I$ with ends glued by a homeomorphism of F, which we call the monodromy of L. The purpose of this paper is to classify all the fibered links of genus zero in S^3 whose monodromy is the identity.

Burde and Zieschang [2] and González-Acuña [3] showed that the only fibered knots of genus one are the trefoil and figure-eight knots. If the number of the boundary component of F is less than or equal to 3, then any homeomorphism of F is isotopic to the identity ([1]), so we have

COROLLARY 1. *The only 2-component fibered link of genus zero in S^3 is the Hopf link of* Fig. 1 *with fiber surface the twisted annulus drawn.*



A_0^2

Fig. 1

COROLLARY 2. *The only 3-component fibered links of genus zero in S^3 are the links of* Fig. 2, *with fiber surfaces as drawn.*

In §1, we give a method for describing a Seifert surface of a link in a 3-manifold using surgery presentation. In §2, we construct every fibered link of genus zero whose monodromy is the identity. In §3, we prove the main theorem.

Professor Y. Matsumoto has independently obtained Corollary 2 using the same method as ours.

1. Surgery on links. Let M be a homology 3-sphere and let $L=L_1 \cup \cdots \cup L_n$

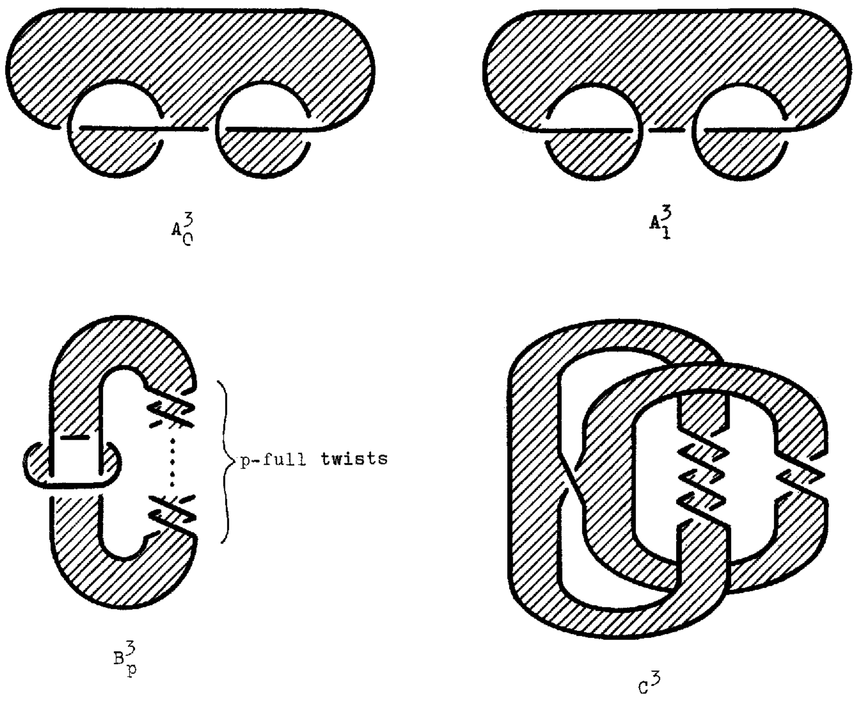


Fig. 2

be a link of oriented simple closed curves in M. Let N_i be disjoint closed tubular neighborhoods of L_i in M with preferred framings, that is, the identifications f_i: $S^1 \times D^2 \to N_i$ such that $f_i(S^1 \times 0) = L_i$ and $f_i(S^1 \times 1)$ is homologically trivial in $M - L$, where D^2 is the disc of R^2 defined by $|x| \leqq 1$ and $S^1 = \partial D^2$. We suppose that the longitude $f_i(S^1 \times 1)$ is oriented in the same way as L_i and the meridian $f_i(1 \times \partial D^2)$ has linking number $+1$ with L_i. Let μ_i and λ_i denote the homology classes of the longitude and meridian respectively in $H_1(\partial N_i)$.

We construct a 3-manifold $M' = (M - (\mathring{N}_1 \cup \cdots \cup \mathring{N}_n)) \cup_h (N_1 \cup \cdots \cup N_n)$ where $\mathring{N}_i$ is the interior of N_i and h is a union of homeomorphisms h_i: $\partial N_i \to \partial N_i \subset M$ such that the induced map h_{i*}: $H_1(\partial N_i) \to H_1(\partial N_i)$ satisfies $h_{i*}(\mu_i) = \mu'_i = a_i \mu_i + b_i \lambda_i$, $h_{i*}(\lambda_i) = \lambda'_i = c_i \mu_i + d_i \lambda_i$, and $a_i d_i - b_i c_i = 1$.

If we further assume M' is a homology 3-sphere, we can choose $\{\mu_1, \dots, \mu_n\}$ and $\{\mu'_1, \dots, \mu'_n\}$ for free bases of $H_1(M - L)$ and $H_1(M' - L)$ respectively. Let v_{ij} and v'_{ij} be the linking numbers of L_i and L_j in M and in M' respectively, and let $v_{ii} = v'_{ii} = 0$. Note that $v_{ij} = v_{ji}$ and $v'_{ij} = v'_{ji}$. Then we have $\lambda_i = \sum_{j=1}^n v_{ij} \mu_j$ and $\lambda'_i = \sum_{j=1}^n v'_{ij} \mu'_j$. Let $r_{ij} = a_i \delta_{ij} + b_i v_{ij}$ and $s_{ij} = d_i \delta_{ij} - b_i v'_{ij}$, where δ_{ij} is Kronecker's delta, and let $R = (r_{ij})$ and $S = (s_{ij})$ be $n \times n$ matrices. Then we have the following propositions.

PROPOSITION 1. $\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = R \begin{bmatrix} \mu'_1 \\ \vdots \\ \mu'_n \end{bmatrix}$ *and* $S = R^{-1}$.

PROPOSITION 2. *Let $\Delta(x_1,\dots, x_n)$ and $\Delta'(x_1,\dots, x_n)$ be the Alexander polynomials of L in M and L in M′ respectively, where x_i corresponds to μ_i or μ'_i. Then*

$$\Delta'(x_1,\dots, x_n) = \Delta(x_1^{s_{11}} x_2^{s_{12}} \cdots x_n^{s_{1n}},\dots, x_1^{s_{n1}} x_2^{s_{n2}} \cdots x_n^{s_{nn}})$$

and

$$\Delta(x_1,\dots,x_n) = \Delta'(x_1^{r_{11}} x_2^{r_{12}} \cdots x_n^{r_{1n}},\dots, x_1^{r_{n1}} x_2^{r_{n2}} \cdots x_n^{r_{nn}}).$$

Since the complements of L in M and L in M' are homeomorphic, as a corollary of Proposition 1, we have the following proposition due to M. Sakuma (unpublished).

PROPOSITION 3. *For links in a homology 3-sphere, the greatest common divisor of the linking numbers of each pair of components is an invariant of link complements. In particular, for links of 2 components in a homology 3-sphere, the absolute value of the linking number of components is an invariant of the link complement.*

Now let $M = S^3$. We consider a Seifert surface F for L in M' such that $F \cap N_i = f(S^1 \times \gamma)$, where γ is a straight-line segment connecting 0 with 1 in D^2. In practice we describe the triple (M', F, L) by drawing a picture of L with orientation and F, and writing the surgery coefficients (a_i, b_i) near L_i if $b_i \neq 0$. If $b_i = 0$, then L_i is a boundary of F in this picture. Two triples (M, F, L) and (M', F', L') are equivalent if there is a homeomorphism $\Phi: M \to M'$ such that $\Phi(F) = F'$.

REMARK. In [5, Chapter 9], M' is described by L and the ratios a_i/b_i.

2. Construction. Let $\tilde{F}$ be a 2-sphere with n holes and $\partial\tilde{F} = S_1^1 \cup \cdots \cup S_n^1$, whose orientations are induced from one for $\tilde{F}$. We identify $\tilde{F} \times S^1$ with the exterior of the link $L = L_1 \cup \cdots \cup L_n$ in S^3 of Fig. 3, and take the preferred framing for the tubular neighborhood N_i of L_i as in §1 so that $\lambda_1, \mu_1, -\mu_j$ and λ_j, $j = 2,\dots, n$, are homology classes of $S_1^1 \times 1$, $1_1 \times S^1$, $S_j^1 \times 1$ and $1_j \times S^1$, respectively, where $1_i \in S_i^1$.

Then every fibered link of genus zero whose monodromy is the identity can be described by the triple (M, F, L) of Fig. 4, where L_1 bounds F and L_j meets F transversely in a point for $j = 2,\dots, n$.

Let $M = M(p_1,\dots, p_n)$. Then (M, F, L) is independent of the order of the p_i, and reversing the orientation of M results in $-M = M(-p_1,\dots, -p_n)$. Now

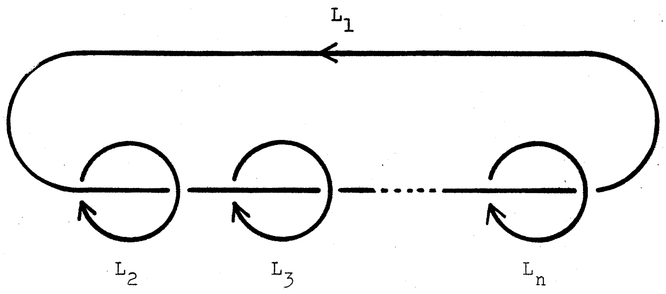


Fig. 3

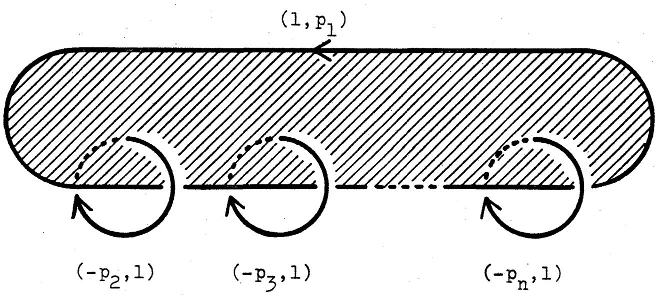


Fig. 4

$\pi_1(\tilde{F}\times S^1)$ has a presentation

$$\langle x_1,\ldots, x_n, y | x_1\cdots x_n=1,\ yx_i=x_iy,\ i=1,\ldots, n\rangle,$$

where x_i is represented by $S_i^1\times 1$ and y by $1_1\times S^1$. Since $y=x_i^{-p_i}$, $i=1,\ldots, n$, in $\pi_1(M)$, $\pi_1(M)$ is presented by

$$G=\langle x_1,\ldots, x_n | x_1^{p_1}=\cdots=x_n^{p_n},\ x_1\cdots x_n=1\rangle.$$

LEMMA 1. *$M(p_1,\ldots, p_n)$, $n\geqq 2$, is homeomorphic to S^3 if and only if $(p_1,\ldots, p_n)$ or $(-p_1,\ldots, -p_n)$ is obtained upon a permutation of:*
(i) *Either $a_m^n(0\leqq m\leqq k)$, $b_p^n(p\geqq 1)$ or c^n if $n=2k+1$.*
(ii) *Either $a_m^n(0\leqq m\leqq k)$ or $d_q^n(q\geqq 1)$ if $n=2k+2$.*
Here $a_m^n=(0, \underbrace{-1,\ldots, -1}_{m}, 1,\ldots, 1)$,

$b_p^n=(p, -1, 1,\ldots, -1, 1)$,
$c^n=(1, -2, -1, 1,\ldots, -1, 1, -3)$ *and*
$d_q^n=(q, -1, 1,\ldots, -1, 1, -q-1)$.

PROOF. Suppose that G is trivial. If $n=2$, by eliminating x_2, G is isomorphic to the group $\langle x_1 | x_1^{p_1+p_2}=1\rangle$. Thus $p_1+p_2=\pm 1$, and we have a_0^2 and d_q^2.

Let us assume for convenience that $|p_1|\geqq|p_2|\geqq\cdots\geqq|p_n|\geqq 0$. If $p_n=0$, then

G is isomorphic to the free product $Z_{|p_1|}*Z_{|p_2|}*\cdots*Z_{|p_{n-1}|}$, where Z_0 means Z. Thus $|p_j|=1$ for $j=1,\dots, n-1$, so we have a^n_m.

Now we suppose $n\geqq 3$ and $p_i\neq 0$ for all i. Then G has the quotient group

$$\langle x_i, x_j, x_k | x_i^{p_i}=x_j^{p_j}=x_k^{p_k}=x_ix_jx_k=1\rangle,$$

where $1\leqq i<j<k\leqq n$. This is the Dyck group denoted by $\sum(|p_i|, |p_j|, |p_k|)$ ([4]) and is known to be nontrivial if $|p_i|\geqq|p_j|\geqq|p_k|\geqq 2$. Therefore we have $|p_1|\geqq |p_2|\geqq|p_3|=\cdots=|p_n|=1$. Since $x_j=x_1^{p_1p_j}$ for $j=3,\dots, n$, G becomes

$$\langle x_1, x_2 | x_1^{p_1}=x_2^{p_2}, x_2x_1^{1+p_1p}=1\rangle,$$

where $p=p_3+\cdots+p_n$. Note that p and n have the same parity. In terms of x_1, with $x_2=x_1^{-1-p_1p}$, this is

$$\langle x_1 | x_1^{p_1+p_2+p_1p_2p}=1\rangle,$$

so we have $p_1+p_2+p_1p_2p=\pm 1$. If $p=0$, we have d^n_q. If $p\neq 0$, using $(p_1p+1)(p_2p+1)=p\pm 1$, we obtain the rest. The converse will be shown in the proof of the theorem. □

3. Main theorem.

THEOREM. *Let L be an n-component fibered link in S^3 whose fiber surface F is of genus zero, $n\geqq 2$. If the monodromy of L is the identity, then the triple (S^3, F, L) is equivalent to one of the following triples drawn in* Fig. 5, *where the dotted bands with "k" or "kq" stand for k or kq bands as shown in* Fig. 6.
(i) $A^n_m(0\leqq m\leqq k)$, $B^n_p(p\geqq 1)$, C^n if $n = 2k + 1$.
(ii) $A^n_m(0\leqq m\leqq k)$, $D^n_q(q\geqq 1)$ if $n = 2k + 2$.
Furthermore these triples are inequivalent to one another.

For each $(p_1,\dots, p_n)$ of Lemma 1, we show that $(M(p_1,\dots, p_n), F, L)$ yields one of the triples of the theorem.

LEMMA 2. *If $p_2=1$, then (M, F, L) yields the triple of* Fig. 7(i), *and if $p_2=-1$, then that of* Fig. 7(ii).

PROOF. If $p_2=1$, then the surgery curve for N_2 is homologous to $\lambda_2-\mu_2$, so we give a meridinal twist in the right-hand sense to the solid torus $S^3-\mathring{N}_2$ of Fig. 8(i) to obtain the solid torus of Fig. 8(ii), whose interior is regarded as the complement of L_2 in S^3. □

PROOF of THEOREM. Using Lemma 2 $(n-1)$-times, we see that $M(a^n_m)$ yields A^n_m. By Lemma 2, $M(b^n_p)$ yields the triple of Fig. 9(i), which is the same as that of Fig. 9(ii). Giving p left-hand twists to the solid torus $S^3-\mathring{N}_1$, in other words,

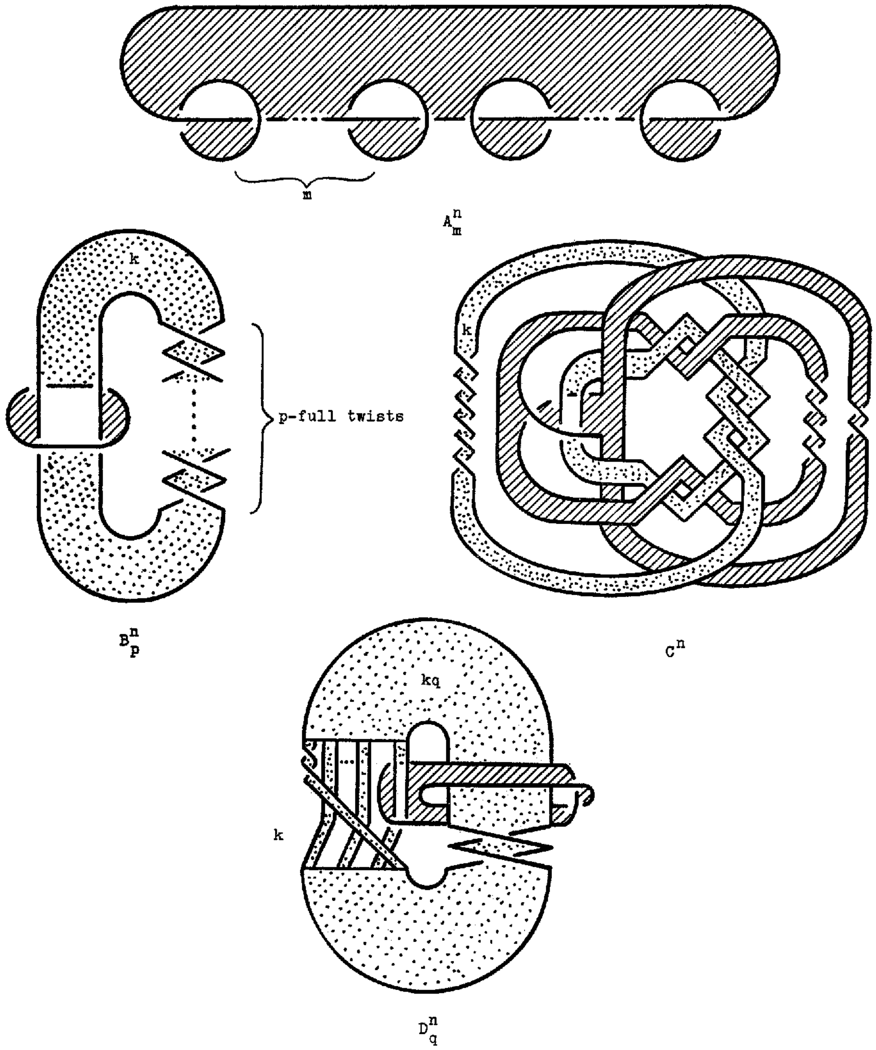


Fig. 5

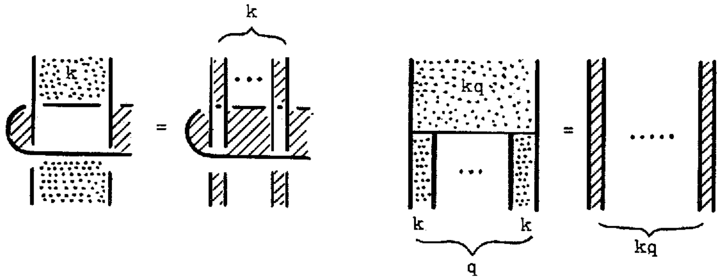


Fig. 6

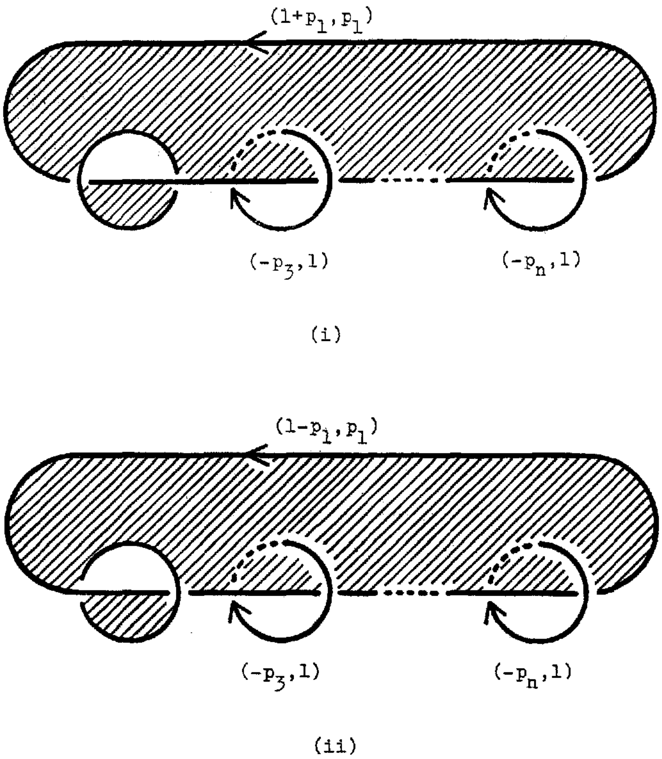


Fig. 7

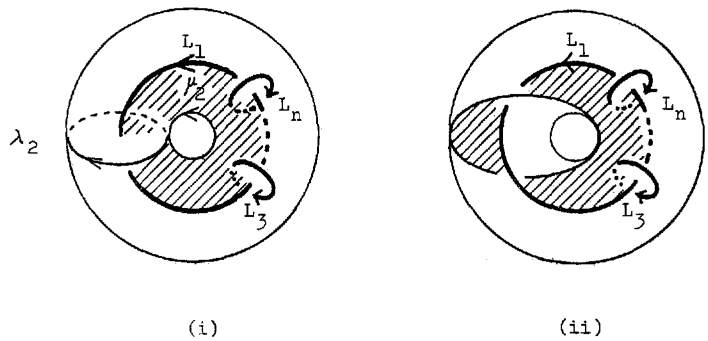


Fig. 8

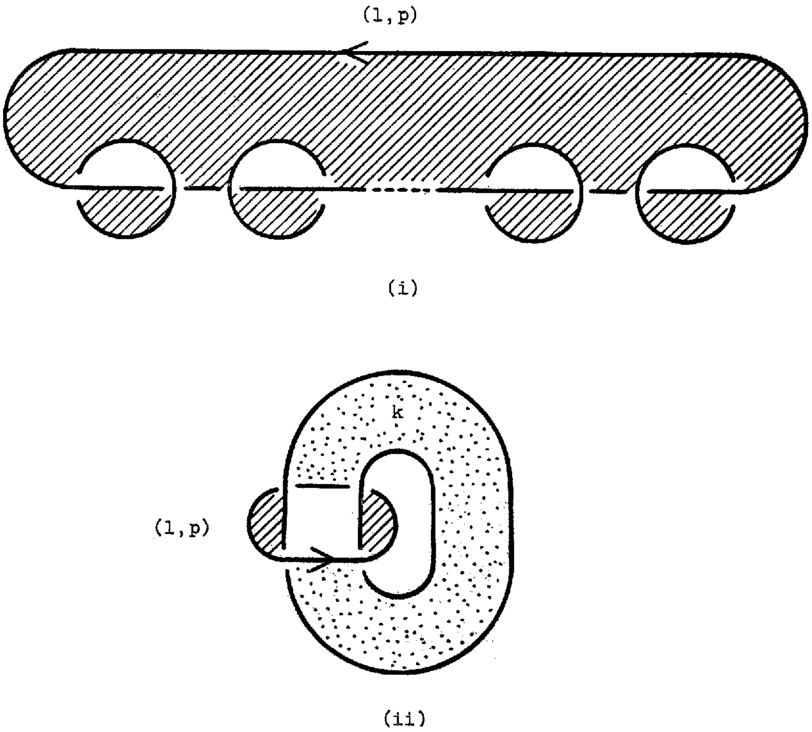


Fig. 9

performing p left-hand twists about L_1, we obtain B_p^n.

By Lemma 2, $M(d_q^n)$ yields the triple of Fig. 10(i). Performing q left-hand twists about L_1, we have the triple of Fig. 10(ii). If we give a left-hand twist to the solid torus $S^3-\mathring{N}_2$ of Fig. 10(iii), we have the solid torus whose interior is the complement of L_2 in S^3, and so we have D_q^n.

By Lemma 2, $M(c^n)$ yields the triple of Fig. 11(i). Performing a left-hand twist about L_1, we have the triple of Fig. 11(ii). Giving a left-hand twist to the solid torus $S^3-\mathring{N}_2$ of Fig. 11(iii), we have the solid torus of Fig. 11(iv), whose interior is the complement of L_2. Then giving a left-hand twist to the solid torus $S^3-\mathring{N}_n$ of Fig. 11(v), we have the solid torus whose interior is the complement of L_n in S^3, and so we obtain C^n.

The inequivalence of these triples can be seen from the linking numbers and the link-types of the sublinks, and the proof is complete. □

REMARK. Using Proposition 2, we can compute the Alexander polynomials of the links of the theorem, obtaing:

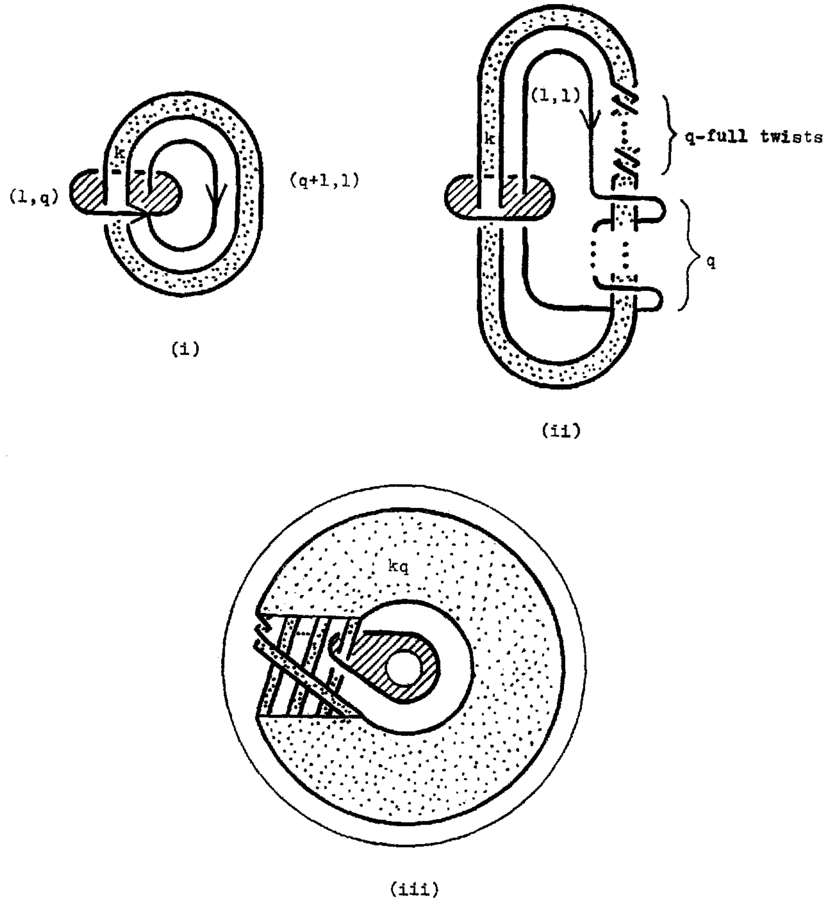


Fig. 10

A_m^n; $(x_1-1)^{n-2}$,

B_p^n; $(x_1x_2^{-p}x_3^{p}\cdots x_{n-1}^{-p}x_n^{p}-1)^{n-2}$,

C_n; $(x_1^6x_2^{-3}x_3^{-6}x_4^{6}\cdots x_{n-2}^{-6}x_{n-1}^{6}x_n^{-2}-1)^{n-2}$,

D_q^n; $(x_1^{q+1}\ x_2^{-q(q+1)}x_3^{q(q+1)}\ldots x_{n-2}^{-q(q+1)}\ x_{n-1}^{q(q+1)}x_n^{-q}-1)^{n-2}$.

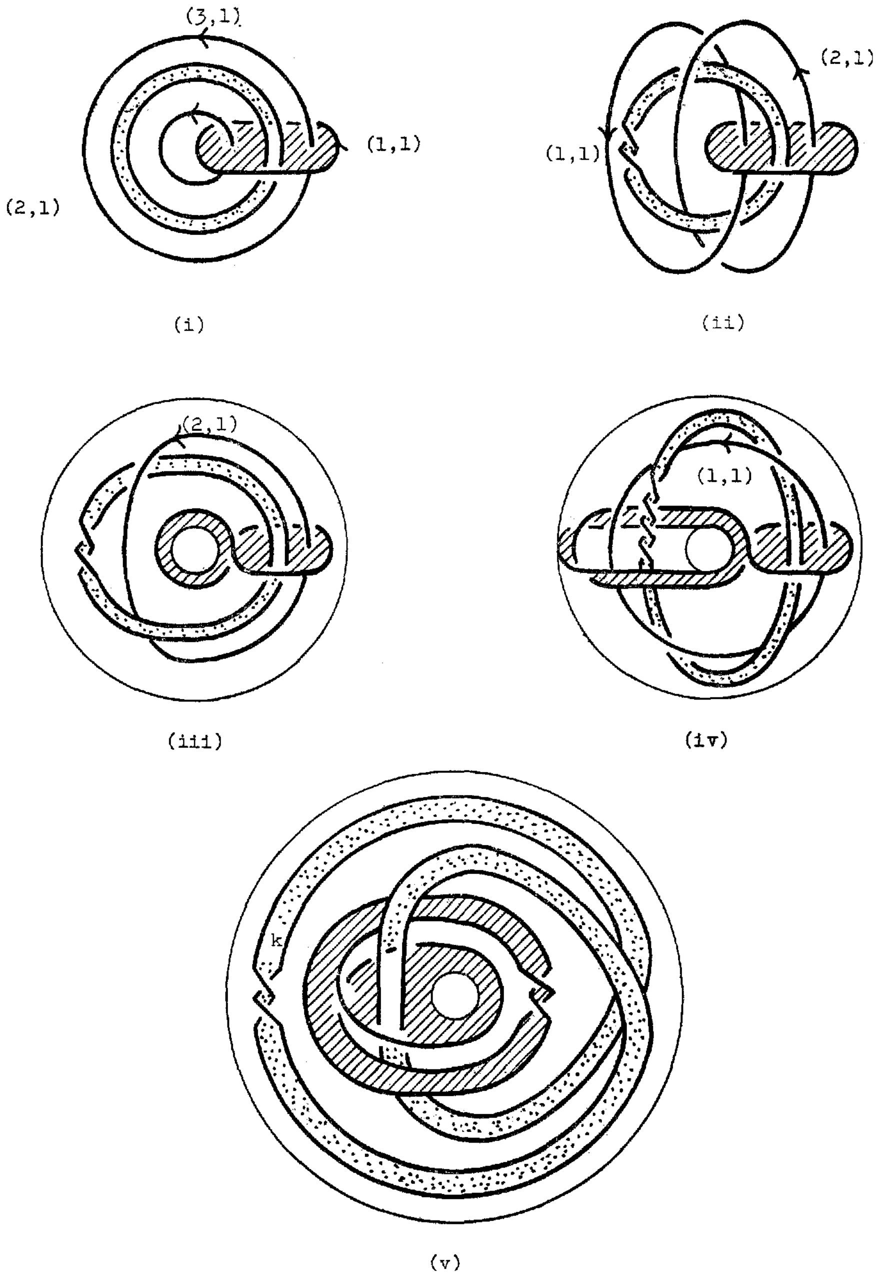


Fig. 11

References

[1] Birman, J. S.: *Braids, Links and Mapping class groups,* Ann. of Math. Studies 81, Princeton University Press (1974).

[2] Burde, G. and Zieschang, H.: Neuwirthsche Knoten und Flächeabbildungen, Abh. Math. Sem. Univ. Hamburg **31**, 239–246 (1967).

[3] González-Acuña, F.: Dehn's constructions on knots, Bol. Soc. Mat. Mexicana (2) **15**, 58–77 (1970).

[4] Milnor, J. W.: *On the 3-dimensional Brieskorn manifolds* $M(p, q, r)$, Ann. of Math. Studies 84, Princeton University Press, 175–225 (1975).

[5] Rolfsen, D.: *Knots and Links,* Mathematics Lecture Series 7, Berkeley, Publish or Perish Inc. 1976.

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