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FIXED POINT THEOREMS IN PROBABILISTIC METRIC SPACES

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Using the probabilistic diameter we shall prove in this paper some fixed point theorems in probabilistic metric spaces. First we shall give some definitions and notations.

1. A pair $(S, \mathscr{F})$ is a *probabilistic metric space*, in [2], if and only if S is an arbitrary set, $\mathscr{F}: S\times S\to\Delta^+$ (Δ^+ is the set of all distribution functions F such that $F(0)=0$) so that the following conditions are satisfied (where $\mathscr{F}(x, y) = F_{x,y}$ for every $x, y\in S$):

1. $F_{x,y}(\varepsilon)=1$, for every $\varepsilon\in R^+$ iff $x=y$.
2. $F_{x,y}=F_{y,x}$, for every $x, y\in S$.
3. $F_{x,y}(\varepsilon)=1$ and $F_{y,z}(\delta)=1$ implies $F_{x,z}(\varepsilon+\delta)=1$, where $x, y, z\in S$, $\varepsilon, \delta\in R^+$.

The (ε, λ)-*topology* in S is introduced by the (ε, λ)-neighbourhoods of $x\in S$:

$$U_x(\varepsilon, \lambda) = \{y: F_{x,y}(\varepsilon)>1-\lambda\} \quad \varepsilon>0,\ \lambda\in(0, 1).$$

A triplet $(S, \mathscr{F}, t)$ is a *Menger space*, see [2], iff $(S, \mathscr{F})$ is a probabilistic metric space and t is a T-norm such that for every $\varepsilon_1, \varepsilon_2\in R^+$

$$F_{x,y}(\varepsilon_1+\varepsilon_2) \geqslant t(F_{x,z}(\varepsilon_1), F_{z,y}(\varepsilon_2))$$

for every $x, y, z\in S$.

A set $M\subset S$ is *probabilistic bounded*, see [2], iff

$$\sup_{\varepsilon} D_M(\varepsilon) = 1$$

where

$$D_M(\varepsilon) = \sup_{\delta<\varepsilon}\ \inf_{x,y\in S} F_{x,y}(\delta)$$

is the *probabilistic diameter* of the set M.

If $f: S\to S$ and $x\in S$ then

$\mathcal{O}_f(x) = \{x, f(x), f^2(x),\ldots\}$.

DEFINITION 1. [3] Let $(S, \mathscr{F})$ be a probabilistic metric space and $f\colon S \to S$. A point $x \in S$ is *regular* for f iff $\sup D_{\mathcal{O}_f(x)}(\varepsilon) = 1$.

DEFINITION 2. [3] Let $(S, \mathscr{F})$ be a probabilistic metric space and $f\colon S \to S$. Two points $x, y \in S$ are *asymptotic* under f iff

$$F_{f^n(x), f^n(y)}(\varepsilon) \to 1 \qquad \text{for} \quad n \to \infty, \quad \text{for every} \quad \varepsilon > 0.$$

DEFINITION 3. [3] Let X be a topological space in which the topology is defined by the family $\{d_n\}_{n \in N}$ of pseudometrics, $f\colon X \to X$ and $x \in X$. The point x is *regular* for f iff for every $n \in N$ there exists M_n such that $D_n(\mathcal{O}_f(x)) < M_n$, where $D_n(A) = \sup\{d_n(x, y)\colon x, y \in A\}$, $A \subseteq S$.

DEFINITION 4. [3] Let X be a topological space in which the topology is defined by the family $\{d_n\}_{n \in N}$ of pseudometrics, $f\colon X \to X$ and $x, y \in X$. The points x, y are *asymptotic* under f iff for every $n \in N$:

$$\lim_{m \to \infty} d_n(f^m(x), f^m(y)) = 0.$$

THEOREM 1. *Let f be a continuous mapping of a complete probabilistic metric space $(S, \mathscr{F})$ into itself such that every point $x \in X$* is regular for f and *every pair of points $x, y \in S$ is asymptotic under f. Let $\phi\colon R^+ \to R^+$ be a one to one mapping such that*

1. $\lim\limits_{n \to \infty} \underbrace{\phi^{-1}\phi^{-1} \cdots \phi^{-1}}_{n-\text{times}}(t) = \lim\limits_{n \to \infty} (\phi^{-1})^n(t) = \infty$ *for all* $t > 0$.

2. $D_{\mathcal{O}_f(f(x))}(\phi(\varepsilon)) \geqslant D_{\mathcal{O}_f(x)}(\varepsilon)$, *for all $x \in S$ and all $\varepsilon > 0$.*

Then there exists one and only one fixed point of the mapping f and for all $x \in S$ the sequence $\{f^n(x)\}_{n \in N}$ converges to this point.

PROOF. From the assumption of the theorem we have

$$D_{\mathcal{O}_f(f^n(x_0))}(\varepsilon) \geqslant D_{\mathcal{O}_f(f^{n-1}(x_0))}(\phi^{-1}(\varepsilon)) \geqslant \cdots \geqslant D_{\mathcal{O}_f(x_0)}((\phi^{-1})^n(\varepsilon))$$

for every $x_0 \in S$.

Now we shall prove that the sequence $\{f^n(x_0)\}_{n \in N}$ is a Cauchy sequence for every $x_0 \in S$, i.e. that for every $\varepsilon > 0$, for every $\lambda \in (0, 1)$ there exists $n_0 \in N$ such that

$$F_{f^n(x_0), f^{n+p}(x_0)}(\varepsilon) > 1 - \lambda$$

for all $n \geqslant n_0$, $p = 1, 2, \ldots$. The point x_0 is regular under f, so we have

$$\sup_{\varepsilon} D_{\mathcal{O}_f(x_0)}(\varepsilon) = 1.$$

This means that for all $\lambda \in (0, 1)$ there exists $\varepsilon(\lambda)$ such that

$$D_{\mathcal{O}_f(x_0)}(\varepsilon(\lambda)) > 1 - \frac{\lambda}{2}.$$

Since $\lim_{n\to\infty} (\phi^{-1})^n(\varepsilon) = \infty$, there exists $n_0 \in N$ such that

$$(\phi^{-1})^{n_0}(\varepsilon) > \varepsilon(\lambda).$$

Then we have that

$$D_{\mathcal{O}_f(x_0)}((\phi^{-1})^{n_0}(\varepsilon)) \geqslant D_{\mathcal{O}_f(x_0)}(\varepsilon(\lambda)) > 1 - \frac{\lambda}{2},$$

i.e. that

$$D_{\mathcal{O}_f(f^n(x_0))}(\varepsilon) > 1 - \frac{\lambda}{2},$$

for all $n \geqslant n_0$. Since a distribution function F is a nondecreasing function, we have that

$$\inf_{x,y\in\mathcal{O}_f(f^n(x_0))} F_{x,y}(\varepsilon) \geqslant \sup_{\delta<\varepsilon} \inf_{x,y\in\mathcal{O}_f(f^n(x_0))} F_{x,y}(\delta) > 1 - \frac{\lambda}{2}$$

which means that

$$F_{x,y}(\varepsilon) > 1 - \frac{\lambda}{2}$$

for all x, $y \in \mathcal{O}_f(f^n(x_0))$ and all $n \geqslant n_0$. Since $n \geqslant n_0$ implies $n+p \geqslant n_0$ we have

$$F_{f^n(x_0),f^{n+p}(x_0)}(\varepsilon) > 1 - \lambda$$

for all $n \geqslant n_0$ and $p = 1, 2, \ldots$. The space S is complete which means that for the sequence $\{f^n(x_0)\}_{n\in N}$, for every $x_0 \in S$, there exists an element $z \in S$ such that

$$\lim_{n\to\infty} f^n(x_0) = z.$$

Since f is a continuous function we obtain

$$f(z) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{n+1}(x_0) = z,$$

and so z is a fixed point of the mapping f.

If we suppose that z is not a unique fixed point of the mapping f, i.e. that there exists $w \in S$, $w \neq z$, such that $f(w) = w$, then we have

$$F_{z,w}(\varepsilon) = F_{f^n(z),f^n(w)}(\varepsilon) \longrightarrow H(\varepsilon) \quad \text{for } n \to \infty,$$

since every pair of points from S is asymptotic under f. So we have $z = w$.

On the Menger space with continuous T-norm t we can define the family of functions $\{t_n(x)\}_{n\in N}$ with

$$t_n(x) = \underbrace{t(t(\cdots(t}_{n\text{-times}}(x, x), x),\dots, x), \quad t_1(x) = x.$$

for all $x \in (0, 1)$ and $n \in N$. An example of T-norm t such that the family $\{t_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ is given in [4].

THEOREM 2. *Let $(S, \mathcal{F}, t)$ be a complete Menger space with continuous T-norm t such that the family $\{t_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$. Let ϕ: $R^+\to R^+$ be a monotone increasing mapping and let f: $S\to S$ be a continuous mapping such that the following conditions are satisfied*

1. $\phi(t)<t, \quad \lim\limits_{n\to\infty}(\phi^{-1})^n(t)=\infty, \quad \lim\limits_{\varepsilon\to\infty}(\phi^{-1}(\varepsilon)-\varepsilon)=\infty$

2. $F_{f(x),f(y)}(\phi(\varepsilon))\geqslant F_{x,y}(\varepsilon)$ *for all* $\varepsilon>0$ *and for all* $x, y\in S$.

Then there exists one and only one fixed point of the mappinf f.

PROOF. As it is done in the preceding theorem we can prove that

$$F_{f^n(x),f^n(y)}(\varepsilon) \geqslant F_{x,y}((\phi^{-1})^n(\varepsilon)) \quad \text{for all } \varepsilon>0.$$

From the condition 1 of the theorem we have

$$\lim_{n\to\infty} F_{f^n(x),f^n(y)}(\varepsilon) = 1 \quad \text{for all } x, y\in S.$$

So we have proved that every two points are asymptotic under f.

Now we shall show that $\sup\limits_{\varepsilon} D_{\mathcal{O}_f(x)}(\varepsilon)=1$ for all $x\in S$, i.e. that all points are regular for f. From the definition of a Menger space we have that for all $\varepsilon>0$

$$\begin{aligned}
F_{f^m(x),x}(\phi^{-1}(\varepsilon)) &\geqslant t(F_{f^m(x),f(x)}(\varepsilon), \quad F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon)) \\
&\geqslant t(F_{f^{m-1}(x),x}(\phi^{-1}(\varepsilon)), \quad F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon)) \\
&\geqslant t(t(F_{f^{m-2}(x),x}(\phi^{-1}(\varepsilon)), \quad F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon)), \quad F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon)) \\
&\geqslant\cdots\geqslant \underbrace{t(t(\cdots t(}_{m-1\text{ times}}F_{f(x),x}(\phi^{-1}(\varepsilon)), \quad F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon),\dots, \ F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon)) \\
&\geqslant t_{m-1}(F_{f(x),x}(\phi^{-1}(\varepsilon)-\varepsilon))
\end{aligned}$$

for all $m\in N$. From the assumption of the theorem t_m is a equicontinuous family of functions at $x=1$ and $\lim\limits_{\varepsilon\to\infty}(\phi^{-1}(\varepsilon)-\varepsilon)=\infty$ which means that for all $\lambda\in(0, 1)$ there exists $\varepsilon(\lambda)$ such that

$$F_{f^m(x),x}(\phi^{-1}(\varepsilon(\lambda))) \geqslant t_{m-1}(F_{f(x),x}(\phi^{-1}(\varepsilon(\lambda))-\varepsilon(\lambda))) > 1 - \lambda$$

for all $m \in N$. Since, for $k<n$, $n, k \in N$

$$F_{f^n(x),f^k(x)}(\varepsilon(\lambda)) \geqslant F_{f^{n-1}(x),f^{k-1}(x)}(\phi^{-1}(\varepsilon(\lambda))) \geqslant \cdots \geqslant F_{f^{n-k}(x),x}(\phi^{-1}(\varepsilon(\lambda))) > 1-\lambda$$

we get, by the definition of the probabilistic diameter, that

$$\sup_{\varepsilon} D_{\mathcal{O}_f(x)}(\varepsilon) = 1$$

which means that every point is regular under f. Now, we shall prove

$$D_{\mathcal{O}_f(f(x))}(\phi(\varepsilon)) \geqslant D_{\mathcal{O}_f(x)}(\varepsilon)$$

for all $x \in S$ and for all $\varepsilon>0$. This fact will be anable us to apply the preceding theorem. The function ϕ is monotone and increasing. This gives the next relation

$$D_{\mathcal{O}_f(x)}(\varepsilon) = \sup_{\delta<\varepsilon} \inf_{n,k \in N\cup\{0\}} F_{f^n(x),f^k(x)}(\delta) \leqslant \sup_{\delta<\varepsilon} \inf_{n,k\in N\cup\{0\}} F_{f^{n+1}(x),f^{k+1}(x)}(\phi(\delta))$$

$$= \sup_{\phi(\delta)<\phi(\varepsilon)} \inf_{n,k\in N} F_{f^n(x),f^k(x)}(\phi(\delta)) \leqslant D_{\mathcal{O}_f(f(x))}(\phi(\varepsilon)).$$

Let $(S, \mathscr{F}, t)$ be a Menger space with continuous T-norm t such that the family $\{t_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$. We know that there exists the sequence $\{a_n\}_{n\in N}$ [4] such that $\lim_{n\to\infty} a_n=1$ and the family of pseudometrics $\{d_n\}_{n\in N}$ defined by

$$d_n(x, y) = \sup\{t\colon F_{x,y}(t)<a_n\}$$

for all $x, y\in S$ and for all $n\in N$, induces (ε, λ)-topology in S. Now we shall prove some theorems for this kind of spaces.

THEOREM 3. *Let $(S, \mathscr{F}, t)$ be a Menger space with continuous T-norm t such that the family $\{t_n(x)\}_{n\in N}$ is equicontinuous at $x=1$. Let $f\colon S\to S$ and $\phi\colon R^+\to R^+$ be a monotone increasing upper-semicontinuous from the right mapping such that following conditions are satisfied*

1. $\phi(t)<t$ for all $t\in R^+$
2. $F_{f(x),f(y)}(\phi(\varepsilon)) \geqslant F_{x,y}(\varepsilon)$ for all $x, y\in S$.

Then for all $x, y\in S$ and all $n\in N$ $d_n(f(x), f(y))\leqslant \phi(d_n(x, y))$ and the mapping f has a fixed point.

PROOF. If we suppose that for some $n\in N$ and some $x, y\in S$ $d_n(f(x), f(y))> \phi(d_n(x, y))$ then there exists $r>0$

$$d_n(f(x), f(y)) > r > \phi(d_n(x, y)).$$

Since

$$d_n(x, y) = \sup\{t: F_{x,y}(t) \leqslant a_n\},$$

from the inequality

$$d_n(f(x), f(y)) > r$$

it follows that

$$F_{f(x),f(y)}(r) \leqslant a_n.$$

Since ϕ is monotone increasing function, the condition

$$\phi(d_n(x, y)) < r$$

is equivalent to the condition

$$d_n(x, y) < \phi^{-1}(r),$$

i.e.

$$F_{x,y}(\phi^{-1}(r)) > a_n \geqslant F_{f(x),f(y)}(r)$$

which is contradiction with the proposition of the theorem. From [1] it follows that the mapping f has a fixed point.

THEOREM 4. *Let $(S, \mathscr{F}, t)$ be a Menger space with a continuous T-norm t such that the family $\{t_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$ and let $f\colon S\to S$, $\phi\colon R^+\to R^+$ and ϕ be a monotone increasing function.*

Then the condition

$D_{\mathcal{O}_f(f(x))}(\phi(\varepsilon)) > D_{\mathcal{O}_f(x)}(\varepsilon)$ *for all $x\in S$ and all $\varepsilon>0$, implies the condition*

$D_n(\mathcal{O}_f(f(x)) \leqslant \phi(D_n(\mathcal{O}_f(x))$ *for all $n\in N$ and for all $x\in S$.*

PROOF. We suppose the oposite, i.e. suppose that exists $x\in S$ and $n\in N$ such that

$$D_n(\mathcal{O}_f(f(x))) > \phi(D_n(\mathcal{O}_f(x))).$$

From this, since ϕ is monotone increasing function, we have

$$\phi^{-1}(D_n(\mathcal{O}_f(f(x)))) > D_n(\mathcal{O}_f(x)).$$

This means that there exists $r>0$ such that

$$D_n(\mathcal{O}_f(x)) < r \quad \text{and} \quad \phi^{-1}(D_n(\mathcal{O}_f(f(x))) > r$$

and so

$$(1) \qquad D_n(\mathcal{O}_f(f(x))) > \phi(r)$$

Then we have

$$\sup\{d_n(f^s(x), f^m(x)): \quad s, m \in N \cup \{0\}\} < r.$$

We can choose $r_1 < r$ such that

$$(2) \qquad d_n(f^s(x), f^m(x)) < r_1 < r \quad \text{for all } s, m \in N \cup \{0\}.$$

The inequality (1) means that there exist i, $k \in N$ such that

$$d_n(f^i(x), f^k(x)) > \phi(r)$$

which implies

$$(3) \qquad F_{f^i(x), f^k(x)}(\phi(r)) \leqslant a_n.$$

From (2) we have that

$$(4) \qquad F_{f^s(x), f^m(x)}(r_1) > a_n \quad \text{for all } s, m \in N \cup \{0\}$$

and by (3) it follows

$$\inf_{s,m \in N} F_{f^s(x), f^m(x)}(\phi(r)) \leqslant a_n.$$

Since, we have that

$$\sup_{t < \phi(r)} \inf_{s,m \in N} F_{f^s(x), f^k(x)}(t) \leqslant \inf_{s,m \in N} F_{f^s(x), f^m(x)}(\phi(r)) \leqslant a_n$$

and so

$$D_{\mathcal{O}_f(f(x))}(\phi(r)) \leqslant a_n.$$

From (4) we get

$$\inf_{s,m \in N \cup \{0\}} F_{f^s(x), f^m(x)}(r_1) \geqslant a_n$$

and so

$$\sup_{t < r} \inf_{m,s \in N \cup \{0\}} F_{f^s(x), f^m(x)}(t) \geqslant a_n,$$

which means that

$$D_{\mathcal{O}_f(x)}(r) \geqslant a_n \geqslant D_{\mathcal{O}_f(f(x))}(\phi(r)) > D_{\mathcal{O}_f(x)}(r).$$

This is a contradiction and the theorem is proved.

Next theorem can be proved similarly as it is done in [6].

THEOREM 5. *Let X be a complete topological space where the topology is defined by a family of pseudometrics $\{d_n\}_{n\in N}$, f: $X\to X$ such that every point $x\in X$ is regular and every two points x, $y\in X$ are asymtotic under mapping f. If there exists ϕ: $R^+\to R^+$ which is monotone, nondecreasing, upersemicontinuous from the right function such that*

$$D_n(\mathcal{O}_f(x)) \leqslant \phi(D_n(\mathcal{O}_f(x)))$$

for all $x\in X$ and for all $n\in N$, then there exists one and only one fixed point z of the mapping f and

$$\lim_{n\to\infty} f^n(x) = z \qquad \text{for all} \quad x\in X.$$

THEOREM 6. *Let $(S, \mathscr{F}, t)$ be a complete Menger space with continuous T-norm t such that the family $\{t_n(x)\}_{n\in N}$ is equicontinuous at the point $x=1$, a mapping f: $S\to S$ be such that every point $x\in S$ is regular and every two points x, $y\in S$ are asymptotic under f and for all $\varepsilon>0$ and for all $x\in S$*

$$D_{\mathcal{O}_f(f(x))}(\phi(\varepsilon)) > D_{\mathcal{O}_f(x)}(\varepsilon)$$

where ϕ: $R^+\to R^+$ is a monotone, increasing, uppersemicontinuous from the right function. Then there exists one and only one fixed point z of the mapping f and for all $x\in S$

$$z = \lim_{n\to\infty} f^n(x)$$

PROOF. From [3] all the points $x\in S$ which are regular under the definition 1 also will be regular under the definition 3 (in topological space S where topology is defined by a family of pseudometrics $\{d_n\}_{n\in N}$). Also from [3] we have that all pairs of the points are asymptotic under f (under the definition 4). From theorem 4 it follows that from the condition

$$D_{\mathcal{O}_f(f(x))}(\phi(\varepsilon)) > D_{\mathcal{O}_f(x)}(\varepsilon)$$

we get the condition

$$D_n(\mathcal{O}_f(f(x)) \leqslant \phi(D_n(\mathcal{O}_f(x))),$$

which means that all the conditions from the preceding theorem are satisfied and the mapping f have one and only one fixed point z and

$$z = \lim_{n\to\infty} f^n(x) \qquad \text{for all} \quad x\in S.$$

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