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## A NOTE ON DEDUCTIVE SYSTEMS OF A HILBERT ALGEBRA

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For a Hilbert algebra  $A$ , we denote by  $Ds(A)$  the lattice of all deductive systems of  $A$  and by  $Spec(A)$  the set of completely irreducible deductive systems of  $A$  (see sections 2 and 3). The paper is organized as follows. In section 1 we recall the basic definitions; also, we put in evidence results which we need in section 2 (especially lemma 1.5. and corollary 1.6.). Section 2 contains the main result of this paper:

*If  $A$  is a bounded Hilbert algebra, then the lattice  $Ds(A)$  is a Boolean lattice iff  $A$  is a finite Boolean lattice (theorem 2.8.).*

As an corollary of this result, in section 3 we deduce the next result:

*If  $A$  is a bounded Hilbert algebra, then every element of  $Ds(A)$  has a unique representations as a intersection of elements of  $Spec(A)$  iff  $A$  is a finite Boolean lattice (theorem 3.4.).*

For distributive lattices with 0 and 1 there exists an analogously result (see [4])

### 1. Definitions and preliminaries

DEFINITION 1.1. [2, p. 4]. A Hilbert algebra is a triple  $(A, \rightarrow, 1)$ , where  $A$  is a nonempty set,  $\rightarrow$  is a binary operation on  $A$ ,  $1 \in A$  is an element such that the next axioms is verified for every  $x, y, z \in A$ :

- (a<sub>1</sub>)  $x \rightarrow (y \rightarrow x) = 1$
- (a<sub>2</sub>)  $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$
- (a<sub>3</sub>) If  $x \rightarrow y = y \rightarrow x = 1$ , then  $x = y$ .

If  $A$  is a Hilbert algebra, then the relation  $x \leq y$  iff  $x \rightarrow y = 1$  is a partial order relation on  $A$  (which will be called *natural ordering* of  $A$ ); with respect to this ordering, 1 is the largest element of  $A$ . A *bounded* Hilbert algebra is a Hilbert algebra with a smallest element 0. In this paper the symbols  $\Rightarrow$  and  $\Leftrightarrow$  is used for logical implication and logical equivalence.

THEOREM 1.2. [2, p. 5]. *If  $x, y, z$  are elements of a Hilbert algebra, then the following hold:*

- (c<sub>1</sub>)  $x \leq y \rightarrow x$
- (c<sub>2</sub>)  $x \rightarrow 1 = 1$
- (c<sub>3</sub>)  $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$
- (c<sub>4</sub>)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
- (c<sub>5</sub>)  $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$
- (c<sub>6</sub>)  $x \leq (x \rightarrow y) \rightarrow y, ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$
- (c<sub>7</sub>)  $1 \rightarrow x = x$
- (c<sub>8</sub>)  $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$
- (c<sub>9</sub>) If  $x \leq y$ , then  $z \rightarrow x \leq z \rightarrow y$  and  $y \rightarrow z \leq x \rightarrow z$ .

EXAMPLES 1. *Classical propositional calculus*, where  $\rightarrow$  denotes the logical implication  $\Rightarrow$ .

2. If  $(A, \vee, \wedge, \neg, 0, 1)$  is a Boolean lattice, then  $(A, \rightarrow, 1)$  is a Hilbert algebra, where  $\rightarrow$  is defined by  $x \rightarrow y = (\neg x) \vee y$  for  $x, y \in A$ .

3. If  $(A, \leq)$  is a partial ordered set with 1, then  $(A, \rightarrow, 1)$  is a Hilbert algebra, where  $x \rightarrow y = 1$  if  $x \leq y$  and  $y$  if  $x \not\leq y$  for  $x, y \in A$ .

4. If  $X$  is a non-empty set and  $\tau$  a topology on  $X$ , then  $(\tau, \rightarrow, X)$  is a Hilbert algebra, where for  $D_1, D_2 \in \tau$ ,  $D_1 \rightarrow D_2 = \text{int}((X - D_1) \cup D_2)$ .

5. A *Heyting algebra* [1, p. 174] is a lattice  $(A, \vee, \wedge)$  with 0 such that for  $x, y, z \in A$ ,  $x \wedge z \leq y$  iff  $z \leq x \rightarrow y$ . If  $(A, \vee, \wedge, 0)$  is a Heyting algebra, then  $(A, \rightarrow, 1)$  is a Hilbert algebra, where  $1 = a \rightarrow a$  for an  $a \in A$ .

In [7, p. 58] a Heyting algebra is called *pseudo-Boolean algebra*. If consider  $A$  only with  $\wedge$ , we obtain the notion of *implicative semilattice* ([5]) which in [3, p. 61] is called *relatively pseudocomplemented meet-semilattice* (in this examples, the element  $x \rightarrow y$  is denoted by  $x * y$ ). If  $A$  is a bounded Hilbert algebra and  $x \in A$ , we denote by  $x^* = x \rightarrow 0$ . From theorem 1.2. we deduce immediately

THEOREM 1.3. If  $A$  is a bounded Hilbert algebra and  $x, y \in A$ , then:

- (c<sub>10</sub>)  $0^* = 1, 1^* = 0$
- (c<sub>11</sub>)  $x \rightarrow y^* = y \rightarrow x^*$
- (c<sub>12</sub>)  $x \rightarrow x^* = x^*, x^* \rightarrow x = x^{**}$
- (c<sub>13</sub>)  $x \rightarrow y \leq y^* \rightarrow x^*$
- (c<sub>14</sub>) If  $x \leq y$ , then  $y^* \leq x^*$ .

REMARK 1.4. If  $(X, \tau)$  is a topological space and  $D \in \tau$ , then  $D^* = \text{int}(X - D)$  and  $D^{**} = \text{int}(\bar{D})$  ( $\bar{D}$  = the adherence of  $D$ ) [6, p. 246]. If  $A$  is a Boolean lattice, then for  $x \in A$ ,  $x^* = \neg x$ .

LEMMA 1.5. If  $A$  is a bounded Hilbert algebra, then the following assertions are equivalent:

- (i)  $A$  is a Boolean lattice (relatively to natural ordering)
- (ii) For every  $x \in A$ ,  $x^{**} = x$ .

PROOF. (i) $\Rightarrow$ (ii). If  $A$  is a Boolean lattice and  $x \in A$ , then  $x^* = x \rightarrow 0 = \neg x$  and consequently  $x^{**} = \neg \neg x = x$ .

(ii) $\Rightarrow$ (i). For each  $x, y \in A$ ,  $x^* \rightarrow y$  is the supremum of  $x$  and  $y$ . Indeed, by  $(c_1)$ ,  $y \leq x^* \rightarrow y$ . Since  $0 \leq y$ , we can infer that  $x^* = x \rightarrow 0 \leq x \rightarrow y$ , and hence  $x \leq x^* \rightarrow y$  by  $(c_3)$ . If  $t \in A$  and  $x \leq t$ ,  $y \leq t$  we obtain successively  $t^* \leq x^*$ ,  $x^* \rightarrow y \leq t^* \rightarrow y$  and  $t^* \rightarrow y \leq t^* \rightarrow t = t^{**} = t$ . It follows that  $x^* \rightarrow y \leq t^* \rightarrow y \leq t$  and so  $x^* \rightarrow y = x \vee y$ . For  $x, y \in A$ ,  $(x \rightarrow y^*)^*$  is the infimum of  $x$  and  $y$ .

Indeed, from  $0 \leq y^*$  we deduce successively  $x^* = x \rightarrow 0 \leq x \rightarrow y^*$ ,  $(x \rightarrow y^*)^* \leq x^{**} = x$ , and  $y^* \leq x \rightarrow y^*$ ,  $(x \rightarrow y^*)^* \leq y^{**} = y$ . If  $t \in A$  and  $t \leq x$ ,  $t \leq y$ , then  $y^* \leq t^*$ ,  $x \rightarrow y^* \leq x \rightarrow t^* \leq t \rightarrow t^* = t^*$  and so  $t^{**} = t \leq (x \rightarrow y^*)^*$ .

It follows that  $(x \rightarrow y^*)^* = x \wedge y$ . If  $x, y, z \in A$ , then by theorems 1.2 and 1.3, we deduce successively  $x \wedge z \leq y \Leftrightarrow (x \rightarrow z^*)^* \leq y \Leftrightarrow y^* \leq x \rightarrow z^* \Leftrightarrow x \leq y^* \rightarrow z^* \Leftrightarrow x \leq z^{**} \rightarrow y^{**} = z \rightarrow y \Leftrightarrow z \leq x \rightarrow y$  and so we deduce that  $(A, \vee, \wedge, \rightarrow, 0, 1)$  is a Heyting algebra. To prove  $A$  is a Boolean lattice, we must show  $x^{**} \leq x$ . But this is trivial by our definition.

COROLLARY 1.6. *If  $A$  is a bounded Hilbert algebra, then  $A$  is a Boolean lattice iff for any elements  $x, y \in A$ ,  $(x \rightarrow y) \rightarrow x = x$  (see also [8, p. 111]).*

PROOF. If  $A$  is a Boolean lattice, then  $(x \rightarrow y) \rightarrow x = (\neg(x \rightarrow y)) \vee x = (\neg(\neg x \vee y)) \vee x = ((\neg \neg x) \wedge \neg y) \vee x = (x \wedge \neg y) \vee x = x$ .

If for every  $x, y \in A$ ,  $(x \rightarrow y) \rightarrow x = x$ , then for  $y = 0$ , we obtain that for every  $x \in A$ ,  $x^* \rightarrow x = x \Leftrightarrow x^{**} = x$  and by lemma 1.5, it results that  $A$  is a Boolean lattice.

If  $A$  is a Hilbert algebra and  $x_1, x_2, \dots, x_n \in A$  ( $n \geq 1$ ), we will define

$$(x_1, x_2, \dots, x_{n-1}; x_n) = \begin{cases} x_n & \text{if } n = 1 \\ x_1 \rightarrow (x_2, \dots, x_{n-1}; x_n) & \text{if } n > 1. \end{cases}$$

REMARK 1.7. From  $(c_3)$  and  $(c_4)$  we deduce immediately

(c<sub>15</sub>) If  $\sigma$  is a permutation of the set  $\{1, 2, \dots, n-1\}$  ( $n \geq 2$ ), then

$$(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}; x_n) = (x_1, \dots, x_{n-1}; x_n)$$

(c<sub>16</sub>) If  $x \in A$ , then  $x \rightarrow (x_1, x_2, \dots, x_{n-1}; x_n) = (x, x_1, \dots, x_{n-1}; x_n) = (x_1, x, x_2, \dots, x_{n-1}; x_n) = \dots = (x_1, \dots, x_{n-1}, x; x_n)$

(c<sub>17</sub>) If  $x, y \in A$ , then  $(x_1, \dots, x_n; x \rightarrow y) = (x_1, \dots, x_n; x) \rightarrow (x_1, \dots, x_n; y)$

(c<sub>18</sub>) If  $A$  is a Heyting algebra, then

$$(x_1, x_2, \dots, x_{n-1}; x_n) = (x_1 \wedge x_2 \wedge \dots \wedge x_{n-1}) \rightarrow x_n \text{ ([1, p. 174])}.$$

## 2. The lattice $Ds(A)$

DEFINITION 2.1. [2, p. 11]. If  $A$  is a Hilbert algebra, a subset  $D$  of  $A$  is a *deductive system* of  $A$  if

$$(a_4) \quad 1 \in D$$

$$(a_5) \quad \text{If } x, x \rightarrow y \in D, \text{ then } y \in D.$$

REMARK 2.2. If  $A$  is a Heyting algebra, then  $D \subseteq A$  is a deductive system iff  $D$  is a filter of  $A$  ([6, p. 249]).

We denote by  $Ds(A)$  the set of all deductive systems of  $A$ . If  $X \subseteq A$ , we denote by  $(X) = \bigcap \{D \in Ds(A) : X \subseteq D\}$  ( $(X)$  is called the deductive system *generated* by  $X$ ). If  $X = \{x_1, x_2, \dots, x_n\}$  we denote by  $(x_1, x_2, \dots, x_n) = (\{x_1, x_2, \dots, x_n\})$ ; the deductive system generated by one element  $a \in A$ , will be denote by  $[a]$  and is easily to verify that  $[a] = \{x \in A : a \leq x\}$  ( $[a]$  is called *principal*).

LEMMA 2.3. If  $A$  is a Hilbert algebra and  $X \subseteq A$  then

$$(X) = \{x \in A : \text{there exist } x_1, x_2, \dots, x_n \in X \text{ such that } (x_1, x_2, \dots, x_n; x) = 1\}.$$

$$\text{In particular, } (x_1, x_2, \dots, x_n) = \{x \in A : (x_1, x_2, \dots, x_n; x) = 1\}.$$

PROOF. See [8, p. 27].

For  $D_1, D_2 \in Ds(A)$  we put

$$D_1 \wedge D_2 = D_1 \cap D_2$$

$$D_1 \vee D_2 = (D_1 \cup D_2)$$

$$D_1 \rightarrow D_2 = \{a \in A : [a] \cap D_1 \subseteq D_2\}.$$

Then  $(Ds(A), \vee, \wedge)$  is a lattice with 0 and 1, where 0 is  $\{1\}$  and 1 is  $A$  (by [2, p. 19], the lattice  $Ds(A)$  is distributive).

LEMMA 2.4. If  $A$  is a Hilbert algebra and  $D_1, D_2 \in Ds(A)$ , then

$$D_1 \vee D_2 = \{x \in A : \text{there exist } x_1, x_2, \dots, x_n \in D_1 \text{ such that } (x_1, x_2, \dots, x_n; x) \in D_2\}.$$

PROOF.  $x \in D_1 \vee D_2$  iff for some  $x_1, x_2, \dots, x_n \in D_1$  and some  $y_1, y_2, \dots, y_m \in D_2$ ,  $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m; x) = 1$  iff for some  $x_1, x_2, \dots, x_n \in D_1$ ,  $(x_1, x_2, \dots, x_n; x) \in D_2$ .

COROLLARY 2.5. If  $A$  is a Hilbert algebra,  $D \in Ds(A)$  and  $a \in A$ , then

$$[a] \vee D = \{x \in A : a \rightarrow x \in D\}.$$

LEMMA 2.6. If  $A$  is a Hilbert algebra and  $D_1, D_2 \in Ds(A)$ , then

- (i)  $D_1 \rightarrow D_2 \in Ds(A)$
- (ii) If  $D \in Ds(A)$ , then  $D \cap D_1 \subseteq D_2$  iff  $D \subseteq D_1 \rightarrow D_2$ .

PROOF. (i). Since  $[1] = \{1\}$  and  $[1] \cap D_1 = \{1\} \subseteq D_2$  we deduce that  $1 \in D_1 \rightarrow D_2$ .

Let  $x, y \in A$  such that  $x, x \rightarrow y \in D_1 \rightarrow D_2$ . If  $t \in [y] \cap D_1$ , then  $y \leq t$  and  $t \in D_1$ . Since  $x \leq (x \rightarrow t) \rightarrow t \in D_1$ , it follows that  $(x \rightarrow t) \rightarrow t \in [x] \cap D_1 \subseteq D_2$  and so

$$(1) \quad (x \rightarrow t) \rightarrow t \in D_2.$$

Analogously we deduce that

$$(2) \quad ((x \rightarrow y) \rightarrow t) \rightarrow t \in D_2.$$

Since  $y \leq t$ , by  $(c_9)$  we deduce succesively  $x \rightarrow y \leq x \rightarrow t$ ,  $(x \rightarrow t) \rightarrow t \leq (x \rightarrow y) \rightarrow t$  and so

$$(3) \quad ((x \rightarrow y) \rightarrow t) \rightarrow t \leq (((x \rightarrow t) \rightarrow t) \rightarrow t = x \rightarrow t \text{ (by } (c_6)).$$

From (2) and (3) it follows that  $x \rightarrow t \in D_2$  and so  $t \in D_2$ ; therefore  $[y] \cap D_1 \subseteq D_2$  and so  $y \in D_1 \rightarrow D_2$ .

(ii)  $(\Rightarrow)$  Let  $a \in D$  and  $t \in [a] \cap D_1$ ; then  $a \leq t$  and  $t \in D_1$ , so that  $t \in D$ .

Therefore  $t \in D_1 \cap D \subseteq D_2$  and so  $t \in D_2$ . It follows that  $[a] \cap D_1 \subseteq D_2$ , and thus  $a \in D_1 \rightarrow D_2$ .

$(\Leftarrow)$  We suppose  $D \subseteq D_1 \rightarrow D_2$  and let  $x \in D_1 \cap D$ ; then  $x \in D_1$ ,  $x \in D_1 \rightarrow D_2$  and thus  $x \in D_2$  since  $x \in [x] \cap D_1 \subseteq D_2$ .

REMARK 2.7. From lemma 2.6. we deduce that  $(Ds(A), \vee, \wedge, \rightarrow, \{1\}, A)$  is a Heyting algebra; for  $D \in Ds(A)$ ,  $D^* = D \rightarrow \{1\} = \{x \in A : [x] \cap D = \{1\}\}$  and so  $[a]^* = \{x \in A : [x] \cap [a] = \{1\}\}$  ( $a \in A$ ). Moreover, if  $A$  is a Heyting algebra, then  $D^* = \{x \in A : x \vee y = 1, \text{ for every } y \in D\}$ .

THEOREM 2.8. If  $A$  is a bounded Hilbert algebra, then the following assertions are equivalent:

- (i)  $(Ds(A), \vee, \wedge, *, \{1\}, A)$  is a Boolean algebra
- (ii)  $A$  is a finite Boolean lattice.

PROOF. (i) $\Rightarrow$ (ii). Let  $x \in A$ ; since  $Ds(A)$  is Boolean lattice, then  $[x] \vee [x]^* = A$ . By corollary 2.5.,  $[x] \vee [x]^* = \{y \in A : x \rightarrow y \in [x]^*\} = \{y \in A : [x \rightarrow y] \cap [x] = \{1\}\}$  (by remark 2.7). So, for every  $y \in A$

$$(1) \quad [x \rightarrow y] \cap [x] = \{1\}.$$

Since  $x \rightarrow y \leq ((x \rightarrow y) \rightarrow x) \rightarrow x$  and  $x \leq ((x \rightarrow y) \rightarrow x) \rightarrow x$  it follows that

$$(2) \quad ((x \rightarrow y) \rightarrow x) \rightarrow x = 1.$$

Since  $x \leq (x \rightarrow y) \rightarrow x$ , by (2) we deduce that  $(x \rightarrow y) \rightarrow x = x$ ; since  $x$  and  $y$  are arbitrary, from corollary 1.6. it follows that  $A$  is a Boolean lattice.

Now, it is sufficient to prove that every filter of  $A$  is principal, in order to show that  $A$  is finite ([3, p. 77]). If  $D \in Ds(A)$ , then  $D \vee D^* = A$  (because  $Ds(A)$  is supposed Boolean lattice) and thus  $0 \in D \vee D^*$ . By lemma 2.4., there exist

$x_1, x_2, \dots, x_n \in D$  such that  $(x_1, x_2, \dots, x_n; 0) \in D^*$  and so, by remark 2.7., for every  $y \in D$ ,  $(x_1, x_2, \dots, x_n; 0) \vee y = 1 \Leftrightarrow x_1 \wedge x_2 \wedge \dots \wedge x_n \leq y$  (by  $(c_{18})$ ) and so  $D = [a]$ , where  $a = x_1 \wedge x_2 \wedge \dots \wedge x_n \in D$ .

(ii) $\Rightarrow$ (i). Suppose that  $A$  is a finite Boolean lattice. By remark 2.7.,  $Ds(A)$  is a Heyting algebra; to prove  $Ds(A)$  is a Boolean lattice, we must show  $D^* = \{1\}$  only for  $D = A$  ([1, p. 175]). Since in finite Boolean lattices every filter is principal, and hence  $D = [a]$  for some  $a \in A$ . By remark 2.7.,  $D^* = [a]^* = \{x \in A: x \vee y = 1, \text{ for every } y \in [a]\}$ . Since, for  $y \geq a$ ,  $a^* \vee y \geq a^* \vee a = 1$  it follows that  $a^* \vee y = 1$  and so  $a^* \in [a]^* = \{1\}$ , therefore  $a^* = 1$ . It follows  $a = 0$  and so  $D = [a] = [0] = A$ .

### 3. The spectrum of a bounded Hilbert algebra

**DEFINITION 3.1.** One element  $D \in Ds(A)$  is called *irreducible (completely irreducible)* iff if  $D = \bigcap_{i \in I} D_i$  for some finite set  $I$  (for every set  $I$ ) then there exists  $i \in I$  such that  $D = D_i$ .

We denote by  $\text{Spec}(A)$  the set of all completely irreducible deductive systems.  $\text{Spec}(A)$  is called the *spectrum* of  $A$ .

**LEMMA 3.2.** For every  $D \in Ds(A)$  and  $a \in A - D$ , there exists  $M \in \text{Spec}(A)$  such that  $a \notin M$  and  $D \subseteq M$ . If  $a = 0$ , we deduce that for every  $D \in Ds(A)$  such that  $D \neq A$  there exists  $M \in \text{Spec}(A)$ ,  $M \neq A$  such that  $D \subseteq M$ .

**PROOF.** See [2, p. 22].

**COROLLARY 3.3.** If  $A$  is a bounded Hilbert algebra, then for every  $D \in Ds(A)$

$$D = \bigcap \{M \in \text{Spec}(A): D \subseteq M\}.$$

**THEOREM 3.4.** If  $A$  is a bounded Hilbert algebra, then the next assertions are equivalent:

- (i) Every  $D \in Ds(A)$  has a unique representation as an intersection of elements of  $\text{Spec}(A)$
- (ii)  $A$  is a finite Boolean lattice.

*Proof.* (i) $\Rightarrow$ (ii). Now let every element of  $Ds(A)$  have a unique representation as a meet of elements of  $\text{Spec}(A)$ ; we claim that  $Ds(A)$  is Boolean lattice. Let  $D \in Ds(A)$  and  $D' = \bigcap \{M \in \text{Spec}(A): D \not\subseteq M\} \in Ds(A)$ . By corollary 3.3.,  $D \cap D' = \bigcap \{M \in \text{Spec}(A)\} = \{1\}$ ; if  $D \vee D' \neq A$ , then by lemma 3.2. there exists  $D'' \in \text{Spec}(A)$  such that  $D \vee D' \subseteq D''$  and  $D'' \neq A$ . Consequently,  $D'$  has two representations  $D' = \bigcap \{M \in \text{Spec}(A): D \not\subseteq M\} = D'' \cap (\bigcap \{M \in \text{Spec}(A): D \not\subseteq M\})$ , which is contradictory. Therefore  $D \vee D' = A$  and so  $Ds(A)$  is a Boolean lattice.

By theorem 2.8.,  $A$  is a finite Boolean lattice.

(ii) $\Rightarrow$ (i). Clearly (see for example [3]).

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