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SOME ORIENTABLE 3-MANIFOLDS CONTAINING KLEIN BOTTLES

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In this paper we study 3-manifolds obtained from two twisted I-bundles over a Klein bottle by identifying their boundaries.

Let K be a Klein bottle and KI be the (orientable) twisted I-bundle over K where I is the unit interval $[0, 1]$. Then KI admits two Seifert fibrations $\mathscr{F}_1$ and $\mathscr{F}_2$ where the orbit manifold of $\mathscr{F}_1$ is a disk with two exceptional points of each index 2, and the orbit manifold of $\mathscr{F}_2$ is a Möbius band without exceptional points (see Ch. VI of [2]). Let α be an oriented fiber of $\mathscr{F}_1$ in ∂KI and β be an oriented fiber of $\mathscr{F}_2$ in ∂KI. In this paper we often regard oriented simple loops with a common base point as elements of a fundamental group, therefore $\pi_1(\partial KI) \cong \langle \alpha, \beta | \alpha\beta = \beta\alpha \rangle$.

For $i=1, 2$ let K_i, K_iI, α_i and β_i be two copies of K, KI, α and β respectively. Let $A=\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ be an element of $GL(2, Z)$ and let $h: \partial K_2I \to \partial K_1I$ be a homeomorphism which induces an isomorphism $h_*: \pi_1(\partial K_2I) \to \pi_1(\partial K_1I)$ such that $h_*(\alpha_2)=r\alpha_1+s\beta_1$ and $h_*(\beta_2)=t\alpha_1+u\beta_1$. Then we obtain a closed orientable 3-manifold from K_1I and K_2I by identifying their boundaries by h. In this paper we call this manifold a sapphire space of type A or type $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ and denote it by $K(A)$ or $K(r, s, t, u)$.

In section 1 we will classify the topological types of sapphire spaces.

THEOREM 1. *Let A, A' be two elements of $GL(2, Z)$. Then $K(A)$ is homeomorphic to $K(A')$ if and only if A' is equal to one of the following matrices: $\pm A^{\pm 1}$, $\pm BA^{\pm 1}$, $\pm A^{\pm 1}B$ or $\pm BA^{\pm 1}B$, where $B=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

In section 2 we will determine the Heegaard genera of sapphire spaces.

THEOREM 3. *Every sapphire space has a Heegaard splitting of genus three. Moreover $K(r, s, t, u)$ has a Heegaard splitting of genus two if and only if $|s|=1$.*

I would like to express my gratitude to Prof. F. Hosokawa and Prof. S. Suzuki and the members of *KOOK* seminar for their helpful suggestions.

1. Classification of spphire spaces. First we study some properties of the twisted I-bundle over a Klein bottle.

Put $F=S^1\times I$ and let f be the self-homeomorphism on F defined by the equality $f(e^{i\theta}, t)=(e^{-i\theta}, 1-t)$ where S^1 is identified with the unit circle of the complex plane. Then KI is obtained from $F\times I$ by identifying $(x, 0)$ with $(f(x), 1)$ for each $x\in F$. Then each element of KI may be represented by $[x, u]$ which is the image of $(x, u)\in F\times I$.

Let $A_{\mathrm{I}}=\{[(e^{i\pi/2}, t), u]|t, u\in I\}\cup\{[(e^{i3\pi/2}, t), u]|t, u\in I\}$ and $A_{\mathrm{II}}=\{[x, 1/2]|x\in F\}$, then A_{I} and A_{II} are properly embedded annuli in KI. The following two lemmas follow from Theorem VI. 34 of [2].

LEMMA 1.1. *Let A be a properly embedded essential annulus in KI (i.e. A is incompressible and not boundary parallel). Then A is ambient isotopic to A_{I} or A_{II}. We say that A is of type* I *or type* II *according as A is ambient isotopic to A_{I} or A_{II}.*

LEMMA 1.2. *Let T be an incompressible torus in KI then T is boundary parallel.*

REMARK 1.3. An annulus of type I is separating and an annulus of type II is non-separating.

The homeotopy group $\mathscr{H}(X)$ of a space X is the group of all self-homeomorphisms of X modulo the subgroup consisting of those homeomorphisms which are isotopic to the identity.

LEMMA 1.4. *$\mathscr{H}(KI)\cong Z_2+Z_2+Z_2$, where the three self-homeomorphisms of KI generating $\mathscr{H}(KI)$ are T, S and R defined by the following equalities; $T[(e^{i\theta}, t), u]=[(e^{i(\theta+2\pi u)}, t), u]$, $S[(e^{i\theta}, t), u]=[(e^{-i\theta}, t), u]$ and $R[(e^{i\theta}, t), u]=[(e^{i\theta}, t), 1-u]$. Namely T is a twist along A_{II} and S is a reflection about A_{I} and R is a reflection about A_{II}.*

PROOF. Let f be a self-homeomorphism of KI. We will prove that f is isotopic to some product of T, S or R. By Lemma 1.1 and Remark 1.3 we may assume that $f(A_{\mathrm{II}})=A_{\mathrm{II}}$. Let c_1, c_2 be two components of ∂A_{II}. By performing a suitable ambient isotopy of KI, if necessary, we may assume that $f(c_i)=c_i$ $(i=1, 2)$. If $f|A_{\mathrm{II}}: A_{\mathrm{II}}\to A_{\mathrm{II}}$ is isotopic to the identity, then put $g=f$, otherwise put $g=S\circ f$. Then $g|A_{\mathrm{II}}: A_{\mathrm{II}}\to A_{\mathrm{II}}$ is isotopic to the identity. Let N be a regular neighborhood of A_{II} in KI, then we may assume that $g(N)=N$. If $g|N: N\to N$ is isotopic to the identity, then put $h=g$, otherwise put $h=R\circ g$. Then $h|N: N\to N$ is isotopic to the identity. Finally if $h|KI\text{-}IntN: KI\text{-}IntN\to KI\text{-}IntN$ is isotopic to the identity relative to $N\cap\partial(KI\text{-}IntN)$, then put $\rho=h$, otherwise put $\rho=T\circ h$. Then $\rho: KI\to KI$ is isotopic to the identity. Hence we obtain that

$\mathscr{H}(KI)$ is generated by T, S and R. Furthermore by the definitions of T, S and R we can easily check that T^2, S^2 and R^2 are all isotopic to the identity and that any two maps of T, S and R are commutative, no two maps of them are isotopic and they are mutually independent over Z_2. This completes the proof.

COROLLARY 1.5. *Let f be a self-homeomorphism of KI* then $(f|\partial KI)^*$: $\pi_1(\partial KI)\to\pi_1(\partial KI)$ is given by $(f|\partial KI)^*(\alpha)=\varepsilon\alpha$ and $(f|\partial KI)^*(\beta)=\delta\beta$ where $|\varepsilon|=|\delta|=1$.

From now on we consider sapphire spaces.

PROPOSITION 1.6. *Let $K(r, s, t, u)=K_1I\cup K_2I$ be a sapphire space. Then $K(r, s, t, u)$ is irreducible and $\partial K_1I=\partial K_2I$ is a separating incompressible torus in $K(r, s, t, u)$ and*

$$H_1(K(r,s,t,u))\cong\begin{cases} Z_{4t}+Z_2+Z_2 & \text{if } s \text{ is even} \\ Z_{4t}+Z_4 & \text{if } s \text{ is odd} \end{cases}$$

$$\pi_1(K(r,s,t,u))\cong\left\langle \begin{matrix} a_1 \\ b_1 \\ a_2 \end{matrix} \;\middle|\; \begin{matrix} a_1b_1a_1^{-1}=b_1^{-1} \\ a_2^2=a_1^{2r}b_1^s \\ a_2a_1^{2t}b_1^u\,a_2^{-1}=b_1^{-1}a_1^{-2t} \end{matrix} \right\rangle.$$

In particular $\pi_1(K_iI)$ can be identified with $\langle a_i, b_i|a_ib_ia_i^{-1}=b_i^{-1}\rangle$ for each $i=1, 2$. Hence a_i, b_i are two oriented simple loops in K_i such that a_i^2 is homotopic to α_i in K_iI and b_i is homotopic to β_i in K_iI $(i=1, 2)$.

PROOF. The first half follows from the irreducibility of KI and the incompressibility of ∂KI in KI. The latter half is easily calculated by Van Kampen Theorem.

LEMMA 1.7. *Let T be a separating incompressible torus in a sapphire space $M=K_1I\cup K_2I$. Then T is ambient isotopic to a torus T' satisfying one of the following three conditions.*

(1) $T'=\partial K_1I=\partial K_2I$

(2) *For $i=1, 2$ each component of $T'\cap K_iT$ is an annulus of type I.*

(3) *$T'\cap K_1I$ are two annuli of type I and $T'\cap K_2I$ are two annuli of type* II *or vice versa.*

PROOF. By the arguments of general position we may assume that each component of $T\cap\partial K_1I$ is a circle. By the incompressibility of T and ∂K_1I and the irreducibility of M we can remove the trivial intersections of $T\cap\partial K_1I$. Therefore we may assume that $T\cap K_1I=A_1\cup A_2\cup\cdots\cup A_n$ and $T\cap K_2I=B_1\cup B_2\cup\cdots\cup B_n$ for some non-negative integer n where A_i is an essential annulus in

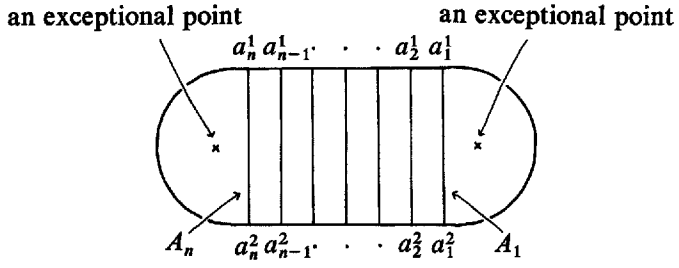
K_1I and B_i is an essential annulus in K_2I. Let h: $\partial K_2I \to \partial K_1I$ be an attaching homeomorphism and $\partial A_i = a_i^1 \cup a_i^2$, $\partial B_i = b_i^1 \cup b_i^2$ be two components of ∂A_i and ∂B_i respectively.

If $n=0$, then by Lemma 1.2 T is ambient isotopic to ∂K_1I. Therefore (1) holds.

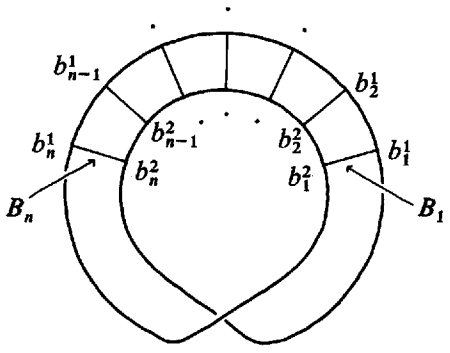
If $n \neq 0$, then we have the following three cases.

Case 1: Both $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are of type I. In this case (2) holds.

Case 2: Either $\{A_i\}_{i=1}^n$ or $\{B_i\}_{i=1}^n$ is of type I and the other is of type II. Without loss of generality we may assume that $\{A_i\}_{i=1}^n$ is of type I and $\{B_i\}_{i=1}^n$ is of type II. By replacing the letters, if necessary, we may assume that $h(b_i^1) = a_i^1$ $(i=1, 2,..., n)$. Then we have $h(b_i^2) = a_{n+1-i}^2$ $(i=1, 2,..., n)$. See Fig. 1.1 and Fig. 1.2, where Fig. 1.1 is the image of $\{A_i\}_{i=1}^n$ in the orbit manifold of $\mathscr{F}_1$ by a projection along $\mathscr{F}_1$ and Fig. 1.2 is the image of $\{B_i\}_{i=1}^n$ in the orbit manifold of $\mathscr{F}_2$ by a projection along $\mathscr{F}_2$.



(Fig. 1.1)



(Fig. 1.2)

Then $A_1 \cup A_n \cup B_1 \cup B_n$ is a separating incompressible torus in M. Hence n must be 2 and (3) holds.

Case 3: Both $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are of type II. We may assume that $h(b_i^1) = a_i^1$ $(i=1, 2,..., n)$. Then we have $h(b_i^2) = a_i^2$ $(i=1, 2,..., n)$. Then

$A_1 \cup B_1$ is a non-separating torus in M (see Fig. 1.2). This is a contradiction.

PROOF OF THEOREM 1.

Let h and h' be two attaching homeomorphisms from $\partial K_2 I$ to $\partial K_1 I$ with $h_* = A$ and $h'_* = A'$.

If A' is equal to one of the matrices in Theorem 1, then it is clear that $K(A)$ is homeomorphic to $K(A')$.

Conversely assume that $K(A)$ is homeomorphic to $K(A')$ and let f: $K(A) \to K(A')$ be a homeomorphism. Then by Lemma 1.7 we have the following three cases.

Case 1: $f(\partial K_1 I)$ is ambient isotopic to $\partial K_1 I$ in $K(A')$. We may assume that $f(\partial K_1 I) = \partial K_1 I$. Then from Corollary 1.5 it is easily calculated that A' is equal to one of the matrices in Theorem 1.

Case 2: $f(\partial K_1 I)$ is ambient isotopic to T' of case (2) in Lemma 1.7. We may assume that $f(\partial K_1 I) = T'$. Then we have $h'_*(\alpha_2) = \pm\alpha_1$. Similarly by observing f^{-1} we have $h_*(\alpha_2) = \pm\alpha_1$. Hence we have $A = \begin{pmatrix} \varepsilon & 0 \\ t & \delta \end{pmatrix}$ and $A' = \begin{pmatrix} \varepsilon' & 0 \\ t' & \delta' \end{pmatrix}$ where $|\varepsilon| = |\delta| = |\varepsilon'| = |\delta'| = 1$. From Proposition 1.6 $H_1(K(A)) \cong Z_{4t} + Z_2 + Z_2$ and $H_1(K(A')) \cong Z_{4t'} + Z_2 + Z_2$, so $|t| = |t'|$. Therefore A' is equal to one of the matrices in Theorem 1.

Case 3: $f(\partial K_1 I)$ is ambient isotopic to T' of case (3) in Lemma 1.7. We may assume that $f(\partial K_1 I) = T'$. Then we have $h'_*(\alpha_2) = \pm\beta_1$. Similarly $h_*(\alpha_2) = \pm\beta_1$. Hence we have $A = \begin{pmatrix} 0 & \varepsilon \\ \delta & u \end{pmatrix}$ and $A' = \begin{pmatrix} 0 & \varepsilon' \\ \delta' & u \end{pmatrix}$ where $|\varepsilon| = |\delta| = |\varepsilon'| = |\delta'| = 1$. Further we may assume that $\varepsilon = \delta = \varepsilon' = \delta' = 1$. In this case both $K(A)$ and $K(A')$ are homeomorphic to a Seifert fibered space whose orbit manifold is a projective plane with two exceptional points of each index 2. The fundamental group G of such a manifold has a presentation as follows: $G \cong \langle v_1, q_1, q_2, h | q_1 h q_1^{-1} = h, q_2 h q_2^{-1} = h, v_1 h v_1^{-1} = h^{-1}, q_1^2 h = 1, q_2^2 h = 1, q_1 q_2 v_1^2 = h^b \rangle$ where b is an integer representing an obstruction class (see Ch. 1 and Ch. 5 of [6]). By a little calculation we have $G \cong \langle v_1, q_1 | v_1 q_1^2 v_1^{-1} = q_1^{-2}, q_1 v_1^2 q_1^{2(b+1)} q_1^{-1} = q_1^{-2(b+1)} v_1^{-2} \rangle$. On the other hand by Proposition 1.6 $\pi_1(K(0, 1, 1, u)) \cong \langle a_1, a_2 | a_1 a_2^2 a_1^{-1} = a_2^{-2}, a_2 a_1^2 a_2^{2u} a_2^{-1} = a_2^{-2u} a_1^{-2} \rangle$. Hence we see that $u-1$ corresponds to b. Then by Theorem 3 of section 2 of Ch. 5 of [6], we get $u-1 = u'-1$ or $u-1 = -(u'-1)-2$, for the number of the exceptional points is 2. Thus we have $|u| = |u'|$ and this completes the proof.

THEOREM 2. *$K(r, s, t, u)$ is a torus bundle over S^1 if and only if $t=0$. At this time the monodromy map is equal to* $\begin{pmatrix} -1 & s \\ 0 & -1 \end{pmatrix}$.

PROOF. Let $K(r, s, t, u)$ be a torus bundle over S^1. Then $H_1(K(r, s, t, u))$ is an infinite group, so by Proposition 1.6 we have $t=0$. Conversely assume

that $t=0$ then it is easy to see that a structure of an annulus bundle over S^1 on K_iI $(i=1, 2)$ extends a structure of a torus bundle over S^1 on $K(r, s, 0, u)$. Furthermore by Proposition 1.6 $\pi_1(K(r, s, 0, u))\cong\langle a_1, b_1, a_2 | a_1b_1a_1^{-1}=b_1^{-1}, a_2b_1a_2^{-1}=b_1^{-1}, a_2^2=a_1^2b_1^s\rangle$. Let $\alpha=a_2a_1^{-1}$ and $\beta=b_1$ and $t=a_1$ then $\pi_1(K(r, s, 0, u))\cong\langle\alpha, \beta, t | \alpha\beta=\beta\alpha, t\alpha t^{-1}=\alpha^{-1}\beta^s, t\beta t^{-1}=\beta^{-1}\rangle$. This shows that the monodromy map is $\begin{pmatrix}-1 & s\\ 0 & -1\end{pmatrix}$ (c.f. Ch. 12 of [1]).

REMARK 1.8. It follows from the Seifert fibrations of KI and Theorem VI .34 of [2] that $K(r, s, t, u)$ is a Seifert fibered space if and only if at least one of r, s, t and u is 0.

2. Heegaard splittings of sapphire spaces

LEMMA 2.1. *Let M be a sapphire space and K be a Klein bottle in M and $N(K)$ be a regular neighborhood of K in M. Then both $N(K)$ and $Cl(M-N(K))$ are homeomorphic to KI.*

PROOF. Since M is orientable $N(K)$ is homeomorphic to KI. Let $T=\partial N(K)$ then T is a separating incompressible torus, so T is ambient isotopic to one of the tori of Lemma 1.7. Therefore $Cl(M-N(K))$ is homeomorphic to KI.

LEMMA 2.2. (Theorem 2 of [5]). *Let M be a closed orientable 3-manifold containing a Klein bottle. Then M has a Heegaard splitting of genus two if and only if M belongs to one of $C(1)$, $C(2)$ or $C(3)$ defined as follows;*

$C(1)$: Let k be a two bridge knot (possibly trivial) and $M(k)$ be its exterior in S^3 (c.f. Ch. 4 of [7]). Let μ_1, μ_2 be two disjoint meridians of k in $\partial M(k)$ and $\bar{\mu}_1$, $\bar{\mu}_2$ be two disjoint simple loops in $IntM(k)$ obtained by pushing μ_1, μ_2 into $IntM(k)$. Let M_1 be a 3-manifold obtained from $M(k)$ by performing arbitraly Dehn surgeries on $M(k)$ along $\bar{\mu}_1$ and $\bar{\mu}_2$. Then $C(1)$ is the family which consists of all 3-manifods obtained from M_1 and KI by identifying ∂KI with ∂M_1 by a homeomorphism which takes β to μ_1.

$C(2)$: Let $M(k)$, μ_1 and $\bar{\mu}_1$ be as in $C(1)$. Let M_2 be a 3-manifold obtained from $M(k)$ by performing an arbitraly Dehn surgery on $M(k)$ along $\bar{\mu}_1$. Then $C(2)$ is the family which consists of all 3-manifolds obtained from M_2 and KI by identifying ∂KI with ∂M_2 by a homeomorphism which takes α to μ_1.

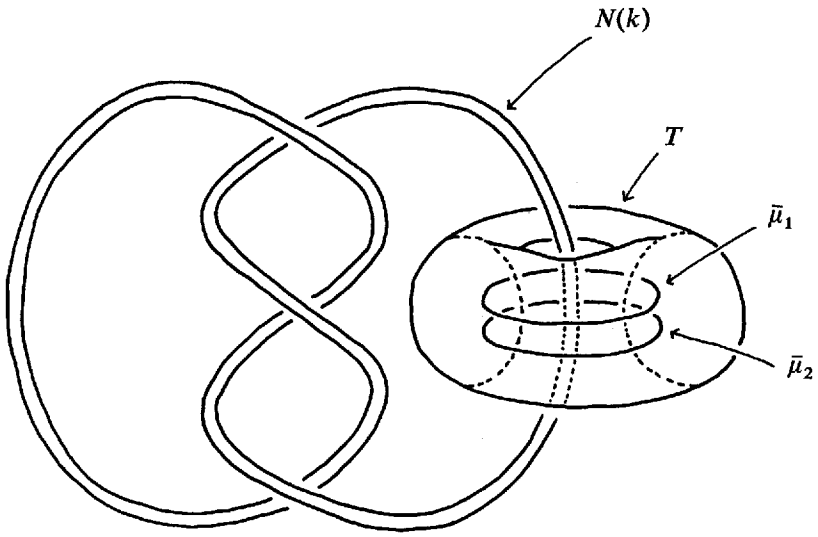
$C(3)$: Let L be a lens space (including $S^2\times S^1$ and S^3) and $L(k)$ be a "one bridge knot" exterior in L (see [4] or [5]). Let μ be a meridian of k in $\partial L(k)$. Then $C(3)$ is the family which consists of all 3-manifolds obtained from $L(k)$ and KI by identifying ∂KI with $\partial L(k)$ by a homeomorphism which takes α to μ.

PROOF OF THEOREM 3.

Let $K(A)=K_1I \cup K_2I$ and $A=\begin{pmatrix} r & s \\ t & u \end{pmatrix}$. The first half of Theorem 3 is proved by using an argument similar to the proof of Theorem 1 of [8]. Furthermore in the case of $|s|=1$, using an argument similar to the proof of Proposition 3 of [8], we can prove that $K(A)$ has a Heegaard splitting of genus two.

Conversely suppose that $K(A)$ has a Heegaard splitting of genus two. Then $K(A)$ belongs to one of $C(1)$, $C(2)$ or $C(3)$ of Lemma 2.2.

If $K(A)$ belongs to $C(1)$, then $K(A)=M_1 \cup KI$ where M_1 is a 3-manifold obtained from a two bridge knot exterior $M(k)$ by performing Dehn surgeries along $\bar{\mu}_1$ and $\bar{\mu}_2$ with surgery coefficients p_i/q_i for coprime integers (p_i, q_i) $(i=1, 2)$ (c.f. Ch. 9 of [7]). By Lemma 2.1, M_1 is homeomorphic to KI, so $H_1(M_1)\cong Z+Z_2$. On the other hand $H_1(M_1)\cong Z+Z_d$ where $d=g.c.d.(p_1, p_2)$. Therefore $d=2$, and so $|p_i|\geq 2$ $(i=1, 2)$. Let T be a torus in $M(k)$ which bounds a solid torus in $M(k)$ and contains $\bar{\mu}_1 \cup \bar{\mu}_2$ as the core of it as in Fig. 2.



(Fig. 2)

Let T' be the image of T after the Dehn surgeries, then T' is a separating incompressible torus in M_1 since $|p_i|\geq 2$ $(i=1, 2)$. By Lemma 1.2 T' is boundary parallel, so k is a trivial knot. Hence M_1 admits a Seifert fibration whose orbit manifold is a disk with two exceptional points of indices p_1 and p_2, and the meridian μ of k is a fiber. Since M_1 is homeomorphic to KI, $|p_1|=|p_2|=2$ and the fibration is $\mathscr{F}_1$. Furthermore the attaching homeomorphism from ∂KI to ∂M_1 takes β to μ, so $u=0$ and $|s|=1$.

If $K(A)$ belongs to $C(2)$, then $K(A)=M_2 \cup KI$ where M_2 is a 3-manifold

obtained from a two bridge knot exterior $M(k)$ by performing a Dehn surgery along $\tilde{\mu}_1$. By Lemma 2.1 M_2 is homeomorphic to KI, so $H_1(M_2)\cong Z+Z_2$. However $H_1(M_2)\cong Z$. This is a contradiction. Therefore $K(A)$ does not belong to $C(2)$.

If $K(A)$ belongs to $C(3)$, then $K(A)=L(k)\cup KI$ where $L(k)$ is a one bridge knot exterior in a lens space L. By Lemma 2.1 $L(k)$ is homeomorphic to KI, so we can regard $L(k)$ as K_1I and KI as K_2I. Since the attaching homeomorphism takes α_2 to μ and $h_*(\alpha_2)=r\alpha_1+s\beta_1$, we have $\mu=r\alpha_1+s\beta_1$ where μ is a meridian of k in $L(k)$. Therefore L is homeomorphic to the 3-manifold $M(|s|, r)$ defined in [3]. Then by Theorem 2.4 of [3] we have $|s|=1$. This completes the proof.

References

[1] J. Hemple: 3-manifolds, Ann. of Math. Studies No. 86 Princeton N. J., Princeton University Press, 1976.

[2] W. Jaco: Lectures of three manifold topology, Conference board of Math. No. 43, 1980.

[3] P. K. Kim: Some 3-manifolds which admit Klein bottles, Trans. Amer. Math. Soc. vol. 244 (1978), 299–312.

[4] T. Kobayashi: Structures of the Haken manifolds with Heegaard splittings of genus two, Osak J. Math. **21** (1984), 437–455.

[5] K. Morimoto: Klein bottles in genus two 3-manifolds, Osaka J. Math. (to appear).

[6] P. Orlik: Seifert manifolds, Lectures Note in Math. 291, Berlin-Heidelberg-New York: Springer verlag, 1972.

[7] D. Rolfsen: Knots and Links, Mathematics Lectures Series 7, Publish or Perish Inc., Berkeley, Ca. 1976.

[8] M. Takahashi and M. Ochiai: Heegaard diagrams of torus bundles over S^1, Comment. Math. Univ. Spancti Pauli, **31** (1982), 63–69.

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