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A CHARACTERIZATION OF THE CLASS OF CONTRACTION TYPE MAPPINGS

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ABSTRACT

In this paper we introduce the concept of a *contraction type mapping* T of a metric space X into itself i.e. of a mapping $T:X\to X$ such that for every $x, y\in X$ there exists a sequence of nonnegative real functions $\Delta_{n,r}(x, y)$ with $\Delta_{n,r}(x, y)\to 0 (r>n\to\infty)$ and positive integer $m(x, y)$ such that $\rho[T^n x, T^r y]\leq \Delta_{n,r}(x, y)$ for all $r>n\geq m(x, y)$. It is proved that necessary and sufficient conditions that a mapping T of a metric space X into itslef has the all conclusions of Banach's contraction principle and $\rho[T^n x, \xi]\leq \Delta_n(x, Tx)$ for $n\geq m(x)$ where $x\mapsto\Delta_n(x, Tx)$ are real nonnegative functions with $\Delta_n(x, Tx)\to 0 (n\to\infty)$ are the following ones: T is a contraction type mapping where $\Delta_{n,r}(x, y)=\max\{\Delta_n(x, Tx), \Delta_r(y, Ty)\}$, T is orbitally continuous and X is T-orbitally complete. *Almost contraction type mappings* are also discussed. This paper gives sufficient conditions for the solution of a integral equation (*IE*).

1. Introduction and results. Let (X, ρ) be a metric space. A mapping T of the space X into itself is said to be a *Bauach coutraction* if there exists a number $\alpha\in[0, 1)$ such that

(B) $$\rho[Tx, Ty]\leq \alpha\rho[x, y], \text{ for any two points } x, y\in X.$$

Every contraction mapping is continuous.

In the fixed point theory of contractive operators (mappings which shrink distance in some manner) on metric space it is well known the result of a Polish mathematician S. Banach [1]. The Banach contraction principle is the following:
If T is a contraction mapping of a complete metric space X into itself, then

(a) *T has a unique fixed point $\xi\in X$,*

(b) *$x_n = T^n x\to\xi$ for every $x\in X$*

and (c) $\rho[T^n x, \xi]\leq \alpha^n(1-\alpha)^{-1}\rho[x, Tx]$, *for every* $x\in X$.

In other words, if T is a contraction mapping on a complete metric space X, then the equation $Tx=x$ has in X a unique solution. Caccioppoli R. [3] observed that a mapping $T:X\to X$ on a complete metric space (X, ρ) has a unique fixed point if for each positive integer n there exists a constant C_n for all $x, y\in X$ such that

(C) $$\rho[T^n x, T^n y]\leq C_n\rho[x, y] \quad \textit{and} \quad \sum_{n=1}^{\infty} C_n < +\infty.$$

M. Edelstein, in [6], has extended the Banach's contraction principle introducing the notion of (ε, λ)-*uniformly locally contractive mapping* i.e., such a mapping $T: X \to X$ of a metric space X into X so that

$$\text{(E)} \qquad p, q \in S(x, \varepsilon) \Longrightarrow \rho[Tp, Tq] < \lambda\rho[p, q]$$

for every $x \in X$, where ε and $\lambda \in [0, 1)$ do not depend on x. The space X is ε-chainable if for every two $a, b \in X$ there is a sequence of elements in X: $x_0, x_1, \ldots, x_n$ such that $x_0 = a$, $x_n = b$ and $\rho[x_i, x_{i+1}] < \varepsilon$ $(i = 0, 1, \ldots, n-1)$. So the Edelstein's extension reads: If X is a complete, ε-chainable metric space and $T: X \to X$ is an (ε, λ)-uniformly locally contractive mapping then there exists a unique point $\xi \in X$ such that $T\xi = \xi$.

Contractive operators have been of interest for many years. We list below some of contractive mappings for which various fixed point theorems have been established. We suppose that T is an operator of a metric space (X, ρ) into itself and that $\alpha \in [0, 1)$.

(1) (Bryant [2]) T is said to be with *contractive iteration* if there exists some positive integer k such that

$$\rho[T^k x, T^k y] \leq \alpha\rho[x, y], \qquad (x, y \in X).$$

(2) (Marjanović [12]) T is said to be (U, k, α)-*contraction* $(U \subset X \times X$, $k \in N)$ if $(x, y) \in U$ implies $(Tx, Ty) \in U$ and

$$\rho[T^k x, T^k y] \leq \alpha \max \{\rho[T^i x, T^i y]: i = 0, 1, \ldots, k-1\}.$$

(3) (Sehgal [15]) T is said to be with *contractive operator* if for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$

$$\rho[T^{n(x)}(y), T^{n(x)}(x)] \leq \alpha\rho[x, y].$$

(4) (Iseki [9]) T is said to be *contractive iterate* if for each $x \in X$, there is an integer $n(=n(x)) \geq 1$ such that for all $y \in X$

$$\rho[T^n x, T^n y] \leq \alpha(\rho[x, T^n x] + \rho[y, T^n y]) + \beta(\rho[x, T^n y] + \rho[y, T^n x]) + \gamma\rho[x, y],$$

where α, β, γ are nonnegative and $4\alpha + 4\beta + \gamma < 1$.

(5) (Istratescu [11]) T is said to be *convex contraction* of order 3 if there exist $a, b, c \in [0, 1)$ such that for all x, y in X,

$$\rho[T^3 x, T^3 y] \leq a\,\rho[T^2 x, T^2 y] + b\rho[Tx, Ty] + c\rho[x, y]$$

and where $a + b + c < 1$.

(6) (Fisher [8], Ray and Rhoades [13]) T is said to be with *quasi contrac-*

tion if for all $x, y \in X$

$$\rho[T^i x, T^j y] \leq \alpha \max \{\rho[T^r x, T^s y], \rho[T^r x, T^n x], \rho[T^s y, T^m y]\}$$

where $0 \leq r$, $n \leq i$, $0 \leq s$, $m \leq j$ and i, j are fixed positive integers.

(7) (Ćirić [4]) T is said to be a *q-contraction* if for every $x, y \in X$ there exist numbers $q(x, y) \in [0, 1)$ and $\delta(x, y) > 0$ such that

$$\rho[T^n x, T^n y] \leq (q(x, y))^n \delta(x, y), \ (n = 1, 2, \ldots).$$

(8) (Rakotch [14]) T is said to be an *ε-weakly contractive mapping* if there exists a function $q: (0, \varepsilon) \to [0, 1)$ with $\sup \{q(t): g \leq t \leq h, 0 < g < h < \varepsilon\} < 1$

$$0 < \rho[x, y] < \varepsilon \quad \textit{implies} \quad \rho[Tx, Ty] \leq q(\rho[x, y])\rho[x, y].$$

(9) (Jaggi D. S. [16]) T is said to be *orbitally contraction* if there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that for all $x, y \in X (x \neq y)$, $\rho[Tx, Ty] \leq \alpha\rho[x, Tx]\rho[y, Ty](\rho[x, y])^{-1} + \beta\rho[x, y]$.

(10) (Iseki [10], Ćirić [5]) T is said to be *generalized contraction* if for all $x, y \in X$ $\min \{\rho[Tx, Ty], \rho[x, Tx], \rho[y, Ty]\}$-$\min \{\rho[x, Ty], \rho[y, Tx]\} \leq \alpha\rho[x, y]$.

In this paper we will investigate a class of mappings from which we can obtain all three good properties of Banach contraction principle. Namely, in connexion with this, we shall introduce the concept of *contraction type mapping* T of a metric space X into itself i.e. of *a mapping* $T: X \to X$ *such that for every* $x, y \in X$ *there exists a sequence of nonnegative real functions* $\Delta_{n,r}(x, y)$ with $\Delta_{n,r}(x, y) \to 0 (r \geq n \to \infty)$ *and positive integer* $m(x, y)$ *such that*

(T) $$\rho[T^n x, T^r y] \leq \Delta_{n,r}(x, y) \quad \textit{for all} \quad r > n \geq m(x, y).$$

It is easy to see that contractions satisfy (T) with $\Delta_{n,r}(x, y) = \alpha^n (1 - \alpha)^{-1} \rho[x, Tx]$. A contraction type mapping need not be continuous, as it is shown by the example 3.

Also, in this section we introduced the concept of a *Σ-contraction* T of a metric space X into itself i.e. *of a mapping* $T: X \to X$ *such that for every* $x, y \in X$ *there exist numbers* $C_n(x, y) > 0$ *and* $K(x, y) > 0$ *such that*

(MT) $$\rho[T^n x, T^n y] \leq C_n(x, y) K(x, y), \quad (n = 1, 2, \ldots),$$

where $\sum_{n=1}^{\infty} C_n(x, y) < +\infty$.

A mapping T of a metric space X into itself is said to be *orbitally continuous* if for every $x \in X$ $T(T^{n_i} x) \to T\xi$ whenever $T^{n_i} x \to \xi \in X$.

A space X is said to be *orbitatty comptete* if every sequence $\{T^{n_i} x\}_{i \in N}, x \in X$, which is a Cauchy sequence, has a limit point in X.

If X is a complete space, then X is T-orbitally complete for any mapping $T: X \to X$.

EXAMPLE 1. Let $X=(0, 1]$ and let f and g be two mappings of X into itself defined as follows $f(x)=(x+1)/2$, $g(x)=x/2$. Then X is f-orbitally complete, but is not g-orbitally complete.

THEOREM 1. *Necessary and sufficient conditions that a mapping T of a metric space X into itself has the following properties* (a), (b) *and*

$$(d) \quad \rho[T^n x, \xi] \leq \Delta_n(x, Tx), \text{ for } n \geq m(x),$$

where $x \mapsto \Delta_n(x, Tx)$ *are real nonnegative functions with* $\Delta_n(x, Tx) \to 0 (n \to \infty)$ *are the following ones:*

(e) *T is a contraction type mapping, where*
$\Delta_{n,r}(x, y) = \max \{\Delta_n(x, Tx), \Delta_r(y, Ty)\}$,

(f) *T is orbitally continuous, and*

(g) *X is T-orbitally complete.*

PROOF. *Necessity.* For arbitrary $x, y \in X$, by (d), we obtain

$$\rho[T^n x, T^r y] \leq \rho[T^n x, \xi] + \rho[\xi, T^r y] \leq \Delta_n(x, Tx) + \Delta_r(y, Ty), \ (n, r \geq \max \{m(x), m(y)\}).$$

Hence

$$\rho[T^n x, T^r y] \leq \Delta_{n,r}(x, y), \ r > n \geq m(x, y) = \max \{m(x), m(y)\},$$

where $\Delta_{n,r}(x, y) = 2 \max \{\Delta_n(x, Tx), \Delta_r(y, Ty)\} \to 0 (n \to \infty)$, i.e. T is a contraction type mapping. From (a) and (b) ((b) is a consequence of (d)) it follows that for any $x \in X$ every sequence $\{T^{n_i} x\}$ is a Cauchy sequence which converges to a point $\xi = T\xi \in X$, i.e. (f) and (g) are satisfied.

Sufficiency. To prove the sufficiency, first we shall construct a sequence $\{x_n\}$ of X in such a way that $x_n = T^n x$ with $x \in X$. Hence, by (e) we obtain

$$(11) \qquad \rho[T^n x, T^{n+r} x] \leq \Delta_{n,n+r-1}(x, Tx), \qquad n + r - 1 > n \geq m(x, Tx).$$

Since $\Delta_{n,n+r-1}(x, Tx) \to 0$ from (11) it follows that the sequence $\{x_n\} = (T^n x\}$ is a Cauchy sequence and, by (g) there exists a point $\xi \in X$ such that $\xi = \lim T^n x$. Then from (f) it follows $T\xi = T(\lim T^n x) = \lim Tx_n = \lim x_{n+1} = \xi$, i.e. ξ is a fixed point under T. Now we shall show that ξ is the unique solution of $Tx = x$. If $p \in X$ is such that $p = Tp$, then from (e) it follows

$$\rho[\xi, p] = \rho[T^n \xi, T^r p] \leq \Delta_{n,r}(\xi, p), \quad r > n \geq m(\xi, p).$$

Since $\Delta_{n,r}(\xi, p) \to 0 (n \to \infty)$, we conclude that $\rho[\xi, p] = 0$, that is $\xi = p$. Therefore we proved (a). Since x was arbitrary, we proved also (b). Relation (d) follows

from (11) when $r \to \infty$. So we proved the theorem.

An immediate corollary (special case) of the preceding statement (Theorem 1) is

THEOREM 2. *Necessary and sufficient conditions that a mapping T of a metric space X into itself has the following properties (a) and (b), are the following ones (f), (g) and*

$$(CT) \qquad \lim_{r>n\to\infty} \rho[T^n x, T^r y] = 0 \qquad \text{for all} \quad x, y \in X.$$

Otherwise, the condition (T) implies the condition (CT).

REMARK 1. The conditions (f), (g) and

$$(12) \qquad \lim_{n\to\infty} \rho[T^n x, T^n y] = 0 \qquad \text{for all} \quad x, y \in X$$

are not sufficient for the (a), i.e. the condition (CT) can be deleted if (a) is to be valid. We show that by the following example.

EXAMPLE 2. Let $X = R$ be the set of real numbers R with usual metric and define $T: R \to R$ by $T(x) = ln(1+\exp x)$. Since R is T-orbitally complete and T is orbitally continuous, thus (f) and (g) holds. Furthermore, for any $x, y \in R$

$$\rho[T^n x, T^n y] = |T^n x - T^n y| = ln \frac{n+\exp x}{n+\exp y} \longrightarrow 0 (n \to \infty).$$

Thus (12) holds, but condition (a) is not satisfied, i.e. T does not have a fixed point.

REMARK 2. A contraction type mapping $T: X \to X$ need not be continuous, even at a point $\xi \in X$ to which all sequences $\{T^n x\}$ converge, $x \in X$. The following example shows it.

EXAMPLE 3. Let $X = [-2, 2]$ and let $T: X \to X$ be defined as follows: $Tx = x/3$ for $x \in [0, 2]$ and $Tx = 1$ for $x \in [-2, 0)$. The mapping T is a contraction type mapping with $\Delta_{n,r}(x, y) = 1/n|x-y|$, has a unique fixed point $\xi = 0$ and is orbitally continuous. But T is not continuous at the point $\xi = 0$.

THEOREM 3. *Let X be a metric space with metric ρ and T a map of X into X. The conditions (a), (b) and for each $x \in X$*

$$(d_1) \quad \rho[T^n x, \xi] \leq (\sum_{k=n}^{\infty} C_k(x, Tx)) K(x, Tx), \ (n = 1, 2, \ldots)$$

are necessary for the conditions (f), (g) and Σ-contraction.

On the other hand, the conditions (a), (b) *and* (d_1) *with* $\sum_{n=1}^{\infty}(\sum_{k=n}^{\infty} C_k)<+\infty$ *are sufficient for* (f), (g) *and* Σ*-contraction.*

PROOF. We shall show that the conditions (f), (g) and (MT) are sufficient for (a), (b) and (d_1).

To prove the sufficiency, first we shall construct a sequence $\{x_n\}$ of X in such a way that $x_n=T^nx$ with $x\in X$. This sequence is a Cauchy sequence:

$$\begin{aligned}\rho[T^nx,\ T^{n+r}x] &\le \rho[T^nx,\ T^{n+1}x]+\cdots+\rho[T^{n+r-1}x,\ T^{n+r}x]\\ &\le C_n(x,\ Tx)K(x,\ Tx)+\cdots+C_{n+r-1}(x,\ Tx)K(x,\ Tx)\end{aligned}$$

and hence

$$\rho[T^nx,\ T^{n+r}x]\le(\textstyle\sum_{k=n}^{\infty}C_k(x,\ Tx))K(x,\ Tx), \tag{13}$$

i.e. the sequence $\{x_n\}=\{T^nx\}$ is a Cauchy sequence and, by (g), there exists a point $\xi\in X$ such that $\xi=\lim x_n=\lim T^nx$. Then from (f) it follows $T\xi=T(\lim x_n)=\lim Tx_n=\lim x_{n+1}=\xi$, i.e. ξ is a fixed point under T. Now we shall show that ξ is the unique solution of $Tx=x$. If $\delta\in X$ is such that $\delta=T\delta$, then from (MT) it follows

$$\rho[\xi,\ \delta]=\rho[T^n\xi,\ T^n\delta]\le C_n(\xi,\ \delta)K(\xi,\ \delta).$$

Since $C_n(\xi,\ \delta)\to 0(n\to\infty)$, we conclude that $\rho[\xi,\ \delta]=0$, that is $\xi=\delta$. Therefore we proved (a). Since x is arbitrary, we proved also (b). Relation (d_1) follows from (13) when $r\to\infty$.

In an analogous to the Theorem 1, we prove that the conditions (a), (b) and (d_1) are sufficient for (f), (g) and Σ-contraction. So we proved the theorem.

2. Further results and applications

COROLLARY 1. (Edelstein [6]) *Let* X *be a complete* ε*-chainable metric space and let* $T\colon X\to X$ *be* $(\varepsilon,\ \lambda)$*-uniformly locally contractive. Then there exists a unique point* $\xi\in X$ *such that* $T\xi=\xi$.

PROOF. For arbitrary $x,\ y\in X$ put

$$l(x,\ y)=\inf\{\sum_{i=1}^{n}\rho[u_{i-1},\ u_i]\colon x=u_0,\ u_1,\ldots,\ u_n=y \text{ an } \varepsilon\text{-}chain\},$$

and let $m=m(x,\ y)$ be the least positive integer so that $\lambda^m l(x,\ y)\le\rho[x,\ y]$. Then, from (E) for $n,\ r\ge m(x,\ y)$ we have,

$$\rho[T^nx,\ T^ry]\le\lambda^{n-m}(1-\lambda)^{-1}\max\{\rho[x,\ y],\ \rho[x,\ Tx]\}:=\Delta_{n,r}(x,\ y)$$

where $\Delta_{n,r}(x, y)\to 0 (n\to\infty)$. Since T is continuous on X, all hyphotheses of Theorem 1 are fulfilled.

COROLLARY 2. (Bryant [2], Sehgal [15]) *Let $T: X\to X$ be a mapping of a complete metric space into itself. If T has a contractive iteration, then T has a unique fixed point.*

PROOF. For arbitrary $x, y\in X$ put

$$t(x, y) = \max\{\rho[T^i x, T^i y]: \quad i = 0, 1,..., k-1\}$$

and let $m=m(x, y)$ be the least positive integer such that $\alpha^{m/k}t(x, y)\leq\alpha\rho[x, y]$. Then, from (1) we have

$$\rho[T^n x, T^r y] \leq \alpha^{(n-m)/k}(1-\alpha)^{-1}\max\cdot\{\rho[x, y], \rho[x, Tx]\}$$
$$:= \Delta_{n,r}(x, y), (n, r \geq m(x, y)).$$

Therefore, T is an contraction type mapping with a $\Delta_{n,r}(x, y)\to 0 (n\to\infty)$. Since for each $x\in X$ the sequence $\{T^n x\}$ converges to a unique fixed point in X, say ξ, then

$$\lim T^{n_i}(x) = \xi \text{ implies } T(\lim T^{n_i}(x)) = T\xi = \xi = \lim T^{n_i+1}x,$$

which means that T is orbitally continuous.

COROLLARY 3. (Iseki [9]) *Let (X, ρ) be a complete metric space and let $T: X\to X$ be a continuous mapping. If T is a contractive iterate, then T has a unique fixed point.*

PROOF. For every $x, y\in X$ and $m=m(x, y)=1$ from (4), we have,

$$\rho[T^n x, T^r y] \leq q^n(1-q)^{-1}s(x), \qquad n, r \geq m(x, y) = 1,$$

where $q=(3\alpha+3\beta+\gamma)(1-\alpha-\beta)^{-1}$ and $s(x)=\sup\{\rho[T^n x, x]: n\in N\}$. Therefore, T is an contraction type mapping with a $\Delta_{n,r}(x, y)=q^n(1-q)^{-1}s(x)\to 0$ $(n\to\infty)$. Since T is continuous on X all hiphotheses of Theorem 1 are fulfilled.

COROLLARY 4. (Marjanović [12]) *Let $T: X\to X$ be a continuous mapping of a complete metric space into itself which is a (U, k, α)-contraction. If $M^2=\cup\{U^n: n\in N$ and $X\times T(X)\subset M^2\}$, then all sequences of iterates $x_n=T^n x, x\in X$ converge to a fixed point of T. When X is U-chainable T has a unique fixed point.*

PROOF. We shall show that T is a contractive mapping on $T(X)$. Let $x, y\in T(X)$ be arbitrary. Since $T(X)\times T(X)\in X\times T(X)\subset M^2$, x and y can be joined by an U-chain. Put $t(u, v)=\max\{\rho[T^i u, T^i v]: i=0, 1,..., k-1\}$ for $(u, v)\in U$ and let

$$p(x, y) = \inf\{\sum_{i=1}^{n} t(u_{i-1}, u_i): x = u_0, u_1, \ldots, u_n = y \text{ an } U\text{-chain}\}.$$

Let $m = m(x, y)$ be the least positive integer so that $\alpha^{m/k} p(x, y) \leq \alpha\rho(x, y)$. Then, from (2) for $n, r \geq m(x, y)$

$$\rho[T^n x, T^r y] \leq \alpha^{(n-m)/k}(1-\alpha)^{-1} \max\{\rho[x, y], \rho[x, Tx]\} := \Delta_{n,r}(x, y).$$

Since $\Delta_{n,r}(x, y) \to 0 (n \to \infty)$ and f is continuous, all hyphotheses of Theorem 1 are fulfilled on $T(X)$. Therefore, T has a unique fixed point in $T(X)$, i.e. in X.

Corollary 5. (Ćirić [4]) *Let T be an orbitally continuous mapping of an T-orbitally complete metric space X into itself. If T is an q-contraction, then (a), (b), (c) holds.*

Proof. For arbitrary $x, y \in X$ from (7) we obtain,

$$\rho[T^n x, T^r y] \leq q^n(1-q)^{-1} \max\{\delta(x, y), \delta(x, Tx)\}, \qquad n, r \geq m(x, y) = 1,$$

where $\Delta_{n,r}(x, y) = q^n(1-q)^{-1} \max\{\delta(x, y), \delta(x, Tx)\} \to 0 (n \to \infty)$. Therefore, T is a contraction type mapping. Since T is orbitally continuous, all hypotheess of Theorem 1 are fulfilled i.e. for T (a), (b) and (c) holds.

Remark 3. Evidentlly, proofs of theorems of Rhoades and Ray, Jaggi, Fisher, Iseki, Istratescu, Rakotch, Sehgal, and some others authors, is analogous to the proof of Corollary 1–5.

Now we shall define a mapping T which satisfies (T) but does not satisfy any of the conditions (B), (C), (E) and (1)–(10).

Example 4. Let $X = [0, +\infty)$ and define $T: X \to X$ by $Tx = x(1+x)^{-1}$, and distance function ρ is the ordinary euclidiean distance on the line. It is easy to see that T satisfies (T) with $\Delta_{n,r}(x, y) = x(1+nx)^{-1}$. Furthermore, for any $x, y \in X$ and $r > n$,

$$\rho[T^n x, T^r y] = \left|\frac{x}{1+nx} - \frac{y}{1+ry}\right| \leq \frac{x}{1+nx} := \Delta_{n,r}(x, y) \longrightarrow 0 (n \longrightarrow \infty).$$

Thus (T) holds. Since X is T-orbitally complete and T is orbitally continuous, it follows by Theorem 1 that T has a unique fixed point it is a point 0. The inequalities (B), (C), (E) and (1)–(9) does not hold for $y = 0$ and $x \in X$.

When, $T: X \to X = [0, 2]$ and define $Tx = 0 (0 \leq x < 1)$, $Tx = \frac{10}{11} x (1 \leq x \leq 2)$, then for any $x, y \in [0, 2]$ and $r \geq n$

$$\rho[T^n x, T^r y] = \left|\left(\frac{10}{11}\right)^n x - \left(\frac{10}{11}\right)^r y\right| \leq \left(\frac{11}{12}\right)^n x := \Delta_{n,r}(x, y) \longrightarrow 0 (n \longrightarrow \infty).$$

Thus (T) holds. Since X is complete and T is orbitally continuous, it follows by Theorem 1 that T has a unique fixed point $\xi=0$. The inequality (10) does not hold for $x=\frac{10}{11}$ and $y=1$.

Therefore, Edelstein, Bryant, Marjanović, Sehgal, Ćiıić, Iseki, Istratescu, Fischer, Ray and Rhoades (and some others) results is in fact a special case of Theorem 1.

In connexion with this, we shall introduce the concept of *almost contraction type mapping* T of a metric space X into itself i.e. of a mapping $T: X\to X$ such that for each $x\in X$, there exists sequence of nonnegative real functions $\Delta_{n,r}(x, Tx)$ with $\Delta_{n,r}(x, Tx)\to 0 (r>n\to\infty)$ and positive integer $m(x)$ such that

$$(AT) \qquad \rho[T^n x, T^r x] \leq \Delta_{n,r}(x, Tx) \qquad \text{for all} \quad r > n \geq m(x).$$

THEOREM 4. *Necessary and sufficient conditions that a mapping T of a metric space X into itself has the following properties* (b), (d) *and*

(h) *T has a fixed point (nonunique) $\xi\in X$,*

are the following ones: (f), (g) *and* (AT) *where*

$$\Delta_{n,r}(x, Tx) = \max\{\Delta_n(x, Tx), \Delta_r(x, Tx)\}.$$

Proof is analogous to the proof of Theorem 1.

An immediate corollary (special case) of the preceding statement is

THEOREM 5. *Necessary and sufficient conditions that a mapping T of a metric space X into itself has the following properties* (b) *and* (h), *are the folowing ones:* (f), (g) *and*

$$(CAT) \qquad \lim_{n,r\to\infty} \rho[T^n x, T^r x] = 0 \qquad \text{for } x\in X.$$

Otherwise, if the condition (AT) *holds, than the condition* (CAT) *holds.*

3. Applications of the Σ-contractive principle to integral equations

Now we shall show how Theorem 3 can be applied for investigation of solvability of an integral equation of the form

$$(IE) \qquad u(x) = v(x) + \lambda\int_\alpha^x K(x, t, u(t))dt \qquad (\alpha\leq x\leq\beta)$$

by successive approximation, where λ is an arbitrary parameter, $v(x)$ is a given continuous function in $[\alpha, \beta]$ and $K(m, s, x)$ is continuous for $m, s\in[\alpha, \beta]$.

THEOREM 6. *Let $K: [\alpha, \beta]\times[\alpha, \beta]\times R\to R$ be continuous and satisfy a Lipschitz condition*

$$(L) \qquad |K(m, s, x) - K(m, s, y)| \leq L|x - y| \qquad (L = const)$$

for all $(s, m) \in [\alpha, \beta] \times [\alpha, \beta]$ *and* $x, y \in R$. *Then for any* $v \in C[\alpha, \beta]$ *the equation* (IE) *has a unique solution* $u \in C[\alpha, \beta]$. *Moreover, defining a sequence of functions* $\{u_n\}$ *inductively by choosing any* $u_0 \in C[\alpha, \beta]$ *and setting*

$$(S) \qquad u_{n+1}(x) = v(x) + \lambda \int_\alpha^x K(x, t, u_n(t))dt,$$

the sequence $\{u_n\}$ *converges (uniformly) on* $[\alpha, \beta]$ *to the unique solution* u.

PROOF. Define an operator $F: C[\alpha, \beta] \to C[\alpha, \beta]$ on a space of continuous function in $[\alpha, \beta]$ as follows

$$(14) \qquad F(g)(x) = v(x) + \lambda \int_\alpha^x K(x, t, g(t))dt.$$

For $g, z \in C[\alpha, \beta]$ we obtain, from (L),

$$\begin{aligned} \rho[Fg, Fz] &= \max_{\alpha \leq x \leq \beta} |Fg(x) - Fz(x)| \\ &\leq |\lambda| L(x-a) \max_{\alpha \leq x \leq \beta} |g(x) - z(x)| \\ &\leq |\lambda| L(b-a)\rho[g, z] \end{aligned}$$

Hence, for $|\lambda| < (L(b-a))^{-1}$, the map $F: C[\alpha, \beta] \to C[\alpha, \beta]$ is a contraction and, by Banach contraction principle, the equation $Fg = g$, i.e. the equation (IE) has a unique solution.

Now we shall show that F is an Σ-contractive operator for every $\lambda \in R$. It is easily seen that

$$\rho[F^n g, F^n z] \leq |\lambda|^n L^n \frac{(x-a)^n}{n!} \rho[g, z], \qquad (n = 1, 2, \ldots),$$

and hence

$$\rho[F^n g, F^n z] \leq C_n(x, y)K(x, y), \qquad (n = 1, 2, \ldots)$$

where $C_n(x, y) = |\lambda|^n L^n (x-a)^n (n!)^{-1}$, $K(x, y) = \rho[g, z]$. Therefore, F is a Σ-contractive operator, on a complete space $C[\alpha, \beta]$. Since F is continuous, we may apply Theorem 3. Therefore, for any real λ the integral equation (IE) has a unique solution, which can be realized as the limit of the sequence of iterates $\{F^n u\}$ i.e. $\{u_n\}$ for each $u_0 \in C[\alpha, \beta]$, where F is defined by (14) and from (S).

REMARK 4. For $K(x, t, u(t)) = K_0(x, t)u(x)$ the above equation (IE) results in an integral equation of Volterra type.

If $K(t, s, u) = f(s, u)$ and $v(x) = 0$ in (IE), the Volterra equation becomes

$$u(x) = \int_0^x f(t, u(t))dt$$

and the solution of this integral equation is precisely the solution of the present initial value problem: *Let* $f: [0, a] \times R \to R$ *satisfy the Lipschitz condition* $|f(s, x) - f(s, y)| \leq L|x - y|$ $(L \in R)$ *for* $s \in [0, a]$, $x, y \in R$. *Then the initial value problem*

$$\frac{du}{dt} = f(t, u), \quad u(0) = 0$$

has exactly one solution u *defined on the entire interval* $[0, a]$.

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