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Funakoshi, Shunsuke

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SOME REMARKS ON REPRESENTATIONS OF NON-TYPE I LOCALLY COMPACT GROUPS

By Shunsuke FUNAKOSHI

Dedicated to Professor Osamu TAKENOUCHI on his 60-th birthday

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In this paper, we shall construct type II and type III factor representations of non-type I groups by direct integral of irreducible representations.

Let G be a separable locally compact group. Let E be a locally compact space on which G acts on the right such that

(1) $(x)g_1g_2 = (xg_1)g_2$,

(2) $xe = x$, where e is the identity of G,

(3) $(x, g)\rightarrow xg$ is a continuous mapping from $E\times G$ to E.

When we shall call (G, E) is a topological transformation group.

For a positive Radon measure μ on E, let μ_g be the measure on E defined by $\mu_g(F)=\mu(Fg)$ for each measurable subset F of E.

Let μ be a positive Radon measure on E which is quasi-invariant under the action of G, i.e., μ_g and μ are absolutely continuous. The triple (G, E, μ) is called a dynamical system.

Let $L=L_{E\times G}$ be the set of all continuous complex valued functions on $E\times G$ with compact support.

A non-negative continuous function $\rho(x, a)$ on $E\times G$ is called a multiplier if it satisfies

$$\rho(x, ab) = \rho(xa, b)\rho(x, a)$$

for each $x\in E$, a, $b\in G$.

Suppose that there exists a positive continuous function $\gamma(x, a)$ on $E\times G$ such that $d\mu(xa)=\gamma(x, a)d\mu(x)$. Then $\gamma(x, a)$ is a multiplier.

For f, $g\in L$, define

$$(f, g) = \iint f(x, a)\overline{g(x, a)}\chi(a)d\nu(a)d\mu(x),$$

where $\chi(a)$ is a non-negative continuous function on G such that $\chi(ab)=\chi(a)\chi(b)$ and ν is a Haar measure on G.

Then L is a pre-Hilbert space. Let $\mathscr{H}$ be the Hilbert space which is the completion of L.

Let ψ be a complex valued, measurable and essentially bounded function on (E, μ). For each $f \in \mathscr{H}$, define

$$L_\psi f(x, a) = \psi(xa) f(x, a)$$
$$(\text{resp. } L'_\psi f(x, a) = \psi(x) f(x, a)).$$

Then L_ψ (resp. L'_ψ) is a bounded operator on $\mathscr{H}$.

Next, we shall define a unitary operator U_d (resp. U'_d), for each $d \in G$, on $\mathscr{H}$ by

$$U_d f(x, a) = \Delta(d)^{-1/2} \chi(d)^{-1/2} f(x, d^{-1}a)$$
$$(\text{resp. } U'_d f(x, a) = \gamma(x, d)^{1/2} \chi(d)^{-1/2} f(xd, ad)),$$

where Δ is the modular function of G.

Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a W^*-subalgebra of $B(\mathscr{H})$ generated by $\{L_\psi | \psi \in L^\infty(E, \mu)\}$ and $\{U_d | d \in G\}$ (resp. $\{L'_\psi | \psi \in L^\infty(E, \mu)\}$ and $\{U'_d | d \in G\}$).

REMARK (cf. [3], [4], [7]). $\mathscr{F}$ (and $\mathscr{F}'$) is isomorphic to $\mathscr{R}_i$ $(i = 1, 2, 3, 4, 5)$:

$\mathscr{R}_1$; the W^*-algebra on $L^2(G) \otimes L^2(S)$ generated gy $\{L(g) \otimes W(g) | g \in G\}$ and $\{I \otimes P(\phi) | \phi \in L^\infty(E)\}$, where $L(g)$ is the left regular representation of G, $(W(g)f)(x) = \lambda_g(x)^{1/2} f(sg)$, $\lambda_g(x)$ is the Radon–Nikodym derivative $d\mu_g/d\mu$, and $(P(\phi)f)(x) = \phi(x) f(x)$.

$\mathscr{R}_2$; the W^*-algebra on $L^2(G \times E)$ generated by $\{V(g) | g \in G\}$ and $\{P(\phi) | \phi \in L^\infty(E)\}$, where $(V(g)f)(t, x) = \lambda_g(x)^{1/2} f(tg, xg)$ and $(P(\phi)f)(t, x) = \phi(x) f(t, x)$.

$\mathscr{R}_3$; the W^*-algebra on $L^2(G \times E)$ generated by the system of imprimitivity $\langle W, P \rangle$ given by $(W(g)f)(t, x) = \Delta(g)^{1/2} f(tg, x)$, $g \in G$, and $(P(\phi)f)(t, x) = \phi(xt) f(t, x)$, $\phi \in L^\infty(E)$.

$\mathscr{R}_4$; the crossed product $G \times_{\alpha_g} L^\infty(E)$ of $L^\infty(E)$ by G, α_g is the action of G on E.

$\mathscr{R}_5$; the W^*-algebra on $L^2(G \times E)$ generated by $\{L^\mu(\phi) | \phi \in \mathfrak{A}(G, E)\}$, where $\mathfrak{A}(G, E)$ is the transformation group C^*-algebra of (G, E) and L^μ is the representation of $\mathfrak{A}(G, E)$ determined by the positive-definite measure $\delta_e \times d\mu$ on $G \times E$.

DEFINITION 1. (1) A dynamical system (G, E, μ) is called *free* if for any $a \in G$ $(a \neq e)$, the set of points x satisfying the condition $x = xa$ $(x \in E)$ is of μ-measure 0.

(2) A dynamical system (G, E, μ) is called *ergodic* if $Fg = F$ for a measurable set F and for every $g \in G$ implies either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$.

LEMMA 1 ([1]). *Let (G, E, μ) be free, $\mathscr{F}$ (and $\mathscr{F}'$) is a factor if and only if (G, E, μ) is ergodic.*

DEFINITION 2. A dynamical system (G, E, μ) is called *measurable* if there exists a σ-finite positive measure ν which is equivalent to μ and $d\nu(xa)=\Delta(a)d\nu(x)$, that is, ν is invariant under G.

LEMMA 2 ([1]). *If (G, E, μ) is free, ergodic and measurable, then $\mathscr{F}$(and $\mathscr{F}'$) is a type* I *or type* II *factor. In particular, if G is not a discrete group, then $\mathscr{F}$ (and $\mathscr{F}'$) is a type I_∞ or type II_∞ factor.*

LEMMA 3 ([1]). *If (G, E, μ) is free, ergodic and non-measurable, then $\mathscr{F}$(and $\mathscr{F}'$) is a type* III *factor.*

A topological transformation group (G, E) is *polonais* if G and E are polonais, i.e., they are separable and metrizable by a complete metric. (G, E) satisfies the *condition* (*) if each neighborhood U of e in G contains a neighborhood W of e such that for all x in E, $Cl[xW]\subseteq xU$. $Cl[xW]$ indicates the closure of xW. If G is locally compact and E is Hausdorff, (G, E) satisfies the condition (*), as one may take W to be any compact neighborhood of e with $W\subseteq U$.

A Borel measure μ on E is *non-trivially ergodic* if it is not concentrated at an orbit.

LEMMA 4 ([2]). *Let (G, E) be a polonais transformation group satisfying the condition (*). Then the orbit space E/G is a T_0-space if and only if E has no non-trivially ergodic measure.*

For a separable locally compact group G, let G^{ir} be the standard Borel space of all irreducible representations of G ([10]).

LEMMA 5. *Let (Γ, Ω) be a polonais transformation group satisfying the condition (*), μ an ergodic Borel measure in Ω, and $\omega\to\Pi^\omega$ a Borel function from Ω to G^{ir}. If for any $\omega\in\Omega$, Π^ω is equivalent to $\Pi^{\gamma(\omega)}$ for all $\gamma\in\Gamma$, then the direct integral $K=\int_\Omega \Pi^\omega d\mu(\omega)$ is a factor representation. Moreover, μ is non-trivially ergodic if and only if K is non-type* I.

PROOF. Suppose K is not a factor representation. Then, there exists a projection $E\neq 0, I$ in $K(G)'\cap K(G)''$, where $K(G)$ is the algebra generated by the representation $\{K(g)|g\in G\}$ and $K(G)'$ is the commutant of $K(G)$.

Since Π^ω, $\omega\in\Omega$, are irreducible, the Boolean algebra of projections associated with the direct integral is maximal in $K(G)'$, and must contain E. Let B be a Borel set with $0\neq\mu(B)\neq\mu(\Omega)$ and $E=\int\chi_B(\omega)Id\mu(\omega)$, where $\chi_B(\omega)$ is the characteristic function of B. K is proper because each Π^ω is irreducible. By the double commutation theorem, the unit ball of $K(G)$ is strongly dense in the unit ball of $K(G)''$. The later being metrizable in the strong topology, there is a sequence $\{g_n\}\subset G$ with $K(g_n)\to E$ strongly i.e., $\int\Pi^\omega(g_n)d\mu(\omega)\to\int\chi_B(\omega)Id\mu(\omega)$. There is a subsequence $\{g_{n_k}\}$ and a null set N of Ω such that $\Pi^\omega(g_{n_k})\to\chi_B(\omega)I$

strongly for all $\omega\in\Omega\setminus N$. Changing notation, we assume that $\Pi^{\omega}(g_n)\to\chi_B(\omega)I$ strongly for all $\omega\in\Omega\setminus N$.

Let $A=\Gamma(B\setminus N)$. As $B\setminus N$ is Borel, A is analytic (it is the image of $\Gamma\times(B\setminus N)$ under a continuous map), and hence measurable. A is invariant, and we claim that $\mu(A)=\mu(B)$. As $\mu(N)=0$ and $B\setminus N\subseteq A$, it suffices to show $A\setminus N\subseteq B$. For any $\omega\in A\setminus N$, there are $\gamma\in\Gamma$ and $\beta\in B\setminus N$ such that $\sigma=\gamma(\beta)$, i.e., $\Pi^{\omega}=\Pi^{\gamma(\beta)}$. Then $\Pi^{\omega}(g_n)\to\chi_B(\omega)I$ implies $\Pi^{\gamma^{-1}(\omega)}(g_n)\to\chi_B(\omega)I$. But $\Pi^{\gamma^{-1}(\omega)}(g_n)=\Pi^{\beta}(g_n)\to\chi_B(\beta)I=I$, hence $\chi_B(\omega)=I$ and $\omega\in B$. Thus μ is not ergodic. Therefore, K is a factor representation.

If K is a factor representation of type I, then μ-almost all the representations Π^{ω} are unitray equivalent, i.e., μ is trivially ergodic under Γ ([8]). Therefore if μ is non-trivially ergodic, then K is a non-type I factor representation.

Conversely, if μ is trivially ergodic under Γ, the representation Π^{ω}, $\omega\in\Omega$, are μ-almost all unitary equivalent and K is a factor representation of type I ([9]). Therefore, if K is a non-type I factor representation, then μ is a trivially ergodic measure under Γ.

By Lemma 4 and Lemma 5, we have

PROPOSITION 1. *Let (Γ,Ω) be a polonais transformation group satisfying the condition (*), μ an ergodic Borel measure on Ω, and $\omega\to\Pi^{\omega}$ a Borel function from Ω to G^{ir}. Moreover, assume that for any $\omega\in\Omega$, Π^{ω} is equivalent to $\Pi^{\gamma(\omega)}$ for all $\gamma\in\Gamma$.*

Here, the orbit space Ω/Γ is a not T_0-space (; non- analytic as Borel space) if and only if there exists a measure μ such that $K=\int_{\Omega}\Pi^{\omega}d\mu(\omega)$ is a non- type I *factor representation of G.*

PROPOSITION 2. *Let (G, E) be an effective (; the identity element e is the only element satisfying $xe=x$ for all $x\in E$) transformation group and let G be abelian.*
If the orbits space E/G is a not T_0-space, then there exists a measure μ(resp. *μ') on an orbit closure X such that (G, X, μ)* (resp. *(G, X, μ')) is a free, ergodic and measurable* (resp. *non-measurable) dynamical system.*

PROOF. Previously, we have constructed such a measure μ (resp. μ') in [5]. Here, we shall only give an outline of the construction.

For the topological transformation group (G, X), we can inductively define, for each integer $n\geqq 0$ and element $g(n)$ in G and for each n-tuple $(i_1,\dots, i_n)$ with $i_k=0$ or 1, an open set $P(i_1,\dots, i_n)$ in X satisfying the following properties: (x is a fixed element of X)

$(1)_n$ $x\in P(0_n)$,

$(2)_n$ if $(i_1,\dots,i_n) \neq (j_1,\dots,j_n)$, then $WP(i_1,\dots,i_n) \cap P(j_1,\dots,j_n)=\emptyset$, where W is a neighborhood of e in G such that $W=W^{-1}$, and if $\{Q_m\}$ is a decreasing basis of open sets at an arbitrary point y in X, $Cl[\cap_{m=1}^{\infty} WQ_m] \subseteqq Gy$,

$(3)_n$ $Cl[P(i_1,\dots,i_n)] \subseteqq P(i_1,\dots,i_{n-1}) \quad (n \geqq 1)$,

$(4)_n$ diameter $P(i_1,\dots,i_n) < 1/n \quad (n \geqq 1)$,

$(5)_n$ $g(k)P(0_k, i_{k+1},\dots,i_n) = P(0_{k-1}, 1, i_{k+1},\dots,i_n) \quad (n \geqq 1)$,

where 0_n is the family of n zeros.

Let M_n $(n=1, 2,\dots)$ be copies of the group $Z/2Z$, the additive group of integers mod 2. Let β_n be a Radon measure on M_n with $\beta_n\{(0)\}=p$ and $\beta\{(1)\}=q$ with $0<p<1$ and $q=1-p$. Then (M_n, β_n) $(n=1, 2,\dots)$ is a measure space. Let (M, β) be the infinite product measure space of (M_n, β_n).

For each $i=\{i_1, i_2,\dots, i_n,\dots\} \in M$, the set $\cap_{n=1}^{\infty} P(i_1,\dots,i_n)$ has precisely one element, say $\Theta(i)$. Θ is a one-to-one mapping of M into X.

Let $M(i_1,\dots,i_n)=\{i_1\} \times \cdots \times \{i_n\} \times \{0, 1\} \times \cdots$, where $i_1,\dots,i_n \in \{0, 1\}$. Then

$$\Theta(M(i_1,\dots,i_n)) = \Theta(M) \cap P(i_1,\dots,i_n).$$

Hence Θ is a homeomorphism. Therefore we may identify M with $\Theta(M)$. Define a measure λ on X by the formulas

$$\lambda(M(i_1,\dots,i_n)) = p^r q^{n-r}$$

and

$$\lambda(X \smallsetminus M) = 0,$$

where r is the number of 0's in $(i_1,\dots,i_n)$.

Let ν be a finite measure on G, equivalent to a right Haar measure. We shall define the convolution product measure $\nu*\lambda$ of ν and λ as follows: If B is a Borel subset of X, let

$$\nu*\lambda(B) = \int_G \lambda(hB)d\nu(h).$$

Denote this convolution product measure $\nu*\lambda$ by μ (resp. μ') if $p=q=1/2$ (resp. $p\neq q\neq 1/2$). Then (G, X, μ) (resp. (G, X, μ')) is a free, ergodic and measurable (resp. non-meaurable) dynamical system.

By Proposition 1 and Proposotion 2, we have the following theorem.

THEOREM 1. *Let G be a separable locally compact group and G^{ir} the standard Borel space of all irreducible representations of G. Assume that*

(1) *Γ is an abelian group,*

(2) *(Γ, Ω) is a polonais effective transformation group satisfying the condition (*) and every orbits are dense in Ω,*

(3) *$\omega \to \Pi^\omega$ is a Borel function from Ω to G^{ir},*

(4) *for any $\omega \in \Omega$, Π^ω is equivalent to $\Pi^{\gamma(\omega)}$ for all $\gamma \in \Gamma$,*

(5) *the orbit space Ω/Γ is a not T_0-space.*

Then there exists an ergodic measure μ (resp. μ') *on Ω such that* $K = \int_\Omega \Pi^\omega d\mu(\omega)$ (resp. $K' = \int_\Omega \Pi^\omega d\mu'(\omega)$) *is a type* II_∞ (resp. *type* III) *factor representation of G.*

Now, let us look at for semi-direct product groups.

Let G be a semi-direct product group $H \times_s N$ of a commutative closed subgroup H and a commutative closed normal subgroup N. H acts on N as an automorphism group, we denote by α_h, $h \in H$, the automorphisms.

Notice that for each h, α_h defines an automorphism in the dual space $\hat{N}$ of N. In fact, for each $\chi \in \hat{N}$ and each $h \in H$ we may consider the function $n \to \chi(\alpha_h(n))$ and verify that it too is a character and denote by the symbol $[\chi]\alpha_h^*$. If we write χh instead of $[\chi]\alpha_h^*$ we deduce easily that $\hat{N}$ becomes an H-space. For each fixed χ in $\hat{N}$ the group H_χ of all $h \in H$ with $\chi h = \chi$ is called the *stability group* of χ.

Here, the following lemma is well pnown.

LEMMA 6 ([11]). *For each $\chi \in \hat{N}$ choose an irreducible representation L of H_χ and consider the subgroup $H_\chi \times N$ consisting of all $(h, n) \in H \times_s N$ with $h \in H_\chi$. Let $(\chi L)(h, n) = \chi(n) L_h$. Then χL is an irreducible representation of $H_\chi \times_s N$ and the induced representation $U^{\chi L} = \operatorname{Ind}_{(H_\chi \times_s N) \uparrow G} (\chi L)$ is an irreducible representation of G.*

In this lemma, suppose that $H_\chi = \{e\}$, e is the identity, for all $\chi \in N$, then $U^{\chi L} = U^\chi = \operatorname{Ind}_{N \uparrow G} \chi$; for $f(x) \in L^2(H, \mu)$

$$U^\chi_{(h,n)} f(x) = \overline{\chi(\alpha_{h^{-1}}(n))} f(hx), \qquad (h, n) \in G.$$

Let X be the closure of $O(\chi)$ in $\hat{N}$, where $O(\chi)$ is the orbit of χ, i.e., $O(\chi) = \{\chi h | h \in H\}$, then X is a standard space as Borel space.

In these terms, by theorem 1, we obtain the following theorem.

THEOREM 2. *Let $G = H \times_s N$ be the semi-direct product group with the following properties;*

(1) *(H, X) is a polonais effective transformation group satisfying the condition (*)*

(2) *$\chi \to U^\chi$ is a Borel function from X to G^{ir}*

(3) *for any $\chi \in X$, U^χ is equivalent to $U^{\chi h}$ for all $h \in H$*

(4) *the orbit space X/H is a not T_0-space,*

then there exists a ergodic measure ν (resp. ν') *on X such that* $\Pi^\nu = \int_X U^\chi d\nu(\chi)$ (resp. $\Pi^{\nu'} = \int_X U^\chi d\nu'(\chi)$) *is a type* II_∞ (resp. *type* III) *factor representation of G.*

REMARK 1. By the way of construction, we obtain at once the following;

$$\{\Pi^{\nu}(g)|g \in G\}'' = \mathscr{R}(H, X, \nu),$$
$$\{\Pi^{\nu'}(g)|g \in G\}'' = \mathscr{R}(H, X, \nu').$$

REMARK 2 (A RELATION OF U^{χ} AND REGULAR REPRESENTATION T).

Let T be the regular representation of $G=H\times_s N$. That is, for $f(h, n)\in L^2(H\times_s N, d\mu\times d\nu)$, μ (resp. ν) is a Haar measure on H (resp. N), $(\tau, \xi)\in H\times_s N$

$$[T(\tau, \xi)f](h, n) = f(\tau h, \alpha_h(\xi)n).$$

We carry out the Fourier transform of $f(h, n)\in L^2(H\times_s N)$ in the variable $n\in N$;

$$\hat{f}(h, \chi) = \int_N f(h, n)\overline{\chi(n)}d\nu(n), \quad (h, \chi)\in H\times\hat{N}$$

Under the action of the Fourier transform, T goes into the representation $\hat{T}$ defined by the equality

$$[\hat{T}(\tau, \xi)\hat{f}](h, \chi) = \overline{\chi(\alpha_{h^{-1}}(\xi))}\hat{f}(\tau h, \chi)\dots\dots\dots(*)$$

Indeed,

$$\begin{aligned}[\hat{T}(\tau, \xi)\hat{f}](h, \chi) &= \int_N f(\tau h, \alpha_h(\xi)n)\overline{\chi(n)}d\nu(n)\\ &= \int_N f(\tau h, \tilde{n})\overline{\chi(\alpha_{h^{-1}}(\xi)\tilde{n})}d\nu(\tilde{n}), \quad \text{where} \quad \tilde{n} = \alpha_h(\xi)n\\ &= \int_N f(\tau h, n)\overline{\chi(n)\chi(\alpha_{h^{-1}}(\xi))}d\nu(n)\\ &= \overline{\chi(\alpha_{h^{-1}}(\xi))}\int_N f(\tau h, n)\overline{\chi(n)}d\nu(n)\\ &= \overline{\chi(\alpha_{h^{-1}}(\xi))}\hat{f}(\tau h, \chi).\end{aligned}$$

On the other hnad,

$$\overline{\chi(\alpha_{h^{-1}}(\xi))}\hat{f}(\tau h, \chi) = U^{\chi}_{(\tau,\xi)}\hat{f}(h, \chi)$$

Therefore, formula (*) shows that T is the direct integral of representation $U^{\chi}=\operatorname{Ind}_{\chi\uparrow G}\chi$, $\chi\in N$, acting in $L^2(H, \mu)$. That is, $T=\int_{\hat{N}} U^{\chi}d\nu(\chi)$.

Now the above results are shown that $T|_X=\int_X U^{\chi}d\nu(\chi)$ is a factor representation of G.

REMARK 3. In the theorem, we have constructed the two different type representations

$$(\Pi^{\nu} = \int_X U^{\chi} d\nu(\chi), \quad H^{\nu} = \int_X H(\chi) d\nu(\chi))$$

and

$$(\Pi^{\nu'} = \int_X U^{\chi} d\nu'(\chi), \ H^{\nu'} = \int_X H(\chi) d\nu'(\chi))$$

by the direct integrals of a family of irreducible representations $(U^{\chi}, H(\chi))_{\chi \in X}$.

The difference of Π^{ν} and $\Pi^{\nu'}$ can be consider with respect to the notions of "rigged Hilbert space" as follows: Where we assume that $G = H \times_s N$ is metrizable and σ-compact. For representations in rigged Hilbert spaces and their related notions, see ([6]).

In general, for a given unitary representation of a metrizable and σ-compact locally compact group in a Hilbert space, we can construct a representation of the group in a rigged Hilbert space.

By the way of the construction of the representation in a rigged Hilbert space, we have that the nuclear spaces corresponding to the two representations (Π^{ν}, H^{ν}) and $(\Pi^{\nu'}, H^{\nu'})$ are same space, denote it by Φ. That is, we obtain the rigged Hilbert space $\Phi \subset H^{\nu} \subset \Phi'$ (resp. $\Phi \subset H^{\nu'} \subset \Phi'$) corresponding to (Π^{ν}, H^{ν}) (resp. $(\Pi^{\nu'}, H^{\nu'})$).

When, the difference of Π^{ν} and $\Pi^{\nu'}$ can be seen the difference of the representation spaces H^{ν} and $H^{\nu'}$ as the subspaces of Φ'.

EXAMPLE (MAUTNER GROUP). Let $\boldsymbol{R}$ denote the additive group of real numbers and $\boldsymbol{C}^2$ the product of two copies of the field of complex numbers. If (z_1, z_2) denotes an element of $\boldsymbol{C}^2$, we will define the action of $t \in \boldsymbol{R}$ on $\boldsymbol{C}^2$ by

$$\alpha_t(z_1, z_2) = (e^{it} z_1, e^{i\gamma t} z_2),$$

where γ is an irrational number.

Let M denote the semi-direct product $\boldsymbol{R} \times_s \boldsymbol{C}^2$, where $\boldsymbol{R}$ acts on $\boldsymbol{C}^2$ as above. M is called the *Mautner group*. For $\zeta = (t; z_1, z_2)$, $\zeta' = (t'; z_1', z_2') \in \boldsymbol{R} \times_s \boldsymbol{C}^2$,

$$\zeta \circ \zeta' = (t + t'; z_1 + e^{it} z_1', z_2 + e^{i\gamma t} z_2').$$

Mautner group is a semi-direct product group with the properties of the Theorem 2.

General case; $G=H\times_s N$	Mautner group; $M=\boldsymbol{R}\times_s \boldsymbol{C}^2$
the action α of H on N	$\alpha_t(z_1, z_2)=(e^{it}z_1, e^{i\gamma t}z_2)$ $(z_1, z_2)\in\boldsymbol{C}^2,\ t\in\boldsymbol{R}$
X is the closure of the orbit $O(\chi)$ through a point $\chi\in N$	the closure of the orbit $\{\alpha_t(z_1, z_2)\mid t\in\boldsymbol{R}\}$ through a point $(z_1, z_2)\in\boldsymbol{C}^2$ is the two-dimentional tourus $\boldsymbol{T}^2$ if $(z_1, z_2)\neq(0, 0)$
transformation group (H, X)	$(\boldsymbol{R}, \boldsymbol{T}^2)$ is a polonais effective transformation group satisfying the condition(*)
$\chi\in\hat{N}$, H_χ is the stability group of χ	$(z_1, z_2)\in\boldsymbol{T}^2$, if $\alpha_t(z_1, z_2)=(e^{it}z_1, e^{i\gamma t}z_2)$ $=(z_1, z_2)$, then $t=0$, i.e., the stability group of (z_1, z_2) is $\{0\}$
$U^\chi=\operatorname*{Ind}_{N\uparrow G}\chi$, $\chi\in\hat{N}$, i.e., $f\in L(H^2, \mu)$, $U^\chi_{(h,n)}f(x)=\overline{\chi(\alpha_{h^{-1}}(n))}f(hn)$ $(h, n)\in G=H\times_s N$	$f(t)\in L^2(\boldsymbol{R}, \mu)$ $U^{(a,b)}_{(s;z_1,z_2)}f(t)=$ $\exp\{-i\operatorname{Re}(e^{-is}az_1+e^{-i\gamma s}bz_2)\}$ $\times f(t+s)$, $(s; z_1, z_2)\in\boldsymbol{R}\times_s\boldsymbol{C}^2$
$\chi\in X\to U^\chi\in G^{ir}$	$(a, b)\in\boldsymbol{T}^2$, $U^{(a,b)}$ is irreducible, $(a, b)\in\boldsymbol{T}^2\to U^{(a,b)}\in G^{ir}$ is a Borel function
$\chi\in X$, $U^\chi\cong U^{\chi h}$ for all $h\in H$	$(a, b)\in\boldsymbol{T}^2$, $U^{(a,b)}\cong U^{\alpha_t(a,b)}$ for all $t\in\boldsymbol{R}$ because that the intertwining operator for $U^{(a,b)}$ and $U^{\alpha_t(a,b)}$ can be only the operator of translation by t for each $t\in\boldsymbol{R}$
the orbit space X/H $\Pi^\nu=\int_X U^\chi d\nu(\chi)$	the orbit space $\boldsymbol{T}^2/\boldsymbol{R}$ is not a T_0-space $\Pi^\nu=\int_{\boldsymbol{T}^2} U^{(a,b)}d\nu(a, b)$, $f(t; a, b)\in L^2(\boldsymbol{R}\times\boldsymbol{T}^2, \mu\times\nu)$, $\Pi^\nu_{(s;z_1,z_2)}f(t; a, b)=$ $\exp\{-i\operatorname{Re}(e^{-is}az_1+e^{-i\gamma s}bz_2)\}$ $\times f(t+s; a, b)$, $(s; z_1, z_2)\in\boldsymbol{R}\times_s\boldsymbol{C}^2$

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References

[1] J. Dixmier: Algèbres quasi-unitaires, Comment Math. Helv., **26** (1952), 275–322.
[2] E. Effros: Transformation groups and C^*-algebras, Ann. of Math., **81** (1965), 38–55.
[3] E. Effros and F. Hahn: Locally compact transformation groups and C^*-algebras, Mem. Amer. Math. Soc., No 75 (1967).
[4] E. C. Gootman: The type of some C^* and W^*-algebras associated with transformation groups, Pacific Jour of Maths., **48**, No. 1 (1973), 93–106.
[5] S. Funakosi: On representations of non-type I groups, Tôhoku Math Journal, **31**, No 2 (1979), 139–150.
[6] S. Funakosi: On a decomposition theory for unitary representations of locally compact groups. Matk. Japon. **28** (1983), 431–458
[7] M. Takesaki: Duality for crossed products and the structure of Von Neumann algebras of type III, Acta Math., **131** (1973), 249–300.
[8] G. W. Mackey: Induced representations of locally compact group II, Ann. of Math., **58** (1953), 193–221.
[9] G. W. Mackey: The theory of group representations, Univ. of Chicago Lecture Notes, 1955.
[10] G. W. Mackey: Borel structures in groups and their duals, Trans. Amer. Math. Soc., **85** (1957), 134–165.
[11] G. W. Mackey: Induced representations of groups and quantum mechanics, W. A. Benjemin, INC, New York, 1968.

Department of Mathematics
Faculty of Education
Kobe University
Nada, Kobe, 657
Japan