Kobe University Repository : Kernel

PDF issue: 2026-11-30

On an intrinsic definition of weakly normal rings

Yanagihara, Hiroshi

(Citation)
Kobe Journal of Mathematics,2(1):89-98

(Issue Date)
1985-06

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0034613

(URL)
https://hdl.handle.net/20.500.14094/E0034613

KOBE UNIVERSITY

YANAGIHARA, H.
KOBE J. MATH.,
2 (1985), 89-98

ON AN INTRINSIC DEFINITION OF WEAKLY NORMAL RINGS

By Hiroshi YANAGIHARA

Dedicated to Professor Mieo Nishi on his 60-th birthday

(Received April 5, 1985)

In the paper [6] C. Traverso defined a commutative ring A to be seminormal if A is equal to the seminormalization ${}^{+}A$ of A in its total quotient ring $Q(A)$. However, R. G. Swan defined seminormal rings intrinsically without using the notion of seminormalizations and discussed certain properties of thus intrinsically defined seminormal rings in his paper [5]. The purpose of this article is to give an analogous theory on weakly normal rings. In other words we shall give a new intrinsic definition of weakly normal rings and obtain some basic properties of these rings.

First we shall recall necessary results on weakly normal ring extensions obtained already in [7] in terms of weakly subintegral extensions, which correspond to subintegral extensions discussed in [5] in the case of seminormality. Then our new intrinsic definition of weakly normal rings is given in close connection with Hamann's criterion for weak normality formulated by S. Itoh in [3], and we show some basic results on these rings. In particular a faithfully flat descent property of weak normality is given, and our new definition of weak normality agrees with usual one for reduced noetherian rings.

All the rings in this paper are commutative with unit and we use the terminology and results in [4].

1. Weakly normal extensions Let B be a commutative ring with unit and let A be a subring of B. If the following conditions are satisfied, then B is said to be *weakly subintegral* over A:

(i) B is integral over A.

(ii) The canonical map Spec $(B)\to$Spec (A) is bijective.

(iii) For all $\mathfrak{q}$ in Spec (B), the field extension $k(\mathfrak{q}\cap A)\hookrightarrow k(\mathfrak{q})$ is purely inseparable, where $k(\mathfrak{q})=B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ and $k(\mathfrak{p})=A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ for $\mathfrak{p}\in$ Spec (A).

A useful characterization of weakly subintegral extensions is given as follows.

LEMMA 1. *Let B be an integral ring extension of a ring A. Then B is weakly subintegral over A if and only if for all fields F and homomorphisms*

$\varphi: A \to F$, there exists at most one homomorphism $\psi: B \to F$ such that the composite of the inclusion $A \subset\!\to B$ and ψ is φ.

PROOF. Assume that B is weakly subintegral over A. If φ is a homomorphism of A to a field F with kernel $\mathfrak{p}$, then we denote by $\mathfrak{q}$ the unique prime ideal of B such that $\mathfrak{p}=A\cap\mathfrak{q}$. Let $\bar{\varphi}$ be the homomorphism of $k(\mathfrak{p})=A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ to F induced by φ. Since $k(\mathfrak{q})$ is a purely inseparable extension over $k(\mathfrak{p})$, there is at most one homomorphism $\bar{\psi}$ of $k(\mathfrak{q})$ to F whose restriction to $k(\mathfrak{p})$ is $\bar{\varphi}$. Therefore it is easy to see that there is at most one homomorphism ψ of B to F whose restriction to A is φ. Conversely assume that for all fields F and homomorphisms φ of A to F there is at most one homomorphism ψ of B to F whose restriction to A is φ. Let $\mathfrak{p}$ be any prime ideal of A, and let F and φ be the algebraic closure $\overline{k(\mathfrak{p})}$ of $k(\mathfrak{p})$ and the composite of the canonical homomorphism $A\to k(\mathfrak{p})$ and the inclusion map of $k(\mathfrak{p})$ into F, respectively. If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of B such that $\mathfrak{q}_1\cap A=\mathfrak{q}_2\cap A=\mathfrak{p}$, then $k(\mathfrak{q}_1)$ and $k(\mathfrak{q}_2)$ are algebraic over $k(\mathfrak{p})$ because B is integral over A. Assume $\mathfrak{q}_1\neq\mathfrak{q}_2$. Let φ be the homomorphism of $k(\mathfrak{p})$ to F induced by φ, that is, the inclusion map of $k(\mathfrak{p})$ into F and $\bar{\psi}_i$ a homomorphism of $k(\mathfrak{q}_i)$ to F whose restriction to $k(\mathfrak{p})$ is $\bar{\varphi}$ for $i=1, 2$. If ψ_i is the composite of the canonical homomorphism $B\to k(\mathfrak{q}_i)$ and $\bar{\psi}_i$ for $i=1, 2$, then we see that ψ_1 is not equal to ψ_2. This is a contradiction. Therefore we see $\mathfrak{q}_1=\mathfrak{q}_2$. This means that the canonical map of Spec (B) to Spec (A) is bijective. Moreover if $k(\mathfrak{q}_1)$ is not purely inseparable over $k(\mathfrak{p})$, then there are at least two homomorphisms of $k(\mathfrak{q}_1)$ to F extending $\bar{\varphi}$ to and hence we would have also a contradiction in the same way as the above. q.e.d.

As a direct consequence of Lemma 1 we have the following.

COROLLARY. *Let C be a ring, and let A and B be subrings of C such that $B\supset A$. Then C is weakly subintegral over A if and only if C and B are so over B and A, respectively.*

LEMMA 2. *Any ring extension $B\supset A$ has a largest subextension weakly subintegral over A.*

PROOF. We may replace B by the integral closure of A in B and hence assume that B is integral over A. Let C be the set of elements b of B satisfying the following condition: for any field F and any ring homomorphisms φ and φ' of B to F with $\varphi|_A=\varphi'|_A$ we have $\varphi(b)=\varphi'(b)$. It is easy to see that C is a subring of B containing A, and that C is the largest subextension weakly subintegral over A by Lemma 1. q.e.d.

Let B be a ring extension of a ring A, and denote by C the largest subextension weakly subintegral over A. If B is integral over A, then we see easily from Remark 1 of §1 in [7] and Lemma 1 that C is the weak normalization ${}^{*}_{B}A$ of A in B.

Henceforth we call C the *weak normalization* of A in B and also denote it by ${}^{*}_{B}A$ for not necessarily integral extensions. Moreover A is said to be *weakly normal* in B if A is equal to ${}^{*}_{B}A$. It is clear by Corollary to Lemma 1 that ${}^{*}_{B}A$ is weakly normal in B, and we see easily that A is weakly normal in B if and only if there is no weakly subintegral subextension in B different from A.

LEMMA 3. *If B is a simple extension $A[b]$ of A such that b^p and pb are contained in A for some prime p, then B is weakly subintegral over A.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of A and assume that p is not contained in $\mathfrak{p}$. Then $b/1=pb/p$ is an element of $A_{\mathfrak{p}}$ and hence $A_{\mathfrak{p}}$ is equal to $B_{\mathfrak{p}}$. This means that there exists a unique prime ideal $\mathfrak{q}$ in B lying over $\mathfrak{p}$ and that we have $k(\mathfrak{p})=k(\mathfrak{q})$. If $\mathfrak{p}$ contains p, then $A/\mathfrak{p}$ is an integral domain of characteristic p. Since B is integral over A, we see easily $\mathfrak{p}B\cap A=\mathfrak{p}$. Therefore, if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of B lying over $\mathfrak{p}$, then we have $\bar{\mathfrak{q}}_1\cap(A/\mathfrak{p})=\bar{\mathfrak{q}}_2\cap(A/\mathfrak{p})=0$ where $\bar{\mathfrak{q}}_i=\mathfrak{q}_i/\mathfrak{p}B$ for $i=1, 2$. By our assumption x^p belongs to $A/\mathfrak{p}$ for any element x in $B/\mathfrak{p}B$. In particular if x is an element of $\bar{\mathfrak{q}}_1$, then x^p is contained in $(A/\mathfrak{p})\cap\bar{\mathfrak{q}}_1=0$. This means that x belongs to $\bar{\mathfrak{q}}_2$, and we see $\bar{\mathfrak{q}}_1\subset\bar{\mathfrak{q}}_2$ and hence $\mathfrak{q}_1\subset\mathfrak{q}_2$. Similarly we have $\mathfrak{q}_2\subset\mathfrak{q}_1$. Therefore there is a unique prime ideal $\mathfrak{q}$ of B lying over $\mathfrak{p}$, and moreover we see that $k(\mathfrak{q})$ is purely inseparable over $k(\mathfrak{p})$. q.e.d.

We call such an extension $B\supset A$ considered in Lemma 3 *an elementary weakly subintegral extension.* Now we shall see that elementary weakly subintegral extensions play similar roles in weakly normal extensions to those played by elementary subintegral extensions in seminormal ones discussed in [5].

First, using elementary weakly subintegral extensions, we give a new proof of an already known result which says that Hamann's crierion works for weakly normal extensions (cf. Theorem 1 in [3] or Theorem 1 in [7]). For this purpose we need the following

LEMMA 4. *Let B be a non-trivial ring extension of a ring A such that A is seminormal in B. If B is weakly subintegral over A, then B contains a non-trivial elementary weakly subintegral extension $A[b]$ of A.*

PROOF. By Corollary to Lemma 1 we may assume that B is a finite A-module. Let $\mathfrak{c}$ be the conductor $A:_A B=\{b\in A\mid bB\subset A\}$ of A in B. If S is a multiplicatively closed subset of A, then $\mathfrak{c}_S$ is the conductor of A_S in B_S, and we see easily that B_S is weakly subintegral over A_S. Moreover A_S is seminormal in B_S by Corollary 2.10 in [5]. We show that if $A_S\neq B_S$, it will suffice to prove our lemma for the extension $B_S\supset A_S$. In fact assume that there exist x in B and s in S such that $(x/s)^p\in A_S$, $p(x/s)\in A_S$ for some prime p but $x/s\notin A_S$. Then there are a and c in A and t in S satisfying $(x/s)^p=a/t^p$ and $p(x/s)=c/t$ and hence there is an element u in S such that $u^p(x^pt^p-as^p)=0$ and $u(xpt-sc)=0$. If we put $b=utx$, we have $b^p\in A$, $pb\in A$ but $b\notin A$. Therefore $A[b]$ is a non-trivial elementary

weakly subintegral extension of A in B.

Now let $\mathfrak{p}$ be a minimal prime divisor of $\mathfrak{c}$ in A and let $S=A-\mathfrak{p}$. Then we have $A_S \neq \mathfrak{c}_S$ and hence $B_S \neq A_S$. By replacing A and B by A_S and B_S, respectively, we may assume that A is quasi-local with maximal ideal $\mathfrak{p}$. In this case $\mathfrak{p}$ is equal to $\mathfrak{c}$ because $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{c}$ which is a radical ideal of A and B as seen easily from Theorem 2.5 in [5]. Since the canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is bijective and $\mathfrak{p}=\mathfrak{c}$ is a radical ideal in B, $\mathfrak{p}$ is a maximal ideal of B. By our assumption $B/\mathfrak{p}$ is purely inseparable over $A/\mathfrak{p}$. If $A/\mathfrak{p}=B/\mathfrak{p}$, then we have $A=B$. This contradicts to $\mathfrak{c} \neq A$. Therefore we have $A/\mathfrak{p} \neq B/\mathfrak{p}$ and there is an element x in B but not in A such that $\bar{x}^p$ is contained in A/p where $\bar{x}$ is the residue class of x in $B/\mathfrak{p}$ and p is the characteristic of $A/\mathfrak{p}$. Then we see that $x^p \in A+\mathfrak{p}=A$, $px \in A$ but $x \notin A$. q.e.d.

COROLLARY. *If B is a non-trivial weakly subintegral extenson of a ring A, then there is a non-trivial elementary subintegral subextension or a non-trivial elementary weakly subintegral subextension of A in B.*

PROOF. If A is not seminormal in B, then there is an element b in B with $b^2 \in A$, $b^3 \in A$ but $b \notin A$ by Theorem 2.5 in [5]. Therefore B contains a non-trivial elementary subintegral subextension $A[b]$ of A. If A is seminormal in B, then there is a non-trivial elementary weakly subintegral subextension of A by Lemma 4. q.e.d.

Now we can state Hamann's criterion for weak normality in terms of elementary subintegral extensions and elementary weakly subintegral extensions.

THEOREM 1. *Let B be a ring and A a subring of B. Then A is weakly normal in B if and only if there is neither any non-trivial elementary subintegral subextension of A nor any non-trivial elementary weakly subintegral subextension of A.*

PROOF. If A is weakly normal in B, then it is clear that A is seminormal in B. Therefore our condition is necessary by Lemma 2.4 in [5] and Lemma 3, because B contains neither non-trivial subintegral subextensions nor non-trivial weakly subintegral subextensions of A. Conversely the condition is sufficient by Corollary to Lemma 4. q.e.d.

COROLLARY. *Let C be a ring, and let A and B be subrings of C such that $B \supset A$. If A is weakly normal in B and B is weakly normal in C, then so is A in C.*

This is a direct consequence of Theorem 1.

THEOREM 2. *Let B be a ring and let A be a subring of B. Then the weak normalization ${}^*_B A$ is the filtered union of all subrings of B which can be obtained*

from A by a finite number of elementary subintegral extensions or elementary weakly subintegral extensions.

PROOF. Let C be the union in question, which is clearly filtered. Since subintegral extensions are weakly subintegral, it is easily seen, by Lemmas 2.1 and 2.4 in [5] and by Lemmas 1 and 3, that C is weakly subintegral over A. Therefore C is contained in ${}^*_B A$. If $C \neq {}^*_B A$, then C is not weakly normal in B and hence, by Theorem 1, there is a non-trivial simple extension $C[b]$ of C which is elementary subintegral or elementary weakly subintgral over C. Then there is a subextension D of A in B obtained from A by a finite number of elementary subintegral extensions or elementary weakly subintegral extensions such that $D[b]$ is elementary subintegral or elementary weakly subintegral over D. However, by the definition of C, $D[b]$ is contained in C. This is a contradiction. Therefore C must be ${}^*_B A$. q.e.d.

The following proposition and its corollary are well known in the case of integral extensions, but we shall give similar proofs to those of Proposition 2.9 and Corollary 2.10 in [5] for convenience' sake.

PROPOSITION 1. *The formation of ${}^*_B A$ commutes with localizations with respect to multiplicatively closed subsets of A.*

PROOF. Let C and D be subrings of B containing A. If D is an elementary subintegral (resp. weakly subintegral) extension of C, D_S is an elementary subintegral (resp. weakly subintegral) one of C_S for any multiplicatively closed subset S of A. Conversely if D is a subring of B_S and is an elementary subintegral extension of C_S, then there exists an elementary subintegral extension D' of C in B with $D = D'_S$ as shown in the proof of Proposition 2.9 in [5]. If D is a subring of B_S and is an elementary weakly subintegral extension of C_S, then there exist x in B and s in S such that $D = C_S[x/s]$, $(x/s)^p \in C_S$ and $p(x/s) \in C_S$ for some prime p. Define t and u for these x and s as in the proof of Lemma 4 and put $b = utx$. Then we see that $D' = C[b]$ is an elementary weakly subintegral extension of C and that $D = D'_S$. Therefore the result follows immediately from Theorem 2. q.e.d.

COROLLARY 1. *Let A be a subring of a ring B and S a multiplicatively closed subset of A. If A is weakly normal in B, so is A_S in B_S. If B is weakly subintegral over A, so is B_S over A_S.*

PROOF. A is weakly normal in B if and only if $A = {}^*_B A$, and B is weakly subintegral over A if and only if $B = {}^*_B A$. Therefore our assertion is a direct consequence of Proposition 1.

COROLLARY 2. *Let A and B be as above. Then A is weakly normal in B (resp. B is weakly subintegral over A) if and only if $A_{\mathfrak{m}}$ is so in $B_{\mathfrak{m}}$ (resp. $B_{\mathfrak{m}}$ is so over $A_{\mathfrak{m}}$) for all maximal ideals $\mathfrak{m}$ of A.*

2. Weak normality A commutative ring A is said to be *weakly normal* if A is reduced and the following conditions are satisfied:
(i) For any elements b, c in A with $b^3=c^2$, there is an element a in A with $a^2=b$ and $a^3=c$.
(ii) For any elements b, c, e in A and any non-zero divisor d in A with $c^p=bd^p$ and $pc=de$ for some prime p, there is an element a in A with $b=a^p$ and $e=pa$.

LEMMA 5. *In any commutative ring $x^p=y^p$ and $px=py$ for some prime p imply $(x-y)^p=0$.*

PROOF. For any positive integer i, it follows from $px=py$ that $px^i=px^{i-1}y=px^{i-2}y^2=\cdots=pxy^{i-1}=py^i$. On the other hand we see $\binom{p}{i}=\binom{p}{p-i}=pn_i$ with some integer n_i for $1\leqq i\leqq p-1$. Therefore we see, for $p\neq 2$,

$$(x-y)^p = x^p - \binom{p}{1}x^{p-1}y + \cdots + (-1)^i\binom{p}{i}x^{p-i}y^i + \cdots - y^p$$

$$= x^p - \binom{p}{1}x^p + \cdots + (-1)^i\binom{p}{i}x^p + \cdots - x^p = 0.$$

If $p=2$, then we see

$$(x-y)^2 = x^2 - 2xy + y^2 = x^2 - 2x^2 + x^2 = 0. \qquad \text{q.e.d.}$$

COROLLARY. *A ring A is weakly normal if and only if the following conditions are satisfied:*
(i) *For any elements b, c in A with $b^3=c^2$, there is a unique element a in A with $a^2=b$ and $a^3=c$.*
(ii) *For any elements b, c, e in A and any non-zero divisor d in A with $c^p=bd^p$ and $pc=de$ for some prime p, there is a unique element a in A with $b=a^p$ and $e=pa$.*

PROOF. This follows immediately from Lemma 3.1 in [5] and Lemma 5.

PROPOSITION 2. *Let $A=\varinjlim A_\alpha$ and assume that each canonical homomorphism $f_\alpha: A\to A_\alpha$ has the property that the image of a non-zero divisor of A under f_α is not a zero-divisor in A_α. If all A_α are weakly normal, then A is weakly normal.*

PROOF. Since each A_α satisfies the condition (i) of Corollary to Lemma 5, so does A by Corollaries 3.2 and 3.3 in [5]. Now let b, c, e be elements in A and d a non-zero divisor in A such that $c^p=bd^p$ and $pc=de$ for some prime p. If we put $b_\alpha=f_\alpha(b)$, $c_\alpha=f_\alpha(c)$, $e_\alpha=f_\alpha(e)$ and $d_\alpha=f_\alpha(d)$, then we see that $c_\alpha^p=b_\alpha d_\alpha^p$ and $pc_\alpha=d_\alpha e_\alpha$ and that d_α is not a zero-divisor in A_α by our assumption. Therefore

there is a unique a_α in A_α with $b_\alpha = a_\alpha^p$ and $e_\alpha = pa_\alpha$, and we see, from the uniqueness of a_α, $f_{\alpha\beta}(a_\alpha) = a_\beta$ for any canonical map $f_{\alpha\beta}: A_\alpha \to A_\beta$ defining the inverse system $\{A_\alpha\}$. This means that there is a unique element a in A with $f_\alpha(a) = a_\alpha$, and it is easy to see $a^p = b$ and $pa = e$. q.e.d.

COROLLARY. *If A_α is weakly normal for all α, then so is the product $A = \prod_\alpha A_\alpha$.*

PROOF. This follows immediately from Proposition 2 and the fact that an element $d = (d_\alpha)$ in A is not a zero-divisor if and only if each d_α is not a zero divisor in A_α. q.e.d.

PROPOSITION 3. *Let A be a reduced ring and let B be a surbing of the total quotient ring $Q(A)$ of A. If A is weakly normal and is contained in B, then A is weakly normal in B.*

PROOF. Let x be an element of B with $x^p \in A$ and $px \in A$ for some prime p. Since B is a subring of $Q(A)$, we may put $x = c/d$ with $c, d \in A$ where d is not a zero divisor of A. If $b = x^p$ and $e = px$, then b and e are in A and we have $c^p = bd^p$ and $pc = de$. Therefore there exists an element a in A such that $a^p = b$ and $pa = e$ by weak normality of A. Then we see $x^p = a^p$ and $px = pa$ and hence $x = a$ by Lemma 5. This means that x is contained in A. Similarly we see easily that an element x of B contained in A if x^2 and x^3 belong to A. Therefore A is weakly normal in B. q.e.d.

PROPOSITION 4. *Let B be a weakly normal ring and let A be a subring of B such that any non-zero divisor in A is also not a zero divisor in B. If A is weakly normal in B, then A is weakly normal.* q.e.d.

PROOF. Sime A is weakly normal in B, A is seminormal in B, and hence, by Corollary 3.4 in [5], A is seminormal. Let b, c, d and e be elements in A where d is not a zero divisor in A, and assume that $c^p = bd^p$ and $pc = de$ for some prime p. Then d is also not a zero divisor in B, and hence there is an element a in B with $b = a^p$ and $e = pa$ because of weak normality of B. Since A is weakly normal in B, this means that a is contained in A. Therefore A is weakly normal.

q.e.d.

COROLLARY. *Let A be a reduced ring and let B be a weakly normal subring of the total quotient ring $Q(A)$ of A. Assume that B contains A. Then A is weakly normal if and only if A is weakly normal in B.*

PROOF. This follows immediately from Proposition 3 and 4. q.e.d.

REMARK 1. *Let A be a reduced ring with only a finite number of minimal prime ideals. Then A is weakly normal if and only if A is weakly normal in the total quotient ring $Q(A)$.* In fact we see easily that any field is weakly normal and hence by Corollary to Proposition 2 that a direct sum of a finite number of

fields is also weakly normal. Therefore our assertion is a direct consequence of Corollary to Proposition 4.

REMARK 2. *In Proposition* 3 *we cannot drop the assumption that B is a subring of* $Q(A)$. Let F be a non-perfect field of characteristic $p>0$. Let A be F and B the algebraic closure $\bar{F}$ of F. Then A and B are both weakly normal, but A is not weakly normal in B. In fact let x be an element in B but not in A such that x^p is contained in A. Since $px=0$, A is not weakly normal in B.

Now we show faithfully flat descent property of seminormality and weak normality.

PROPOSITION 5. *Let B be a seminormal (resp. weakly normal) ring and A be a subring of B such that A has only a finite number of minimal prime ideals. If B is faithfully flat over A, then A is seminormal (resp. weakly normal).*

PROOF. Since B is faithfully flat over A, we see $aB \cap A = aA$ for any element a in A by Chap. 1, §3, Prop. 9 in [1]. Therefore the following commutative diagram of rings is a pull-back diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ Q(A)=A_S & \longrightarrow & B_S \end{array}$$

where S is the set of non zero-divisors in A. By our assumption $Q(A)$ is a direct product of a finite number of fields and has Krull dimension 0. For any prime ideal $\mathfrak{p}$ of $Q(A)=A_S$, $Q(A)_{\mathfrak{p}}$ is a field and hence $(A_S)_{\mathfrak{p}}$ is seminormal in $(B_S)_{\mathfrak{p}}$. Therefore A_S is seminormal in B_S by Corollary 2.10 in [5]. This means that A is seminormal in B as shown in the proof of Proposition 1 in [7]. Therefore if B is seminormal, then A is seminormal by Corollary 3.4 in [5]. On the other hand assume that B is weakly normal. Let b, c, d and e be elements in A with $c^p=bd^p$ and $pc=de$ for some prime p, where d is not a zero divisor in A. Since B is faithfully flat over A, d is not a zero divisor in B and so there exists an element a in B such that $b=a^p$ and $e=pa$ by weak normality of B. Then we see that $c^p=bd^p=(ad)^p$ and $pc=de=p(ad)$ and hence $c=ad$ by Lemma 5 and reducedness of B. Since we have $dB \cap A=dA$, there is an element a' in A such that $c=a'd$. Since d is not a zero divisor in B, we have $a=a'$. This means that a is an element of A, and hence A is weakly normal. q.e.d.

COROLLARY. *Let A be a ring and let B be a weakly normal subring of the total quotient ring* $Q(A)$ *of A. If B is faithfully flat over A and A has only a finite number of minimal prime ideals, then A is weakly normal in B.*

PROOF. This is a direct consequence of Propositions 3 and 5. q.e.d.

PROPOSITION 6. *Let A be a weakly normal ring and S a multiplictively closed subset of A which contains no zero-divisors in A. Then A_S is weakly normal.*

PROOF. If A is weakly normal, then A is seminormal and hence A_S is also seminormal by Proposition 3.7 in [5]. Let x, y, z and w be in A_S with $y^p=xz^p$ and $py=zw$ for some prime p, where z is not a zero-divisor in A_S. Then there are elements a, b, c, d in A and s in S such that $x=b/s^p$, $y=c/s^2$, $z=d/s$ and $w=e/s$. Then we see $c^p/s^{2p}=bd/s^{2p}$ and $p(c/s^2)=de/s^2$. Since S does not contain any zero-divisor in A, we have $c^p=bd^p$ and $pc=de$. Moreover it is easy to see that d is not a zero-divisor in A. Since A is weakly normal, there is an element a in A with $b=a^p$ and $e=pa$. If we put $u=a/s$, then we have $u^p=a^p/s^p=b/s^p=x$ and $pu=pa/s=e/s=w$. This means that A_S is weakly normal. q.e.d.

LEMMA 6. *If A is a reduced ring, then A is weakly normal in $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over all maximal ideals of A.*

PROOF. By Lemma 3.8 in [5] it is known that A is seminormal in $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$. Therefore it suffices to show that there exist no elementary weakly subintegral subextensions of A in $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$. We show this fact by the same idea as in the proof of Lemma 3.8 in [5]. Let $\alpha=(a_{\mathfrak{m}})$ be an element of $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$ such that $b=\alpha^p$ and $e=p\alpha$ are contained in A for some prime p. Let $\mathfrak{I}$ be the set of s in A satisfying the following condition: there is an element a in A_s such that, for any $\mathfrak{m}\not\ni s$, $a_{\mathfrak{m}}$ is the image of a under the canonical map: $A_s\to A_{\mathfrak{m}}$. Now we shall show that $\mathfrak{I}$ is an ideal of A and then that $\mathfrak{I}$ contains 1. Let s and t be in $\mathfrak{I}$. To show $s+t\in\mathfrak{I}$, we may replace A by A_{s+t} and hence assume $As+At=A$. Then the following diagram of commutative rings is a pull-back diagram:

$$\begin{array}{ccc} A & \longrightarrow & A_s \\ \downarrow & & \downarrow \\ A_t & \longrightarrow & A_{st}\,. \end{array}$$

In fact Spec (A_s) and Spec (A_t) are affine open subsets of an affine scheme Spec(A), and hence the intersection of them is Spec (A_{st}) (cf. Chap. II, Prop. 2.2b in [2]). If $a\in A_s$ and $a'\in A_t$ have the required property in the definition of $\mathfrak{I}$, then we see easily by Lemma 5 and reducedness of A that a and a' have the same image in A_{st}. Therefore A contains an element a'' which has the images a and a' in A_s and A_t, respectively. It is clear that a'' has the required property for $s=1$ and hence $s+t$ is also an element of $\mathfrak{I}$. Moreover let s be in $\mathfrak{I}$ and x in A. Since $sx\notin\mathfrak{m}$ implies $s\notin\mathfrak{m}$, we see easily that sx is also cotained in $\mathfrak{I}$. Therefore $\mathfrak{I}$ is an ideal of A. Finally we shall show that $\mathfrak{I}$ is not contained in any maximal ideal $\mathfrak{m}$ of A. If we put $a_{\mathfrak{m}}=a/s$ with $a\in A$ and $s\notin\mathfrak{m}$, then we have $(a/s)^p=b/1$ and $pa/s=e/1$ in $A_{\mathfrak{m}}$. This means that there exists $t\in A-\mathfrak{m}$

such that $t^p(a^p-s^pb)=0$ and $t(pa-se)=0$. If $\mathfrak{m}'$ is any maximal ideal of A not containing st, then we have $(a/s)^p=(at/st)^p=b/1$ and $p(a/s)=p(at/st)=e/1$ in $A_{\mathfrak{m}'}$ and hence we see $a/s=a_{\mathfrak{m}'}$ by Lemma 5. This means that st belongs to $\mathfrak{I}$ by definition. But $\mathfrak{m}$ does not contain st and so $\mathfrak{m}$ does not contain $\mathfrak{I}$. Thus we have $\mathfrak{I}=A$ and hence α is an element of the image of A in $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$. q.e.d.

PROPOSITION 7. *Let A be a ring. If $A_{\mathfrak{m}}$ is weakly normal for any maximal ideal $\mathfrak{m}$ of A, then A is weakly normal.*

PROOF. It is easy to see that if a is not a zero-divisor of A, then the image of a in $\prod_{\mathfrak{m}} A_{\mathfrak{m}}$ is also not a zero-divisor. Therefore our assertion is a direct consequence of Lemma 6, Proposition 4 and Corollary to Proposition 2. q.e.d.

References

[1] N. Bourbaki: Algèbre commutative, Chap. 1, 2, Hermann, Paris, 1961.
[2] R. Hartshorne: Algebraic Geometry, Springer-Verlag, Berlin, 1977.
[3] S. Itoh: On weak normality and symmetric algebras, J. Algebra, **85** (1983), 40–50.
[4] H. Matsumura: Commutative algebra, Benjamin Inc., New York, 1970.
[5] R. G. Swan: On seminormality, J. Algebra, **67** (1980), 210–229.
[6] C. Traverso: Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa, **24** (1970), 585–595.
[7] H. Yanagihara: Some results on weakly normal ring extensions, J. Math. Soc. Japan, **35** (1983), 649–661.

Department of Mathematics
Hyogo University of Teacher Education
Shimokume, Yashiro-cho
Hyogo, 673-14 Japan