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(Citation)

Kobe Journal of Mathematics, 2(2):103-115

(Issue Date)

1985-12

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0034615>

(URL)

<https://hdl.handle.net/20.500.14094/E0034615>



ON THE FUNDAMENTAL THEOREM OF LATTICE-PRIMAL ALGEBRAS

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(Received January 9, 1984)

(Revised February 13, 1985)

ABSTRACT

It is known that in a variety generated by a lattice-primal algebra $\mathfrak{A}$ (i.e., an ordered algebra with a lattice ordering, with respect to which the operations are monotonic and in which every finitary monotone function is a term-function) every element is isomorphic to a distributive extension $\mathfrak{A}[\mathfrak{D}]$ (i.e., extension by a bounded distributive lattice $\mathfrak{D}$) of the generator algebra. Moreover, the variety as a category is equivalent to distributive lattices and antiequivalent to the category of all totally order-disconnected spaces, the so-called Priestley-spaces. This was essentially proved first by B. A. Davey, D. Duffus, R. W. Quackenbush and I. Rival. The results of our approach were proved independently two years later in 1980, and were announced at the Czechoslovakian Summer School in the same year. Our proofs seem to be more traditional and direct as they follow the lines of the corresponding earlier analogous results concerning varieties generated by primal algebras and Boolean extensions; but they are surely less elegant. Nevertheless, we still think that they reflect certain details better. These investigations were suggested by A. P. Huhn.

We introduce *normal subdirect powers* of ordered algebras, they contain the diagonal and are closed under every monotone finitary function performed componentwise. After this we show, by Jónsson's lemma that in a lattice-primal generated variety every member is essentially (i.e., up to isomorphism) a (normal) subdirect power of the generator algebra. The crucial result is to prove that normal subdirect powers are all distributive extensions; we succeeded in giving a constructive proof for this.

We also give two applications, first, we show that in a lattice variety generated by an order-functionally complete finite lattice $\mathfrak{L}$, every member containing $\mathfrak{L}$ as a "0, 1-sublattice" is isomorphic to a distributive extension of $\mathfrak{L}$. Secondly, we give a proof for a result of G. Gierz telling that given a lattice $\mathfrak{L}$ as before, gluing a distributive lattice $\mathfrak{D}$ into a prime interval $[a, b]$ of $\mathfrak{L}$, the free lattice over the variety generated by $\mathfrak{L}$, with the resulting partial lattice $L_{a,b}^{\mathfrak{D}}$ as a free generating set, is isomorphic to $\mathfrak{L}[\mathfrak{D}]$, the extension of $\mathfrak{L}$ by $\mathfrak{D}$.

A. L. Foster [1] introduced the concept of extending an algebra by a Boolean algebra: if $\mathfrak{A} = \langle A; F \rangle$ is an algebra, $\mathfrak{B} = \langle B; \wedge, \vee, ', 0, 1 \rangle$ a Boolean algebra, then the extension of $\mathfrak{A}$ by $\mathfrak{B}$ is the algebra, whose underlying set consists of all maps $\xi: A \rightarrow B$, which are of finite range, $a, b \in A, a \neq b$ implies $\xi(a) \wedge \xi(b) = 0$, and $\bigvee_{a \in A} \xi(a) = 1$. The operations are defined as follows: let f be an n -ary operation symbol from the type of $\mathfrak{A}$, then for any maps $\xi_1, \dots, \xi_n$ satisfying the conditions just mentioned, $f(\xi_1, \dots, \xi_n)$ is defined to be the map $A \rightarrow B$ for which

$$f(\xi_1, \dots, \xi_n)(a) = \bigvee_{\substack{a_1, \dots, a_n \in A \\ f(a_1, \dots, a_n) = a}} (\xi_1(a_1) \wedge \dots \wedge \xi_n(a_n)) \quad (a \in A).$$

For p -ary f , picking the element $a \in A$, let the value of f in $A[B]$ be the map sending a to 1 and all other elements of A to 0. The resulting algebra is denoted by $\mathfrak{U}[\mathfrak{B}]^* = \langle A[B]^*; F \rangle$. Foster has proved the following fundamental theorem: if $\mathfrak{U}$ is (finite) primal (i.e. every finitary function on A is a term-function on $\mathfrak{U}$), then in $HSP(\mathfrak{U})$, the variety generated by $\mathfrak{U}$, every algebra is isomorphic to an algebra of form $\mathfrak{U}[\mathfrak{B}]^*$ for a suitable Boolean-algebra $\mathfrak{B}$. Our aim is to extend these concepts and to give certain generalizations of these results. Namely, we will introduce the extensions of partially ordered algebras by bounded distributive lattices. Let a triple $\mathfrak{U} = \langle A; F, \leq \rangle$ be given, where $\langle A; F \rangle$ is a (finitary) universal algebra, $\leq$ is a partial order on A and the operations from F are all isotone with respect to this order; then it will be called a *partially ordered algebra* (or simply, *ordered algebra*). Moreover, let $\mathfrak{D}$ be a nontrivial bounded distributive lattice (now and in the sequel we always assume that the distributive lattices considered are all bounded, nontrivial, and all homomorphisms between them preserve bounds). By the *Priestley-space* of $\mathfrak{D}$ we mean the set $H_{\mathfrak{D}}$ of all dual prime ideals of $\mathfrak{D}$, endowed with the set-theoretic inclusion as a partial order, and with the topology, $\mathfrak{T}$ for which all the sets of form $\mathfrak{P}_d = \{P \mid P \in H_{\mathfrak{D}}, d \in P\}$ and $\mathfrak{D}_d = \{P \mid P \in H_{\mathfrak{D}}, d \notin P\}$, $d \in \mathfrak{D}$, constitute a sub-basis. It can be seen that $(H_{\mathfrak{D}}; \mathfrak{T}, \leq)$ is a compact and *totally order-disconnected* space (i.e., there exists a U clopen increasing subset of $H_{\mathfrak{D}}$ containing x and not containing y whenever x, y are elements of $H_{\mathfrak{D}}$ such that $x \not\leq y$), and that all the sets of the form $\mathfrak{P}_{d_1} \cap \mathfrak{D}_{d_2}$, $d_1, d_2 \in \mathfrak{D}$, give a basis of $\mathfrak{T}$. $\mathfrak{D}$ can be reobtained from its Priestley-space as the lattice of all clopen increasing subsets. (By an *increasing subset* we mean a subset U for which $x \in U$, $x \leq y$ always implies $y \in U$.) If $\mathfrak{D}$ is a Boolean lattice (or algebra), then its Priestley-space is just its Stone-space, for in this case the ordering in the Priestley-space is trivial (we call the equality relation on every set the *trivial order*), and the dual prime ideals not containing a given element d are just the dual prime ideals containing d' (the complement of d). (See Priestley [6]).

But $\mathfrak{U}[\mathfrak{B}]^*$ can also be defined in the following way: let $S_{\mathfrak{B}}$ be the Stone-space of $\mathfrak{B}$, and take all $S_{\mathfrak{B}} \rightarrow A$ continuous maps (A is considered as a discrete space); the operations are taken pointwise. (One can easily see that these maps form a subalgebra of $\mathfrak{U}^{S_{\mathfrak{B}}}$). This algebra is isomorphic to $\mathfrak{U}[\mathfrak{B}]^*$; and isomorphism is given by the correspondence $f \mapsto \xi_f$, where

$$\xi_f(a) = r^{-1}(\{a\}f^{-1}) \quad (a \in A);$$

r denotes the canonical isomorphism of $\mathfrak{B}$ onto the Boolean-algebra of all clopen subsets of $S_{\mathfrak{B}}$. (In the paper some map symbols are written to the left, while

others to the right of their argument, but this need not lead to any confusion.) Corresponding to this, if $\mathfrak{A}$ is an ordered algebra, $\mathfrak{D}$ is a 0, 1-distributive (= bounded distributive) lattice, then we define $\mathfrak{A}[\mathfrak{D}]$, the *extension of $\mathfrak{A}$ by $\mathfrak{D}$* , as the subalgebra of the direct power $\mathfrak{A}^{H_{\mathfrak{D}}}$, consisting of all $(H_{\mathfrak{D}}; \mathfrak{T}, \leq) \rightarrow \mathfrak{A}$ continuous monotone mappings ($\mathfrak{A}$ is endowed with discrete topology). This concept is not entirely new: its roots can already be found in Wille, Mitschke [9] and E. T. Schmidt [7] for certain lattices in place of $\mathfrak{A}$ (considered as ordered algebras in the natural way), so our generalization is very natural. Taking a universal algebra $\mathfrak{A}$ and a Boolean algebra $\mathfrak{B}$, giving $\mathfrak{A}$ the trivial order, the *distributive extension* $\mathfrak{A}[\mathfrak{B}]$ essentially gives back the Boolean extension $\mathfrak{A}[\mathfrak{B}]^*$.

Many fact suggests, that distributive lattices play a similar role in the theory of ordered algebras, as Boolean algebras do in the case of universal algebras. For example, it is natural to ask, whether in a variety generated by an *order-primal algebra* (i.e., a finite ordered algebra in which every order-preserving finitary function is expressible by a term-function) every algebra is isomorphic to a suitable distributive extension of the generator algebra. We given an affirmative answer under the assumption that the order of the latter algebra is a lattice-order or trivial. We also mention some lattice-theoretic application of the general results: first, if $\mathfrak{Q}$ is an order-functionally complete lattice (see Wille [8]), then every lattice in the variety generated by $\mathfrak{Q}$, in which $\mathfrak{Q}$ is a “0, 1-sublattice” (i.e. the zero and unit of $\mathfrak{Q}$ is the zero and unit of the lattice considered), is isomorphic to some $\mathfrak{Q}[\mathfrak{D}]$. As a second application, we show that — denoting $HSP(\mathfrak{Q})$ by $\mathfrak{B}$, and by $\mathfrak{Q}_{a,b}^{\mathfrak{Q}}$ the partial lattice obtained by gluing to the interval $[a, b]$ of $\mathfrak{Q}$ in a certain way, by $\mathfrak{Q}_{a,b}^{\mathfrak{Q}} - \mathfrak{F}_{\mathfrak{B}}(\mathfrak{Q}_{a,b}^{\mathfrak{Q}})$ (the free lattice over $\mathfrak{B}$ generated by $\mathfrak{Q}_{a,b}^{\mathfrak{Q}}$) is isomorphic to $\mathfrak{Q}[\mathfrak{D}]$ (see G. Gierz [3]).

1. Further Preliminaries In this section we develop some further general concepts and concerning ordered algebras.

Let $\mathfrak{A}, \mathfrak{B}$ be ordered algebras of the same similarity type. An isotone, operation-preserving map $f: A \rightarrow B$ is said to be a *homomorphism*; it is a *Q-homomorphism*, if the restriction of the order of $\mathfrak{B}$ is the smallest order on $Im(f)$, with respect to which the operations of $\mathfrak{B}$ are monotone and f is isotone. $\mathfrak{B}$ is called a *homomorphic (Q-homomorphic) image* of $\mathfrak{A}$, if there exists an $f: \mathfrak{A} \rightarrow \mathfrak{B}$ surjective *homomorphism (Q-homomorphism)*. Let H (respectively Q) denote the corresponding (closure) operator on classes of ordered algebras, let S and P be the operators of taking *subalgebras* and *direct products*; the order is restricted in the former case to the subuniverse, and is taken componentwise, in the latter.

The kernels of homomorphisms give the *congruences*. It would be possible to characterize directly congruences and also *Q-homomorphisms*, but this will be not needed in the sequel.

A class of ordered algebras is called *variety (Q-variety)*, if it is closed under

H , S and P (Q , S and P). For any class K , $HSP(K)$ will be the smallest variety containing K , i.e. the variety generated by K . Unfortunately, the Q -variety generated by K cannot be described in such an easy way. $QSP(K)$ is not a Q -variety in general, because the operator-inequality $SQ \leq QS$ need not hold, as trivial examples show. But the concepts of the variety and Q -variety coincide in the most important case treated by us, as it can be seen from

LEMMA 1.1. *If in every element of a class K of ordered algebras the order is a lattice-order, and the corresponding lattice operations, $\wedge$ and $\vee$ are term functions given by common terms in all K -algebras, then the order in every algebra from $HSP(K)$ is a lattice-order, every homomorphism in $HSP(K)$ is a Q -homomorphism, and $HSP(K) (= QSP(K))$ is the Q -variety generated by K .*

PROOF. Under the given assumptions it is clear that the orders in $P(K)$ are lattice-orders and the corresponding $\wedge$ and $\vee$ are expressed by the same terms. Hence subalgebras of elements of $P(K)$ are closed under $\wedge$ and $\vee$, and so orders in $SP(K)$ are all lattice-orders. Let the two fixed terms for $\wedge$ and $\vee$ be denoted by these symbols themselves. Now let $f: \mathfrak{B} \rightarrow \mathfrak{C}$ be a surjective homomorphism, where $\mathfrak{B} \in SP(K)$. In $\mathfrak{C}$ $\wedge$ and $\vee$ also determine a lattice (because homomorphic images preserve identities); let $\leq$ denote the ordering of this lattice. The operations of $\mathfrak{C}$ are isotone with respect to $\leq$: indeed, if μ and n -ary operation ($n \geq 1$), $a_i \leq b_i$ in $\mathfrak{C}$, $i=1, \dots, n$, then —by the $\wedge$, $\vee$ -preserving property of f and by a well-known property of lattice homomorphisms— there exist elements $u_i, v_i \in B$, $i=1, \dots, n$, such that $u_i f = a_i$, $v_i f = b_i$, $u_i \leq v_i$. By the monotonicity of μ in $\mathfrak{B}$, $\mu(u_1, \dots, u_n) \rightarrow \mu(v_1, \dots, v_n)$, i.e., $\mu(u_1, \dots, u_n) \wedge \mu(v_1, \dots, v_n) = \mu(u_1, \dots, u_n)$; applying f we get $\mu(a_1, \dots, a_n) \wedge \mu(b_1, \dots, b_n) = \mu(b_1, \dots, b_n)$, and this means that $\mu(a_1, \dots, a_n) \leq \mu(b_1, \dots, b_n)$. The term-functions $\wedge$ and $\vee$ are also monotone with respect to the original order of $\mathfrak{C}$. If $c \leq d$ in $\mathfrak{C}$, then —as we have just seen above— there exist $a \leq b \in B$, such that $af = c$, $bf = d$. By the monotonicity of f , $c \leq d$. This means, that $\leq$ is finer than $\leq$. But it cannot be properly finer, because any proper refinement of a lattice order leads to losing of the monotonicity of the corresponding lattice-operations. Thus $\leq$ is same as $\leq$.

Similarily, if $\mathfrak{B}, \mathfrak{C}$ are arbitrary elements of $HSP(K)$, and f is a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$, then the restriction of the order to $\mathfrak{C}$ to $Im(f)$ is just the ordering of the lattice “carried over by f ”. The proof is complete.

If all orderings of elements of K are trivial, then the Q -variety generated by K can be obtained from the ordinary (unordered) variety of K by endowing all algebras in it with the trivial order. We get the (ordered) variety generated by K from the latter variety, if we endow every element from it with an operation-compatible partial order in every possible way.

In the sequel by a variety we always mean a Q -variety. This agreement is supported by two circumstances: on one hand, due to the preceding remarks, every variety of unordered algebras is essentially a Q -variety, on the other hand, the formulation of the statement concerning order-primal generated varieties raised in the Introduction shows, that if this statement is true, then the order of $\mathfrak{A}$ totally, determines all possible orderings on elements of the Q -variety generated by $\mathfrak{A}$, but this is obviously false in $HSP(\mathfrak{A})$ in general.

Now we prove an analogue of the famous lemma of B. Jónsson (see [5]).

LEMMA 1.2. *Under the assumptions of Lemma 1.1, all subdirectly irreducible elements of $QSP(K)$ are in $QSP_u(K)$ (P_u denotes the operator of forming ultraproducts), and $QSP(K) = IP_s QSP_u(K)$, with P_s as the operator of taking subdirect products.*

PROOF. First note that $HSP(K) = QSP(K)$ by Lemma 1.1, and, using the lattice-structures, it is easily checked that the congruences on elements of $HSP(K)$ are just the operation-compatible equivalences. Moreover, by the presence of lattice-operations, $HSP(K)$ is congruence-distributive (see [5], or in [4], p. 354). Now we can conclude the proof in the same way as in [4], Chapter 6, Theorem 6 in §39, where the expression “prime product” is used for the ultraproduct.

DEFINITION 1.3. Let $\mathfrak{A}$ be an ordered algebra and I any nonvoid index-set, $\tilde{\mathfrak{A}}$ a subalgebra of the direct power $\mathfrak{A}^I$. $\tilde{\mathfrak{A}}$ is called a *normal subdirect power*, provided

- (i) it contains the diagonal (i.e. the set of all $I \rightarrow A$ constant maps);
- (ii) It is closed under every finitary function defined on A and monotone with respect to the ordering of $\mathfrak{A}$, if applied pointwise (i.e. if $\alpha_1, \alpha_2, \dots, \alpha_n$ are elements of $\tilde{\mathfrak{A}}$ and f is an n -ary function on A as mentioned, then the choice function from A^I defined by $f(\alpha_1, \alpha_2, \dots, \alpha_n)(i) = f(\alpha_1(i), \alpha_2(i), \dots, \alpha_n(i))$ for every $i \in I$, belongs to $\tilde{\mathfrak{A}}$).

Note that if $\mathfrak{A}$ has the trivial order, then we essentially get the classical concept of normal subdirect powers, using the fact that Grätzer's quaternary function T with all constants generates all functions (see Grätzer [4], p. 150).

2. The Main Theorem For a finite ordered algebra $\mathfrak{A}$, let $\mathcal{U}(\mathfrak{A})$ denote the set of all increasing subsets of A (including $\emptyset$ and A). To any normal subdirect power $\tilde{\mathfrak{A}} \leq \mathfrak{A}^I$ we can assign a 0, 1-distributive sublattice $\mathfrak{D}$ of the lattice of all subsets of I , in the following way: let its base set be $D = \{H \mid H \subseteq I \text{ and } H = Bf^{-1} \text{ for suitable } B \in \mathcal{U}(\mathfrak{A}) \text{ and } f \in \tilde{\mathfrak{A}}\}$; we show that D is closed under the set-theoretic union and intersection, and contains $\emptyset$ and I . Take $B = \emptyset$ and any $f \in \tilde{\mathfrak{A}}$, for the $\emptyset$, and $B = A$ and any $f \in \tilde{\mathfrak{A}}$ for I . The case $|A| = 1$ being trivial, assume $|A| \geq 2$.

Fix any $a, b \in A$ such that $a < b$ in $\mathfrak{A}$ (if there are no such elements, then we may take any a, b with $a \neq b$). For any increasing subset B of A (i.e., $B \in \mathfrak{B}(\mathfrak{A})$) introduce the following functions on A :

$$\mathfrak{B}_B(x) = \begin{cases} b, & \text{if } x \in B \\ a, & \text{otherwise,} \end{cases}$$

$$\delta(x, y) = \begin{cases} b, & \text{if } x \geq b \text{ or } y \geq b \\ a, & \text{otherwise,} \end{cases}$$

$$\hat{\delta}(x, y) = \begin{cases} b, & \text{if } x \geq b \text{ and } y \geq b \\ a, & \text{otherwise.} \end{cases}$$

Trivial computation shows that these are all monotone. Moreover, for any poset $(P, \leq)$ and $x \in P$ let, in general, $[x] = \{y \in P \mid y \geq x\}$.

Taking any element d_1, d_2 of D , there exist $B_i \in \mathfrak{B}(\mathfrak{A})$, $f_i \in \tilde{\mathfrak{A}}$ such that $d_i = B_i f_i^{-1}$ ($i=1, 2$). Now the maps $g(t) = \delta(\mathfrak{B}_{B_1}(f_1(t)), \mathfrak{B}_{B_2}(f_2(t)))$, and $h(t) = \hat{\delta}(\mathfrak{B}_{B_1}(f_1(t)), \mathfrak{B}_{B_2}(f_2(t)))$, where t runs over I , obviously belong to $\tilde{\mathfrak{A}}$ by normality, and $[b]g^{-1} = d_1 \cup d_2$, $[b]h^{-1} = d_1 \cap d_2$. Hence $d_1 \cup d_2, d_1 \cap d_2 \in D$, which was to be shown. Putting $d_1 = d_2$ we also see that

$$D = \{H \mid H \subseteq I \text{ and } H = [b]f^{-1} \text{ for an } f \in \tilde{\mathfrak{A}}\}.$$

In the case of trivially ordered $\mathfrak{A}$, D becomes a Boolean lattice, because every subset of A is increasing, so —as can be seen by the original definition of D — D contains the complement of every its element.

$$\text{Let } \mathfrak{D} = (D; \cup, \cap).$$

LEMMA 2.1. $\tilde{\mathfrak{A}}$ is isomorphic to a subalgebra of $\mathfrak{A}[\mathfrak{D}]$.

PROOF. Consider $(X; \mathfrak{I}, \leq)$, the Priestley-space of $\mathfrak{D}(X = H_{\mathfrak{D}})$. We assign a well-determined element F_f of $\mathfrak{A}[\mathfrak{D}]$ to any $f \in \tilde{\mathfrak{A}}$; the correspondence $f \mapsto F_f$ will give an embedding. Take arbitrary $f \in \tilde{\mathfrak{A}}$, and fix it. For any $B \in \mathfrak{B}(\mathfrak{A})$ let d_B denote the set Bf^{-1} ; write simply d_b for $d_{[b]}$. Now let F_f assign to the dual prime ideal P the value $a \in A$ iff $P \in \mathfrak{P}_{d_a}$ but $P \notin \mathfrak{P}_{d_{[a]} \setminus \{a\}}$; i.e. for any $a \in A$,

$$\{a\}F_f^{-1} = \mathfrak{P}_{d_a} \setminus \mathfrak{P}_{d_{[a]} \setminus \{a\}} \quad (1)$$

F_f is well-defined: Let us assume, that $a, b \in A$, $a \neq b$ both correspond to P . $a < b$ is impossible, for then $P \in \mathfrak{P}_{d_b}$, $d_{[a] \setminus \{a\}} \supseteq d_b$ implies $d_{[a] \setminus \{a\}} \in P$, so $P \in \mathfrak{P}_{d_{[a]} \setminus \{a\}}$, a contradiction, because F_f sends P to a by assumption. Hence $a \parallel b$ (incomparable). $d_a, d_b \in P$, so $d_a \cap d_b \in P$, but $d_a \cap d_b = [a]f^{-1} \cap [b]f^{-1} = ([a] \cap$

(b)) $f^{-1} = d_{[a] \cap [b]} \subseteq d_{[a] \setminus \{a\}}$, hence $d_{[a] \setminus \{a\}} \in P$, that is, $P \in \mathfrak{P}_{d_{[a] \setminus \{a\}}}$, again a contradiction.

F_f is defined everywhere; Let $A_1, \dots, A_k$ be all the increasing subsets of A (A is finite!) the f -inverse image of which belongs to P . Then $d_{A_1} \cap \dots \cap d_{A_k} \in P$, and so, by $d_{A_1 \cap \dots \cap A_k} = d_{A_1} \cap \dots \cap d_{A_k}$, $d_{A_1 \cap \dots \cap A_k} \in P$. From this we see that between the A_i there is a smallest, say A_1 . If $a_1, \dots, a_m$ are all the minimal elements of A_1 , then $A_1 = [a_1] \cup \dots \cup [a_m]$, so $d_{A_1} = d_{a_1} \cup \dots \cup d_{a_m}$. $d_{A_1} \in P$ implies (using the fact, that P is prime), that some d_{a_i} belongs to P , say, d_{a_1} . If we had $m > 1$, then —being d_{a_1} a proper subset of d_{A_1} — we would get a contradiction. This means that $m = 1$, $A_1 = [a_1]$, $P \in \mathfrak{P}_{d_{a_1}}$, but $P \notin \mathfrak{P}_{d_{[a_1] \setminus \{a_1\}}}$ by the minimality of A_1 . Hence F_f assigns a_1 to P , by (1).

F_f is continuous: Indeed, it is sufficient to show that the right-hand side of (1) is always clopen. Taking any $d \in D$. $\mathfrak{P}_d$ belongs to the canonical sub-basis of $\mathfrak{A}$, which consists only of clopen sets; observe that $[a] \setminus \{a\} = \bigcup_{b > a} [b]$, so $d_{[a] \setminus \{a\}} = ([a] \setminus \{a\})f^{-1} = \bigcup_{b > a} [b]f^{-1} = \bigcup_{a > a} d_b$, hence $\mathfrak{P}_{d_{[a] \setminus \{a\}}} = \bigcup_{a > b} \mathfrak{P}_{d_b}$ and the assertion follows, because this union is finite.

F_f is monotone: let $P_1 \subseteq P_2$, $P_1 F_f = a$, $P_2 F_f = b$, but $a \not\leq b$. By definition, $d_a \in P_1$, but then also $d_a \in P_2$; this, together with $d_b \in P_2$, implies $d_a \cap d_b = d_{[a] \cap [b]} \in P_2$. $a \not\leq b$ shows, that $[b] \setminus \{b\} \supseteq [a] \cap [b]$, hence $d_{[b] \setminus \{b\}} \in P_2$, contradiction with $P_2 F_f = b$. The facts just proved show that $F_f \in \mathfrak{U}[\mathfrak{D}]$.

Now consider the correspondence $\varphi: f \mapsto F_f$ with f changing. φ is operation-preserving: Let μ an n -ary operation, $n > 0$, $f_1, \dots, f_n \in \tilde{\mathfrak{U}}$. We have to show that $F_\mu(f_1, \dots, f_n)$ assigns $\mu(F_{f_1}(P), \dots, F_{f_n}(P))$ to any P . Let $F_{f_i}(P) = a_i$, $i = 1, \dots, n$, $\mu(f_1, \dots, f_n) = h$, $\mu(a_1, \dots, a_n) = a$. By the monotonicity of μ , $[a_1]f_1^{-1} \cap \dots \cap [a_n]f_n^{-1} \subseteq [a]h^{-1}$; $F_{f_i}(P) = a_i$ implies $[a_i]f_i^{-1} \in P$, but then the left-hand side of the inclusion, consequently, also its right-hand side, belongs to P . The proof will be complete if we point out that $([a] \setminus \{a\})h^{-1} \notin P$. Suppose, that contrary, $([a] \setminus \{a\})h^{-1} \in P$.

$$\begin{aligned} ([a] \setminus \{a\})h^{-1} &= \{x \in I \mid \mu(f_1(x), \dots, f_n(x)) > a\} \\ &= \bigcup_{\mu(b_1, \dots, b_n) > a} (\{b_1\}f_1^{-1} \cap \dots \cap \{b_n\}f_n^{-1}) \\ &= \bigcup_{\mu(b_1, \dots, b_n) > a} ([b_1]f_1^{-1} \cap \dots \cap [b_n]f_n^{-1}), \end{aligned}$$

since if $\hat{b}_i \geq b_i$, $f_i(x) = \hat{b}_i$, $i = 1, \dots, n$, the monotonicity of μ implies $\mu(f_1(x), \dots, f_n(x)) = \mu(\hat{b}_1, \dots, \hat{b}_n) \geq \mu(b_1, \dots, b_n) > a$. The latter union in the above computation is finite, so some its member must be in P , that is, for some $b_1, \dots, b_n \in A$ with $\mu(b_1, \dots, b_n) > a$, $[b_1]f_1^{-1} \cap \dots \cap [b_n]f_n^{-1} \in P$, and then $[b_i]f_i^{-1} \in P$, $i = 1, \dots, n$. For an i we necessarily have $b_i \not\leq a_i$, otherwise $\mu(b_1, \dots, b_n) \leq \mu(a_1, \dots, a_n) = a$ would hold. Let, for example, $b_1 \not\leq a_1$. We have $[b_1]f_1^{-1} \in P$ and $[a_1]f_1^{-1} \in P$, so $[b_1]f_1^{-1} \cap$

$[a_1]f_1^{-1} = ([b_1] \cap [a_1])f_1^{-1} \in P$. But, by $b_1 \not\leq a_1$, $[a_1] \cap [b_1] \subseteq [a_1] \setminus \{a_1\}$, hence $([a_1] \setminus \{a_1\})f_1^{-1} \in P$, a contradiction.

φ is monotone: Given any $f, g \in \tilde{\mathfrak{A}}$, with $f \leq g$ and a dual prime ideal P , let $F_f(P) = a$, i.e., $[a]f^{-1} \in P$, $([a] \setminus \{a\})f^{-1} \notin P$. $f \leq g$ implies $[a]g^{-1} \supseteq [a]f^{-1}$, therefore $[a]g^{-1} \in P$. Again, there is a smallest ($\emptyset \neq$) $H \subseteq [a]$ increasing subset with the property $Hg^{-1} \in P$, and this is of the form $H = [b]$. Obviously, $F_g(P) = b$, and — by $H \subseteq [a] - a \leq b$. Being P arbitrary, this means that $F_f \leq F_g$.

φ is “inverse-monotone”: Let us suppose that $F_f \leq F_g$, but still $a = f(t) \not\leq g(t)$ for some $t \in I$. Then $t \in [a]f^{-1} = H$, $t \notin [a]g^{-1} = K$. Choose a dual prime ideal P of $\mathfrak{D}$ containing H and not containing the set $H_0 = ([a] \setminus \{a\})f^{-1} \cap K$ (this is possible, because $H_0 \in D$, and $H_0 \not\supseteq H$, since $t \in H$, $t \notin H_0$). $H_0 \notin P$ implies $([a] \setminus \{a\})f^{-1} \notin P$, so $F_f(P) = a$. On the other hand, $[\hat{a}]g^{-1} \in P$ for no $\hat{a} \geq a$, otherwise $[a]g^{-1} = K \in P$, therefore $H_0 \in P$ would hold. Hence $F_g(P) \neq \hat{a}$ for these $\hat{a}$, which means $F_g(P) \not\geq F_f(P)$. But this is impossible by assumption. Thus, φ is an embedding.

The second important result towards the main theorem is

THEOREM 2.2. *If $\mathfrak{A}$ is a finite ordered algebra with lattice-order or the trivial order, and $\tilde{\mathfrak{A}} \subseteq \mathfrak{A}^I$ is a normal subdirect power, then the φ of the preceding lemma is also surjective, that is, an isomorphism.*

PROOF. Take arbitrary $F \in \mathfrak{A}[\mathfrak{D}]$, i.e. $(\mathfrak{X}; \mathfrak{A}, \leq) \rightarrow \mathfrak{A}$ continuous monotone mapping. Let $\{a_1, \dots, a_m\}$ be the range of F , with the a_i are pairwise distinct. The sets $A_i = [a_i]F^{-1}$ are clopen increasing, so they represent elements d_i of D , hence consist of all dual prime ideals containing d_i : $A_i = \mathfrak{P}_{d_i}$. $d_i \in D$ means that $d_i = A_i^? f_i^{-1}$ for suitable $f_i \in \tilde{\mathfrak{A}}$ and $A_i^? \in \mathcal{U}(\mathfrak{A})$ (increasing subset of A), thus $A_i = \mathfrak{P}_{A_i^? f_i^{-1}}$.

First suppose $\mathfrak{A}$ to be lattice-ordered, and consider the functions

$$\eta_{A_i^?}^{a_i}(t) = \begin{cases} a_i, & \text{if } t \in A_i^?, \\ 0, & \text{otherwise} \end{cases} \quad i = 1, \dots, m,$$

and

$$\theta(t_1, \dots, t_m) = \bigvee_{i: t_i \geq a_i} a_i,$$

where 0 is the zero and $\vee$ is the join corresponding to the lattice-ordering of $\mathfrak{A}$, the empty join means 0. These are all monotone, so $\tilde{\mathfrak{A}}$ is closed under them.

We claim that F is the φ -image of $h = \theta(\eta_{A_1^?}^{a_1}(f_1), \dots, \eta_{A_m^?}^{a_m}(f_m)) \in \tilde{\mathfrak{A}}$: $F_h = F$. Suppose $F(P) = a_i$. For any j , P belongs to $[a_j]F^{-1} = A_j$ iff $a_j \leq a_i$, that is,

$$A_j^? f_j^{-1} \in P \quad \text{iff} \quad a_j \leq a_i. \quad (2)$$

Consider $[a_i]h^{-1}$. Of course $[a_i]h^{-1} \supseteq \bigcap_{j: a_j \leq a_i} A_j^o f_j^{-1}$ by the definition of $\mathfrak{B}$ and the $\eta_{A_j^o}^j$, and the meet is in P by (2). Hence $[a_i]h^{-1} \in P$. Now it suffices to prove that $([a_i] \setminus \{a_i\})h^{-1} \notin P$, since this means $F_h(P) = a_i$, being P arbitrary. Assume $([a_i] \setminus \{a_i\})h^{-1} \in P$. Then

$$\begin{aligned} ([a_i] \setminus \{a_i\})h^{-1} &\subseteq \bigcup_{l: a_l \not\leq a_i} \{s \mid \eta_{A_l^o}^{a_l}(f_l(s)) \geq a_l\} = \\ &= \bigcup_{l: a_l \not\leq a_i} \{s \mid f_l(s) \in A_l^o\} = \bigcup_{l: a_l \not\leq a_i} A_l^o f_l^{-1}, \end{aligned}$$

since if a join is strictly greater than a_i , then some of its member is not less than or equal to a_i . So $\bigcup_{l: a_l \not\leq a_i} A_l^o f_l^{-1} \in P$, and since P is prime, there exists $A_l^o f_l^{-1} \in P$, and for this l we have $a_l \not\leq a_i$, but this is a contradiction with (2).

In the case of trivially ordered $\mathfrak{A}$ replace θ by

$$\hat{\theta}(t_1, \dots, t_m) = \begin{cases} a_i, & \text{if } t_j = a_j \text{ holds exactly when } j = i, \\ b \text{ (a fixed element),} & \text{otherwise;} \end{cases}$$

the argument will be correct.

REMARKS. 1. The definition of the map F_f seems to be very complicated as it is put in (1), but still it is natural, because f is determined by the sets $\{a\}f^{-1} = [a]f^{-1} \setminus ([a] \setminus \{a\})f^{-1}$, and in fact the definition (1) “transplants” from I to $X = H_{\mathfrak{D}}$. Indeed, $\mathfrak{B}_{a_i}$ and $\mathfrak{B}_{[a_i] \setminus \{a_i\}}$ are the images of the D -elements $[a]f^{-1}$ and $([a] \setminus \{a\})f^{-1}$ under the canonical isomorphism of $\mathfrak{D}$ onto the lattice of clopen increasing subsets of $(0; \mathfrak{A}, \leq)$.

2. It is an open question yet, whether Theorem 2.2 remains valid, if we weaken the assumptions on the ordering of $\mathfrak{A}$.

3. The proof of Lemma 2.1 and Theorem 2.2 cannot be simplified so that I be itself the underlying set of a Priestley space and the maps $f \in \tilde{\mathfrak{A}}$ are all the continuous monotone mappings of this space into $\mathfrak{A}$. For example, if I is infinite, $|A| \geq 2$, $\tilde{\mathfrak{A}} = \mathfrak{A}^I$, then the continuity of all f yields that *every* one-element subset of I is clopen, but this is impossible by compactness.

Now we are able to prove the main theorem:

THEOREM 2.3. (The Fundamental theorem). *Let $\mathfrak{A}$ be a nontrivial order-primal algebra with lattice order or trivial order. Then every element of $QSP(\mathfrak{A})$ is, up to isomorphism, of form $\mathfrak{A}[\mathfrak{D}]$ for a 0, 1-distributive lattice $\mathfrak{D}$. $\mathfrak{D}$ is Boolean if $\mathfrak{D}$ is trivially ordered.*

PROOF. It is a matter of routine to check that in both cases $\mathfrak{A}$ is simple and has no proper subalgebras. By lemma 1.2 (resp. the classical Jónsson-lemma) the nontrivial subdirectly irreducible members of $QSP(\mathfrak{A})$ are the isomorphic copies

of $\mathfrak{A}$, since $QSP_u(\mathfrak{A}) = QS(\mathfrak{A}) = Q(\mathfrak{A})$. So any $\mathfrak{B} \in QSP(\mathfrak{A})$ is essentially a subdirect power of $\mathfrak{A}$, or trivial. In the latter case $\mathfrak{B}$ can be considered as the algebra of all continuous monotone maps of the *empty* Priestley-space into $\mathfrak{A}$ (the only such map is $\emptyset$). In the former case $\mathfrak{B}$ is necessarily a normal subdirect power, and then Theorem 2.2 applies.

Obviously, the part of this theorem concerning trivially ordered $\mathfrak{A}$ essentially tells the same as Foster's fundamental theorem, mentioned in the introduction.

We do not know, whether the restrictions on the ordering of $\mathfrak{A}$ can be weakened or not. But any nontrivial ordering necessarily must have a connected graph, otherwise we would get a nontrivial congruence on $\mathfrak{A}$ by considering the partition consisting of the connected components; this would yield a $\mathfrak{B} \in QSP(\mathfrak{A})$ with $1 < |B| < |A|$, and no $\mathfrak{A}[\mathfrak{D}]$ satisfies such inequalities in place of $\mathfrak{B}$.

3. Two Lattice-Theoretic Applications In this section we prove those lattice-theoretic consequences of the main results, which were mentioned in the last part of the introduction.

THEOREM 3.1. *Let $\mathfrak{Q}$ be a finite, order-functionally-complete lattice (that is, every monotone function is an algebraic function). If $\mathfrak{R} \in HSP(\mathfrak{Q})$, $\mathfrak{R}$ is bounded and $\mathfrak{Q}$ is isomorphic to a sublattice of $\mathfrak{R}$ so that the zero [resp. unit] of $\mathfrak{Q}$ corresponds to zero [resp. unit] of $\mathfrak{R}$, then $\mathfrak{R}$ is isomorphic to some $\mathfrak{Q}[\mathfrak{D}]$, where $\mathfrak{D}$ is a 0, 1-distributive lattice.*

PROOF. By Jónsson's lemma, we can assume that $\mathfrak{R}$ is a subdirect product of homomorphic images of sublattices of $\mathfrak{Q}$, let I be the index set. Denote the bounds of $\mathfrak{Q}$ by 0 and 1, and let φ be the embedding $\mathfrak{Q} \rightarrow \mathfrak{R}$ in question. Since $\mathfrak{R}$ is subdirect, 0φ and 1φ are necessarily the constant maps $I \rightarrow \{0\}$ and $I \rightarrow \{1\}$, respectively. Denote by $\bar{\pi}_i$ the restriction of π_i (the i -th projection of $\mathfrak{R}$ onto $\mathfrak{Q}$) to $L\varphi$ ($i \in I$). By the previous remark the range of $\bar{\pi}_i$ contains 0 and 1, the zero and the unit of the i th factor, but then the simplicity ^{*)} of $\mathfrak{Q}$ (see Wille [8]) yields, that $\ker \bar{\pi}_i$ is the identity relation on $L\varphi$, so $\varphi\pi_i$ is an automorphism of $\mathfrak{Q}$. Using this family of automorphisms $\{\varphi\pi_i | i \in I\}$, one can easily see that $\mathfrak{R}$ is isomorphic to a *diagonal* subdirect power of $\mathfrak{Q}$, which is normal by the order-functional-completeness of $\mathfrak{Q}$. Now, in view of Theorem 2.2, the assertion follows.

^{*)} We also give here a simple proof. If $a \equiv b(\theta)$, $a < b$ and c, d are arbitrary with $c < d$ (for example, $c=0$ and $d=1$), then consider the (monotonic) function $\mu: L \rightarrow L$, $\mu(x)=d$ if $x \geq b$ and $\mu(x)=c$, otherwise. Obviously, $\mu(a)=c$, $\mu(b)=d$, and by order-functional-completeness, μ can be considered as an algebraic function, but this implies that also $c \equiv d(\theta)$. For lattices this means that the equality relation and the universal relation are the only congruences, i.e. $\mathfrak{B}$ is simple (for the behaviour of lattice-congruences, see G. Grätzer, General Lattice Theory, Birkhäuser, 1978).

Finally, we reprove the theorem of G. Gierz mentioned in the beginning of the paper.

THEOREM 3.2. (G. Gierz [3]). *Let $\mathfrak{Q}$ be as in the previous theorem. Given a 0, 1-distributive lattice $D = (D; \wedge, \vee, 0, 1)$, a prime interval $[a, b]$ of $\mathfrak{Q}$, consider the partial lattice $L_{a,b}^D$, which arises by gluing D into the interval $[a, b]$ (i.e., a and b is identified with 0 and 1, respectively, the partial order remains the same inside of L and D as it was before; moreover, for $d \in D$ and $c \in L$, $d \leq c$ iff $a \leq c$ and $d \leq c$ iff $c \leq b$), and let $\mathfrak{B}$ denote $\text{HSP}(\mathfrak{Q})$, the (lattice-) variety generated by $\mathfrak{Q}$. Then $\mathfrak{F}_{\mathfrak{B}}(L_{a,b}^D)$, the free lattice over $\mathfrak{B}$ generated by $L_{a,b}^D$ exists, and it is isomorphic to $\mathfrak{Q}[\mathfrak{D}]$.*

PROOF. First we note, that $\mathfrak{B}$ essentially consists of all subdirect powers of $\mathfrak{Q}$ by congruence-distributivity via Jónsson's lemma. So if we are given any "regular" map of $L_{a,b}^D$ into a $\mathfrak{B}$ -lattice (ie. a monotone map preserving the lattice-structure both in $\mathfrak{Q}$ and $\mathfrak{D}$, the latter glued into $[a, b]$), then we can decompose it into (regular) maps $L_{a,b}^D \rightarrow \mathfrak{Q}$ (the original map followed by the corresponding projections in the subdirect representation of the lattice in question). Consider now any regular map $\varphi: L_{a,b}^D \rightarrow \mathfrak{Q}$. Apart from the case in which φ is constant, $\ker(\varphi \upharpoonright L)$ is a congruence on L with at least two congruence classes. But as we have already seen in Theorem 3.1, $\mathfrak{Q}$ is simple, and this means, that $\varphi \upharpoonright L$ is essentially an automorphism of $\mathfrak{Q}$ (identifying L in $L_{a,b}^D$ with that L , into which φ maps $L_{a,b}^D$). Obviously, $\varphi \upharpoonright D$ sends a dual ideal of D to $b\varphi$, while the other elements of D to $a\varphi$. (Here we utilize the trivial fact that $[a\varphi, b\varphi]$ is a prime interval of $\mathfrak{Q}$). Consider now all nonconstant regular maps $\varphi_\gamma: L_{a,b}^D \rightarrow \mathfrak{Q}$ ($\gamma \in \Gamma$, an index set for indexing all these mappings). Take the subalgebra $\mathfrak{F}$ of $\mathfrak{Q}^\Gamma$ generated by all elements ("choice functions") of the form $\hat{l} = \{l\varphi_\gamma \mid \gamma \in \Gamma\}$, $l \in L_{a,b}^D$ arbitrary. It is easily seen that $\mathfrak{F}$ contains $L_{a,b}^D$ as a relative partial sublattice (identifying any $l \in L_{a,b}^D$ with $\hat{l}$), this $L_{a,b}^D$ generates $\mathfrak{F}$ by definition; moreover, it is clear, that any regular map of this $L_{a,b}^D$ (as inside of $\mathfrak{F}$) into an arbitrary $\mathfrak{B}$ -lattice can be extended to a homomorphism of $\mathfrak{F}$ into that lattice (consider any representation of the latter as a subdirect power of $\mathfrak{Q}$, factorize the given regular map as above, from these factors keep only one copy for each collection of equal maps, and then notice, that the remaining factors all occur in the construction of $\mathfrak{F}$). Next, observe that we can omit all the φ_γ , for which $\varphi_\gamma \upharpoonright L$ is not the identity map of L . Indeed, all these φ_γ can be obtained from such a $\varphi_{\gamma'}$, for which $\varphi_{\gamma'} \upharpoonright L$ is the identity map, multiplying it with a suitable automorphism of $\mathfrak{Q}$ (see above), and a straightforward computation shows that after dropping these factors from the construction of $\mathfrak{F}$, the new $\mathfrak{F}$ still remains isomorphic with the old one*). Hence we can assume from now on that for all $\gamma \in \Gamma$, $\varphi_\gamma \upharpoonright L$ is the

*) Of course, the φ_γ 's with only one value (i.e., constants) can also be ignored.

identity map of L , and we have proved, that $\mathfrak{F}$ is the free lattice over $\mathfrak{B}$ generated by $L_{a,b}^D$. We shall prove about this $\mathfrak{F}$, making use of the principle just before Lemma 2.1 and Theorem 2.2, that it is isomorphic to $\mathfrak{L}[\mathfrak{D}]$.

For this we first note that $\mathfrak{F}$ is obviously a normal subdirect power of $\mathfrak{L}$ because of the order-functional-completeness ($\mathfrak{F}$ contains the diagonal: the l 's ($l \in L$) constitute it). So, by Theorem 2.2, $\mathfrak{F}$ is isomorphic to a distributive extension $\mathfrak{L}[\hat{\mathfrak{D}}]$, and it is sufficient to show, that $\mathfrak{D} \approx \hat{\mathfrak{D}}$. $\hat{D}$, the underlying set of $\hat{\mathfrak{D}}$ is obtained by taking the inverse images of increasing subsets of $\mathfrak{L}$ under the choice functions belonging to $\mathfrak{F}$ (i.e. the sets of those components, which belong to a given increasing subset of $\mathfrak{L}$). But we can restrict ourselves to increasing subsets of the form $[c]$, $c \in L$; see the remarks before Lemma 2.1. Now we finish the proof in two steps: first, we show that the elements of D can be found in $\hat{D}$ in a natural way (Step 1), second, we see that every element of $\hat{D}$ is in fact an element of D in this interpretation (Step 2.).

Step 1. Given any element $d \in D$, take the element $\hat{d} = \{d\varphi_\gamma | \gamma \in \Gamma\}$ of $\mathfrak{F}$ (this is a choice function over Γ from $\mathfrak{L}$, so it is really an element of $\mathfrak{F}$), and then consider the subset of Γ $b \hat{d}^{-1} = \{\varphi | d\varphi = b\}$. The latter belongs to $\hat{\mathfrak{D}}$ by definition, because $\{\varphi | d\varphi = b\} = \{\varphi | d\varphi \geq b\}$. The correspondence

$$d \mapsto b \hat{d}^{-1}$$

then obviously embeds $\mathfrak{D}$ into $\hat{\mathfrak{D}}$ (using the fact that the $\varphi \in \Gamma$'s are homomorphisms on D). So we can regard D as a subset of $\hat{D}$ and Step 1 is ready.

Step 2. We show, that in fact the elements of the form $b \hat{d}^{-1}$ exhaust $\hat{D}$. $\hat{D}$ consists of all the sets $[c]\mu(\hat{l}_1, \dots, \hat{l}_n)^{-1}$, where μ is any n -ary lattice term, and $\hat{l}_1, \dots, \hat{l}_n$ belong to $L_{a,b}^D$. We proceed by induction on the height of μ . If μ is the projection e_i^n , then

$$\begin{aligned} \mu(\hat{l}_1, \dots, \hat{l}_n) &= e_i^n(\hat{l}_1, \dots, \hat{l}_n) = \hat{l}_i, \text{ and so} \\ [c]\mu(\hat{l}_1, \dots, \hat{l}_n)^{-1} &= [c]\hat{l}_i^{-1} = \{\varphi | \hat{l}_i \varphi \geq c\} \\ &= \begin{cases} \emptyset \text{ or } \Gamma & \text{if } \hat{l}_i \notin D, \text{ or } \hat{l}_i \in D \text{ and } b \not\geq c \text{ or } a \geq c, \\ \{\varphi | d\varphi = b\} = b \hat{d}^{-1}, & \text{if } \hat{l}_i \in D, b \geq c \text{ and } a \not\geq c. \end{cases} \end{aligned}$$

In each case, the resulting subset of Γ represents an element of D (0, 1 and d , respectively). Now let for example $\mu(x_1, \dots, x_n) = \mu_1(x_1, \dots, x_n) * \mu_2(x_1, \dots, x_n)$, where $*$ is one of the symbols $\vee$ and $\wedge$,

$$\begin{aligned} [c]\mu(\hat{l}_1, \dots, \hat{l}_n)^{-1} &= \{\gamma | \mu(\hat{l}_1 \varphi_\gamma, \dots, \hat{l}_n \varphi_\gamma) \geq c\} \\ &= \{\gamma | \mu_1(\hat{l}_1 \varphi_\gamma, \dots, \hat{l}_n \varphi_\gamma) * \mu_2(\hat{l}_1 \varphi_\gamma, \dots, \hat{l}_n \varphi_\gamma) \geq c\} \\ &= \bigcup_{\substack{e, f \in L \\ e * f \geq c}} (\{\gamma | \mu_1(\hat{l}_1 \varphi_\gamma, \dots, \hat{l}_n \varphi_\gamma) = e\} \cap \{\gamma | \mu_2(\hat{l}_1 \varphi_\gamma, \dots, \hat{l}_n \varphi_\gamma) = f\}) \end{aligned}$$

(by the monotonicity of μ_i in L)

$$= \bigcup_{\substack{e, f \in L \\ e * f \geq c}} (\{\gamma | \mu_1(l_1\varphi_\gamma, \dots, l_n\varphi_\gamma) \geq e\} \cap \{\gamma | \mu_2(l_1\varphi_\gamma, \dots, l_n\varphi_\gamma) \geq f\}).$$

The sets in this last expression belong to D by the induction hypothesis, so $[c]\mu(l_1, \dots, l_n)^{-1}$ also belongs to D , being D a lattice under $\cap$ and $\cup$ as a subset of $\hat{D}$. This means that $D = \hat{D}$, which was to be proved.

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