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ON THE GENUS OF TORUS LINKS

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Let ℓ be a tame oriented link in a 3-space R^3 satisfying that there is a solid torus V in R^3 whose boundary, ∂V, contains ℓ. For a longitude λ, a meridian μ on ∂V, if the number of points of $\lambda \cap \ell$ and $\mu \cap \ell$ are minimum, λ and μ are said to be *nice with respect to* ℓ. Let λ and μ be nice with respect to ℓ and let p, q be the number of $\mu \cap \ell$, $\lambda \cap \ell$ respectively. Assume that $pq \neq 0$. Then if V is unknotted in R^3, ℓ is said to be *a torus link of type* (p, q) and if V is knotted in R^3, ℓ is said to be *a cable link of type* (p, q) *with a core* c of V and V is said to be *associated to* ℓ. And ℓ is denoted by $\ell_{p,q}$ [3]. If p coincides with the absolute value of the intersection number of μ and ℓ, ℓ is said to be *a link with same orientation* (*on* ∂V).

If $k = k_{p,q}$ is a torus knot, the genus of k, $g(k) = \frac{1}{2}(p-1)(q-1)$ in [3], [4] and if $k = k_{p,q}$ is a cable knot with a core c, $g(k) = \frac{1}{2}(p-1)(q-1) + pg(c)$ in [3], [5].

In this paper, we will consider the genus of torus links and cable links and show the followings.

THEOREM. *Let ℓ be a torus link of type (p, q) with same orientation and let d be the greatest common divisor of p and q. Then $g(\ell) = \frac{1}{2}\{(p-1)(q-1) - d + 1\}$.*

COROLLARY. *Let ℓ be a cable link of type (p, q) with same orientation on ∂V and let d be the greatest common divisor of p and q. Then $g(\ell) = \frac{1}{2}\{(p-1)(q-1) - d + 1\} + pg(c)$* for the core c of V.

We study in the piecewise linear category throughout this paper.

PROOF OF THEOREM

To prove Theorem, we prepare the following special case for Theorem.

LEMMA 1. *Let ℓ_0 be a torus link of type (p, np) with same orientation for an integer n. Then $g(\ell_0) = \frac{1}{2}(p-1)(np-2)$.*

PROOF. Let V be an unknotted solid torus associated to ℓ_0 and let μ, c^* be a meridian, a core of V respectively. For a link $L=pc^* \cup \mu$, we easily see that the Alexander polynomial $\Delta_L(t_1,\dots,t_p,t)=(t-1)^{p-1}$, where pc^* means non-twisted p parallel loops to c in the interior of V hence pc^* is a trivial link in R^3. So $\Delta_{\ell_0}(t_1,\dots,t_p)=(t_1^n\cdots t_p^n-1)^{p-1}(t_1\cdots t_p-1)^{-1}$ by Theorem 3.2 in [1]. Hence $\deg.\Delta_{\ell_0}(t,\dots,t)=p(p-1)n-p$. By the way, for the reduced Alexander polynomial $\Delta_{\ell_0}(t)$ of ℓ_0, deg. $\Delta_{\ell_0}(t)\leqq 2g(\ell_0)+p-1$ and deg. $\Delta_{\ell_0}(t,\dots,t)+1=$ deg. $\Delta_{\ell_0}(t)$ [2]. Therefore $2g(\ell_0)\geqq p(p-1)n-2p+2=(p-1)(np-2)$.

On the other hand, we can easily construct an orientable surface F with $\partial F=\ell_0$ such that $g(F)=\frac{1}{2}(p-1)(np-2)$. So $g(\ell_0)\leqq\frac{1}{2}(p-1)(np-2)$.

LEMMA 2. *Let $\ell=\ell_{p,q}$ be a torus link such that the greatest common divisor of p and q is d. Then ℓ is a cable link of type $\left(d,\frac{pq}{d}\right)$ with a core $k_{\frac{p}{d},\frac{q}{d}}$, a torus knot of type $\left(\frac{p}{d},\frac{q}{d}\right)$.*

PROOF. As each component of ℓ is a torus knot of type $\left(\frac{p}{d},\frac{q}{d}\right)$, the core of V associated to a cable link ℓ is also a torus knot of type $\left(\frac{p}{d},\frac{q}{d}\right)$. If ℓ is a knot, Lemma 2 is trivial. So assume that $d\geqq 2$. As the linking number of each two components of ℓ is $\pm\frac{pq}{d^2}$, ℓ is a cable link of type $\left(d,d\frac{pq}{d^2}\right)=\left(d,\frac{pq}{d}\right)$.

PROOF OF THEOREM. By Lemma 2, $g(\ell_{p,q})=g(\ell_{d,\frac{pq}{d}})$. Let f be a faithful homeomorphism of V associated to $\ell_{d,\frac{pq}{d}}$ onto an unknotted solid torus V^*. As f is faithful, $f(\ell_{d,\frac{pq}{d}})$ is a torus link of type $\left(d,\frac{pq}{d}\right)$ ([6]), denoted by $\ell^*_{d,\frac{pq}{d}}$. Since $\ell_{d,\frac{pq}{d}}$ is a link with same orientation, $\ell^*_{d,\frac{pq}{d}}$ is also a link with same orientation and

$$g(\ell_{d,\frac{pq}{d}}) = g(\ell^*_{d,\frac{pq}{d}})+dg(k_{\frac{p}{d},\frac{q}{d}})$$

by Lemma 2 and Corollary 1.9.3 in [5]. By the way,

$$g(\ell^*_{d,\frac{pq}{d}}) = \frac{1}{2}(d-1)\left(\frac{pq}{d}-2\right)$$

by Lemma 1 for $\frac{pq}{d}$ is devided by d and

$$g(k_{\frac{p}{d},\frac{q}{d}}) = \frac{1}{2}\left(\frac{p}{d}-1\right)\left(\frac{q}{d}-1\right).$$

Therefore we obtain that $g(\ell_{p,q})=g(\ell_{d,\frac{pq}{d}})=\frac{1}{2}\{(p-1)(q-1)-d+1)\}$.

Corollary is obtained by the above Theorem and Corollary 1.9.3. in [5].

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