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ON KNOTS WITH COMPANIONS

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1. Introduction and notation.

In his paper [4], Shibuya proposed a conjecture about knots with companions. In this paper, we will give an affirmative answer.

Let V be a solid torus in 3-sphere S^3 and K be a *geometrically essential* knot in V i.e. any meridian disk of V intersects K. Then we call that $core(V)$ is a companion of K if $K \not\approx core(V)$, where "$\approx$" means the same knot type and $core(V)$ is the core of solid torus V. Let V^* be a solid torus in S^3. We assume that there is an orientation preserving homeomorphism f from V to V^* (orientations of V and V^* are induced from that of S^3). Then Shibuya's conjecture is as follows.

SHIBUYA'S CONJECTURE

If $K \approx f(K)$ *then* $core(V) \approx core(f(V)) = core(V^*)$.

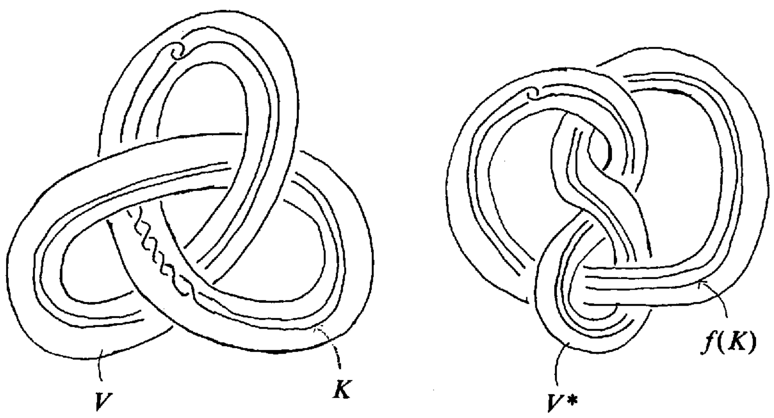

Fig. 1

In this paper we will prove the following.

THEOREM.
$core(V)$ *and* $core(V^*)$ *are knotted in* S^3.

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If $K \approx f(K)$ *then* $core(V) \approx core(f(V)) = core(V^*)$

REMARK.

Recently T. Soma (Kyushu Inst. Univ.) proved the following by using the Gromov invariant:

"$core(V)$ is trivial and $core(V^*)$ is knotted. Then $K \not\approx f(K)$."

Connecting this with Theorem, we obtain an affirmative answer about Shibuya's Conjecture.

Let X be a 3-ball and X^* be a 3-ball in X such that $\partial X^* \cap \partial X$ consists of two disks. We call that 3-ball pair (X, X^*) is knotted if $core(X^*)$ is a knotted 1-ball in X. Then we call that X^* is a knotted 3-ball in X. $U(A; B)$ is a regular neighborhood of A in B and for $A \subset S^3$, we denote $U(A) = U(A; S^3)$. $Int(A)$ or $\mathring{A}$ means the interior of A and $cl(A)$ means the closure of A. For the general notations, we refer to [1]. We work in the PL-category through this paper.

2. Lemmas.

In this section we will prove some lemmas for proving Theorem.

LEMMA 1

Let K be a knot in 3-sphere S^3. Suppose that V and V^* are knotted solid tori in S^3 which contain K and that K is geometrically essential in V and V^*. Then by an ambiant isotopy of S^3 which leaves K fixed, it can be arranged that either

(1) $\partial V \cap \partial V^* = \emptyset$, or

(2) *there exist meridian disks D_1 and D_2 of both V and V^* such that one component of $cl(V^* - \bigcup_{i=1}^{2} D_i)$ is a knotted 3-ball in some component of $cl(V - \bigcup_{i=1}^{2} D_i)$* (see Fig. 2).

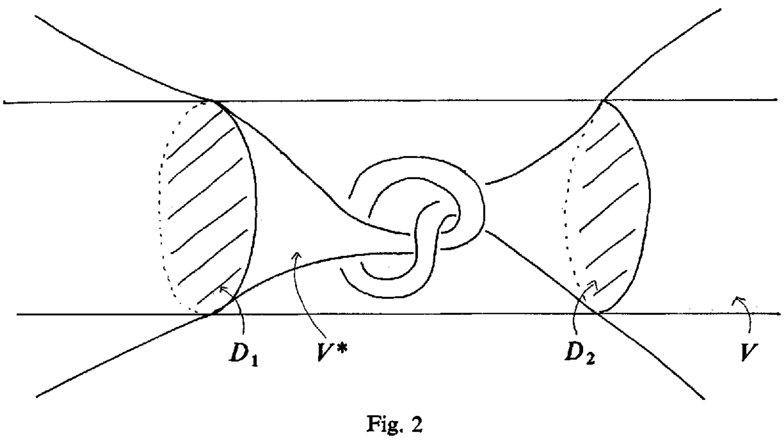

Fig. 2

This lemma is essentially same as Satz 1 (p. 216) in [2], so we omit the proof.

LEMMA 2 (Haken's finiteness theorem)

Let M be a compact 3-manifold. Then there exists the non-negative integer $h(M)$ which is satisfying the following conditions:

(1) *For any set of 2-sided disjoint incompressible surfaces $F_1,\dots, F_n$ in M, if $n>h(M)$, there are F_i and F_j $(i\neq j)$ which are parallel.*

(2) *$h(M)$ is the smallest non-negative integer which satisfies above condition (1).*

(ROOF) see [1].

LEMMA 3.

Let V, V^* and K be the same as in lemma 1. Set f is a homeomorphism from V to V^* with $f(K)=K$. Then by an ambiant isotopy of S^3 which leaves K fixed, it can be arranged that either

(1) *$core(V)\approx core(V^*)$, or*

(2) *there is the solid torus V' satisfying the following conditions.*

  (a) *V' contains K and K is geometrically essential in V'.*

  (b) *$core(V')$ is knotted in S^3.*

  (c) *There exist meridian disks D_1 and D_2 of both V and V' such that one component of $cl(V'-\bigcup_{i=1}^{2} D_i)$ is a knotted 3-ball in some component of $cl(V-\bigcup_{i=1}^{2} D_i)$.*

(PROOF). By lemme 1 the following two cases occur.

case (1) $\partial V\cap\partial V^*=\emptyset$, or

case (2) there exist meridian disk D_1 and D_2 of both V and V^* such that one component of $cl(V^*-\cup D_i)$ is a knotted 3-ball in some component of $cl(V-\cup D_i)$.

In case (2), we put $V'=V^*$. So we may assume that $\partial V\cap\partial V^*=\emptyset$. We divide this into three cases:

case (i) $V^*\subset Int(V)$

case (ii) $V\subset Int(V^*)$

case (iii) $V\cup V^*=S^3$

Set $T=\partial V$ and $T_1=\partial V^*$. If we can prove this lemma in the case (i), we can also prove this in the case (ii) by exchanging V and f for V^* and f^{-1}. So we will prove this in the cases (i) and (iii).

Case (i). If T is parallel to T_1, we obtain $core(V)\approx core(V^*)$. So we may assume that T is not parallel to T_1. Because $V^*=f(V)\subset V$, we can define f^n, so we put $V_n=f^n(V)$ and $T_n=\partial V_n$.

SUBLEMMA 1.

(1) *For any non-negative integer i, T_i is incompressible in $S^3-\mathring{U}(K)$.*

(2) *For any distinct non-negative integers i and j, T_i and T_j are not parallel.*

(Proof of sublemma 1). The proof of (1) is done by induction. When $i=1$ T_1 is incompressilbe in $S^3-\mathring{U}(K)$. We assume that (1) is true when $i=p$ and assume that T_{p+1} is not incompressible in $S^3-\mathring{U}(K)$. There is a compressing disk D^2 in $S^3-\mathring{U}(K)$ i.e. $D^2\cap T_{p+1}=\partial D^2$ and ∂D^2 is not homotopic to zero in T_{p+1}. If $T_p\cap D^2\neq\emptyset$, we consider the innermost disk d i.e. $d\cap T_p=\partial d$ and $d\subset D^2$. Since T_p is incompressible in $S^3-\mathring{U}(K)$, ∂d is homotopic to zero in T_p. So we can cancel the intersection $d\cap T_p$. Thus we may assume $T_p\cap D^2=\emptyset$. Since $T_{p+1}\subset V_p$, $D^2\subset V_p$. Then $f^{-1}(D^2)$ is a compressing disk of T_p. This is a contradiction that T_p is incompressible.

By definition, T_1 is not parallel to T. So $T_2=f(T_1)$ is not parallel to $T_1=f(T)$ and $T_{n+1}=f^n(T_1)$ is not parallel to $T_n=f^n(T)$. Thus for any i and j $(i\neq j)$, T_i and T_j is not parallel. This completes the proof of sublemma 1.

Put $N=h(S^3-\mathring{U}(K))+1$. Then $T_1,\dots,T_N$ is a set of 2-sided disjoint incompressible non-parallel surfaces and $N>h(S^3-\mathring{U}(K))$. This is a contradiction by lemma 2. This completes the proof of the case (i).

Case (iii). For $T_1\subset V$, we can define $T_2=f(T_1)$. Then after deformation by an ambient isotopy, there are possible two cases:

case (a) $T\cap T_2=\emptyset$,

case (b) $T\cap T_2\neq\emptyset$.

Case (a). Set $B=V\cap V^*$ and $A=V_1-\mathring{B}$. Then we have $T_2\subset B$ or $T_2\subset A$. We may assume $T_2\subset B$ by exchanging V for V^* if $T_2\subset A$. Furthermore we divide into two cases:

case (a-i) T_2 is not parallel to T,

case (a-ii) T_2 is parallel to T.

Case (a-i). Since $T_2\subset B$ we can define $T_3=f(T_2)$ and T_3 is not parallel to T_1. Then $T_3\subset B$, so we can define $T_4=f(T_3)$ and T_4 is not parallel to T_2. Thus we can define $T_n=f^n(T)$.

Sublemma 2.

(1) *For any i, T_i is incompressible in $S^3-\mathring{U}(K)$.*

(2) *For any i and j, T_i and T_j are not parallel.*

The proof of this sublemma is essentially same as that of sublemma 1, so we omit the proof.

Put $N=h(S^3-\mathring{U}(K))+1$. Then $T_1,\dots,T_N$ is a set of 2-sided disjoint incompressible non-parallel surfaces and $N>h(S^3-\mathring{U}(K))$. This is a contradiction by lemma 2.

Case (a-ii). We may assume that $T_2=T$. We put $A'=V-\mathring{B}$. Since *core*(V) is knotted, we may assume that a meridian disk D_1 of V_1 is in B. So $f(D_1)$ is a meridian disk of V, f maps a meridian of T_1 to that of T. By the same argument f maps a longitude of T_1 to that of T. A' is homeomorphic to $S^3-\mathring{V}_1=S^3-\mathring{U}$

$(core(V^*))$ and A is homeomorphic to $S^3-\mathring{V}=S^3-\mathring{U}(core(V))$. So $f|_{A'}$ is an orientation preserving homeomorphism from A' to A which maps a meridian of T_1 and a longitude of T_1 to those of T, $core(V)\approx core(V^*)$.
Case (b). Since T_2 is incompressible in $S^3-\mathring{U}(K)$ and not incompressible in S^3, A component of $S^3-\mathring{U}(T_2)$ which contains K is homeomorphic to solid torus. We denote this by V'. After deformation by an ambiant isotopy, we may assume that the possible case in this lemma is the case (2). This completes the proof.

3. Proof of Theorem.

Now we begin the proof of Theorem. We recall the situation. Let V be a solid torus in S^3 and K be a geometrically essential knot in V. f is an orientation preserving homeomorphism from V to V^*. Because $K\approx f(K)$, by ambiant isotopy, we may assume that $K=f(K)$. The proof is done by induction about $h(V-\mathring{U}(K))$.
(1) $h(V-\mathring{U}(K))=1$. Since K is geometrically essential, $K\approx core(V)$. So $core(V)\approx K\approx f(K)\approx core(f(V))=core(V^*)$.
(2) $h(V-\mathring{U}(K))=n\geqq 2$. We assume that Theorem is ture for the case $h(V-\mathring{U}(K))<n$. By lemma 3, the possible cases are the case (1) and case (2) in lemma 3. Since the case (1) is a conclusion of Theorem, we assume that there is the solid torus V' satisfying the following conditions: (a) V' contains K and K is geometrically essential in V'. (b) $core(V')$ is knotted in S^3. (c) There exist meridan disks D_1 and D_2 of both V and V' such that one component of $cl(V'-\bigcup_{i=1}^{2} D_i)$ (say X') is a knotted 3-ball in some component of $cl(V-\bigcup_{i=1}^{2} D_i)$ (say X). We put $Y=V-X$, $W'=Y\cup X'$ and $W=W'-\mathring{U}(\partial W'\colon W')$. Then W is a knotted solid torus in V, $core(W)$ is geometrically essential in V and K is geometrically essential in W. We dived into two cases:

case (a) $core(W)\approx K$,

case (b) $core(W)\not\approx K$.

Case (a). We denote, by α, a knot obtained from a knotted ball pair $(X, core(X'))$ attaching a trivial ball pair. Then $K\approx core(W)\approx core(V)\sharp\alpha$, where "$\sharp$" means a connected sum of knots [3]. Thus

$$core(V)\sharp\alpha \approx core(W) \approx K = f(K) \approx core(f(W)) \approx core(f(V))\sharp\alpha.$$

By the unique decomposition theorem of knots [3], we obtain $core(V)\approx core(f(V)) = core\ (V^*)$.
Case (b). Because $core(W)\not\approx K$, we obtain $h(V-\mathring{U}(K))>h(W-\mathring{U}(K))$ and $h(V-\mathring{U}(K))>h(V-\mathring{U}(core(W)))$. By the assumption of induction, $core(W)\approx core(f(W))$. Combining this with $h(V-\mathring{U}(K))>h(V-\mathring{U}(core(W)))$, we obtain $core(V)\approx core(V^*)$. This completes the proof of Theorem.

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