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THE COUNTABLE-OPEN TOPOLOGY IN THE LOCALLY CONVEX SETTING

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Let X be a completely regular Hausdorff space and E a real or complex Hausdorff locally convex space whose topology is generated by a family $\{p: p \in I\}$ of continuous semi-norms on E. Let $C_b(X, E)$ denote the vector space of all bounded continuous E-valued functions on X. When E is the real or complex field, this space is denoted by $C_b(X)$. The notion of the countable-open topology on $C_b(X)$ was briefly introduced by Gulick and Schmets in ([3], p. 256). In this paper, we define the countable-open topology on the space $C_b(X, E)$ and discuss its relation with some other locally convex topologies on $C_b(X, E)$. We also consider its completeness and characterize its separability in terms of X and E. Many properties of this topology resemble with those of the σ-compact-open topology studied in [2].

DEFINITION. The *countable-open topology* σ_0 (resp. *σ-compact-open topology* σ, *compact-open topology* κ, *point-open topology* ρ) on $C_b(X, E)$ is defined by the family $\{\|\cdot\|_{A,p}: A \in \mathscr{A},\ p \in I\}$ of semi-norms, where $\mathscr{A}$ consists of all countable (resp. σ-compact, compact, finite) subsets of X and $\|f\|_{A,p} = \sup \cdot \{p(f(x)): x \in A\}, f \in C_b(X, E)$. The *uniform topology* u on $C_b(X, E)$ is given by the semi-norms $\|\cdot\|_p = \|\cdot\|_{X,p},\ p \in I$.

We shall denote by $C_b(X) \otimes E$ the vector space spanned by the set of all functions of the form $g \otimes a$, where $g \in C_b(X)$, $a \in E$, and $(g \otimes a)(x) = g(x)a\ (x \in X)$.

THEOREM 1. (i) *$\rho \leq \sigma_0 \leq \sigma \leq u$ and $\rho \leq \kappa \leq \sigma$.*
(ii) *$\sigma_0 = u$ iff X is separable.*
(iii) *$\sigma_0 = \sigma$ iff every σ-compact subset of X has separable closure in X.*
(iv) (a) *$\kappa \leq \sigma_0$ iff every compact subset of X is separable.*
(b) *$\sigma_0 \leq \kappa$ iff every countable subset of X is relatively compact.*
(v) *σ_0 and u have the same bounded sets in $C_b(X, E)$.*

PROOF. (i) This follows easily from the definition.
(ii) Suppose there is a countable subset $A \subseteq X$ such that $\bar{A} = X$. For any $P \in I$, let $W = \{f \in C_b(X, E): \|f\|_p \leq 1\}$ be a u-neighbourhood of 0 in $C_b(X, E)$. Then $V = \{f \in C_b(X, E): \|f\|_{A,p} \leq 1\}$ is a σ_0-neighbourhood of 0 contained in W; hence

$u \leq \sigma_0$. Conversely, suppose $u \leq \sigma_0$ but $X \neq \bar{B}$ for every countable $B \subseteq X$. Choose a $p_1 \in I$ and a $c \in E$ with $p_1(c)=2$. There exists a countable $D \subseteq X$ and a $p_2 \in I$ such that

$$W_2 = \{f \in C_b(X, E): \|f\|_{D,p_2} \leq 1\} \subseteq W_1 = \{f \in C_b(X, E): \|f\|_{p_1} \leq 1\}.$$

Let $y \in X \setminus \bar{D}$. By complete regularity of X, we can choose a $g \in C_b(X)$ with $0 \leq g \leq 1$, $g(y)=1$, $g(\bar{D})=0$. Then $g \otimes c \in W_2$ but $g \otimes c \notin W_1$, a contradiction.

(iii) and (iv) follow by the argument simmilar to the one used in (ii).

(v) Since $\sigma_0 \leq u$, every u-bounded set in $C_b(X, E)$ is σ_0-bounded. Now, suppose there is a set $H \subseteq C_b(X, E)$ which is σ_0-bounded but not u-bounded. Then there exist sequences $\{f_n\} \subseteq H$, $\{x_n\} \subseteq X$, and a semi-norm $p \in I$ such that $p(f_n(x_n)) \geq n^2$ for all $n \geq 1$. Let $A = \{x_n\}$. Then H is not absorbed by the σ_0-neighbourhood $\{f \in C_b(X, E): \|f\|_{A,p} \leq 1\}$, a contradiction.

THEOREM 2. (i) *If $(C_b(X, E), \sigma_0)$ is bornological, then $\sigma_0 = u$.*
(ii) *If E is metrizable, then the following are equivalent: (a) $\sigma_0 = u$; (b) $(C_b(X, E), \sigma_0)$ is metrizable; (c) $(C_b(X, E), \sigma_0)$ is bornological.*

PROOF. (i) Suppose $(C_b(X, E), \sigma_0)$ is bornological. Since σ_0 and u have the same bounded sets, the identity map i: $(C_b(X, E), \sigma_0) \to (C_b(X, E), u)$ is continuous ([4], Ch. II, Theorem 8.3). Hence $u \leq \sigma_0$.
(ii) If E is metrizable, then $(C_b(X, E), u)$ is also metrizable, and so (a)⇒(b). Next, (b)⇒(c) follows from ([4], Ch. II, Theorem 8.1), and (c)⇒(a) follows from part (i).

We now consider the completeness and sequential completeness of the countable-open topology.

THEOREM 3. (i) *If X is a k-space, E is complet and $k \leq \sigma_0$, then $(C_b(X, E), \sigma_0)$ is complete.*
(ii) *If E is sequentially complete, then so is $(C_b(X, E), \sigma_0)$.*

PROOF. (i) Let $\{f_\alpha\}$ be a σ_0-Cauchy net in $C_b(X, E)$, and let $f(x) = \lim_\alpha f_\alpha(x)$ $(x \in X)$. For any countable set $A \subseteq X$ and $p \in I$, there exists an index α_0 such that $p(f_\alpha(x) - f_\lambda(x)) \leq 1$ for all $x \in A$ and $\alpha, \lambda \geq \alpha_0$. Then, for any fixed $x \in A$ and $\alpha \geq \alpha_0$, $p(f_\alpha(x) - f(x)) \leq 1$. Hence $f_\alpha \xrightarrow{\sigma_0} f$. Next, suppose f is not bounded. Then, there exist a $p_1 \in I$ and a sequence $\{x_n\} \subseteq X$ such that $p_1(f(x_n)) \geq n^2$ for all $n \geq 1$. If $B = \{x_n\}$, we can choose an index α_1 such that $p_1(f_\alpha(x) - f(x)) \leq 1$ for all $x \in B$ and $\alpha \geq \alpha_1$. Choose $r \geq 2$ such that $\sup\{p_1(f_{\alpha_1}(x)): x \in X\} \leq r$. Then, for any $n > r$,

$$p_1(f(x_n)) \leq p_1(f(x_n) - f_{\alpha_1}(x_n)) + p_1(f_{\alpha_1}(x_n)) \leq 1 + r < n^2,$$

a contradiction. Hence f is bounded. Since $\kappa \leq \sigma_0$, $f_\alpha \to f$ uniformly on every

compact subset of X. Since X is a k-space, it follows that f is continuous. Thus $f \in C_b(X, E)$, as required.
(ii) Let $\{f_n\}$ be a σ_0-Cauchy sequence in $C_b(X, E)$. Then, as in part (i), there exists a bounded E-valued function f on X such that $f_n \xrightarrow{\sigma_0} f$. Suppose f is not continuous at some $x_0 \in X$. Then there exist a net $\{x_\lambda: \lambda \in \Lambda\} \subseteq X$, a $p \in I$, and an $\varepsilon > 0$ such that $x_\lambda \to x_0$ but $p(f(x_\lambda) - f(x_0)) > \varepsilon$ for all $\lambda \in \Lambda$. Correspondence to each f_n, choose an index $\lambda_n \in \Lambda$ such that $p(f_n(x_\lambda) - f(x_0)) < \varepsilon/4$ for all $\lambda \geq \lambda_n$. If $A = \{x_0\} \cup \{x_{\lambda_n}\}$, there exists an integer N such that $\|f_n - f\|\|_{,p} < \varepsilon/4$ for all $n \geq N$. Then, for any $n \geq N$,
$p(f(x_{\lambda_n}) - f(x_0)) \leq p(f(x_{\lambda_n}) - f_n(x_{\lambda_n})) + p(f_n(x_{\lambda_n}) - f_n(x_0)) + p(f_n(x_0) - f(x_0)) < \varepsilon,$
a contradiction. Thus $(C_b(X, E), \sigma_0)$ is sequentially complete.

Finally, we characterize the separability of $(C_b(X, E), \sigma_0)$.

THEOREM 4. *The following are equivalent.*
(a) *X is a compact metric space and E is separable.*
(b) *$(C_b(X, E), \sigma_0)$ is separable.*

PROOF. (a)$\Rightarrow$(b) If X is a compact metric space, then, by a classical result of M. and S. Krein (see [3], Theorem 1), $(C_b(X), \|\cdot\|)$ is separable. Let $\{g_m\}$ and $\{a_n\}$ be countable dense subset of $(C_b(X), \|\cdot\|)$ and E respectively, and let G be the countable subspace generated by $\{g_m \otimes a_n: m, n = 1, 2, \ldots\}$ over rationals. Then G is u-dense in $C_b(X) \otimes E$ as follows. Let $f = \sum_{i=1}^{q} f_i \otimes b_i$ ($f_i \in C_b(X)$, $b_i \in E$) be in $C_b(X \otimes E$ and $p \in I$. Let $r = \max\{p(b_i): 1 \leq i \leq q\}$ and $s = \max\{\|f_i\|: 1 \leq i \leq q\}$. Put $M = 2q(r+1)(s+1)$. Choose $g_{m_i} \in \{g_m\}$ and $a_{n_i} \in \{a_n\}$ such that $\|g_{m_i} - f_i\| \leq 1/M$ and $p(a_{n_i} - b_i) \leq 1/M$ $(1 \leq i \leq q)$. Let $g = \sum_{i=1}^{q} g_{m_i} \otimes a_{n_i}$. Then $g \in G$ and, for any $x \in X$,

$$p(g(x) - f(x)) \leq \sum_{i=1}^{q} |g_{m_i}(x)| p(a_{n_i} - b_i) + \sum_{i=1}^{q} |g_{m_i}(x) - f_i(x)| p(b_i)$$

$$< 1/2(r+1) + r/2(r+1)(s+1) < 1.$$

Now, by ([1], Ch. III.1, Proposition 1 and Lemma 2), $C_b(X) \otimes E$ is u-dense in $C_b(X, E)$. Since $\sigma_0 \leq u$, it follows that $(C_b(X, E), \sigma_0)$ is separable.
(b)$\Rightarrow$(a) Let $\{f_n\}$ be a countable σ_0-dense subset of $C_b(X, E)$. Choose a nonzero $\varphi \in E'$, the topological dual of E. Then it is easy to show that $\{\varphi \circ f_n\}$ is σ_0-dense in $C_b(X)$. Hence $(C_b(X), \sigma_0)$ is separable and so, by ([3], Theorems 4 and 7), X is a compact metric space. Next, for any fixed $z \in X$, $\{f_n(z)\}$ is dense in E. This completes the proof.

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